A Very Brief Note on the Riemann Hypothesis

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Abstract. Robin's criterion states that the Riemann Hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \cdot n \cdot \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. We require the properties of superabundant numbers, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. In this note, using Robin's inequality on superabundant numbers, we prove that the Riemann Hypothesis is true. This proof is an extension of the article "Robin's criterion on divisibility" published by The Ramanujan Journal on May 3rd, 2022.

Keywords: Riemann Hypothesis · Robin's inequality · Sum-of-divisors function · Superabundant numbers · Prime numbers.

1 Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859). The Riemann Hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer d divides n. Define $f(n)$ as $\frac{\sigma(n)}{n}$. We say that $\mathsf{Robin}(n)$ holds provided that

$$f(n) < e^{\gamma} \cdot \log \log n,$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The Ramanujan's Theorem states that if the Riemann Hypothesis is true, then the previous inequality holds for large enough n. Next, we have the Robin's Theorem:

Proposition 1. $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann Hypothesis is true [6, Theorem 1 pp. 188].*

It is known that $\mathsf{Robin}(n)$ holds for many classes of natural numbers n.

Proposition 2. *$\mathsf{Robin}(n)$ holds for all natural numbers $n > 5040$ such that $q \leq e^{31.018189471}$, where q is the largest prime divisor of n [7, Theorem 4.2 pp. 4].*

Superabundant numbers were defined by Leonidas Alaoglu and Paul Erdős (1944). In 1997, Ramanujan's old notes were published where he defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers. Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer [3, pp. 367]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

We know the following property for the superabundant numbers:

Proposition 3. *If n is superabundant, then n is a Hardy-Ramanujan integer [1, Theorem 1 pp. 450].*

Proposition 4. *[1, Theorem 7 pp. 454]. Let n be a superabundant number such that p is the largest prime factor of n, then*

$$p \sim \log n, \quad (n \to \infty).$$

Proposition 5. *[1, Theorem 9 pp. 454]. For some constant $c > 0$, the number of superabundant numbers less than x exceeds*

$$\frac{c \cdot \log x \cdot \log \log x}{(\log \log \log x)^2}.$$

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \;\; for \;\; (m > 1).$$

There is a close relation between the superabundant and colossally abundant numbers.

Proposition 6. *Every colossally abundant number is superabundant [1, pp. 455].*

Several analogues of the Riemann Hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann Hypothesis might be false.

Proposition 7. *If the Riemann Hypothesis is false, then there are infinitely many colossally abundant numbers $n > 5040$ such that $\mathsf{Robin}(n)$ fails (i.e. $\mathsf{Robin}(n)$ does not hold) [6, Proposition pp. 204].*

In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n.

Proposition 8. *[5, Theorem 4.4 pp. 12]. Let n be a superabundant number such that p is the largest prime factor of n and $2 \leq q \leq p$, then*

$$\left\lfloor \frac{\log p}{\log q} \right\rfloor \leq \nu_q(n).$$

Putting all together yields the proof of the Riemann Hypothesis.

2 Main Results

The following is a key Lemma.

Lemma 1. *If the Riemann Hypothesis is false, then there are infinitely many superabundant numbers n such that $\mathsf{Robin}(n)$ fails.*

Proof. This is a direct consequence of Propositions 1, 6 and 7. □

For every prime number $q_k > 2$, we define the sequence

$$Y_k = \frac{e^{\frac{0.2}{\log^2(q_k)}}}{(1 - \frac{0.01}{\log^3(q_k)})}.$$

As the prime number q_k increases, the sequence Y_k is strictly decreasing [7, Lemma 6.1 pp. 6]. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x. We know the following Propositions:

Proposition 9. *[2, Lemma 2.7 pp. 19]. For $x \geq 7232121212$:*

$$\theta(x) \geq (1 - \frac{0.01}{\log^3(x)}) \cdot x.$$

Proposition 10. *[2, Lemma 2.7 pp. 19]. For $x \geq 2278382$:*

$$\prod_{q \leq x} \frac{q}{q-1} \leq e^{\gamma} \cdot (\log x + \frac{0.2}{\log^2(x)}).$$

We prove another important inequality:

Lemma 2. *Let $q_1, q_2, \ldots, q_k$ denote the first k consecutive primes such that $q_1 < q_2 < \ldots < q_k$ and $q_k > 7232121212$. Then*

$$\prod_{i=1}^{k} \frac{q_i}{q_i - 1} \leq e^{\gamma} \cdot \log \left(Y_k \cdot \theta(q_k) \right).$$

Proof. By the Proposition 9, we know that

$$\theta(q_k) \geq (1 - \frac{0.01}{\log^3(q_k)}) \cdot q_k.$$

In this way, we can show that

$$\begin{aligned}\log(Y_k \cdot \theta(q_k)) &\geq \log\left(Y_k \cdot (1 - \frac{0.01}{\log^3(q_k)}) \cdot q_k\right)\\ &= \log q_k + \log\left(Y_k \cdot (1 - \frac{0.01}{\log^3(q_k)})\right).\end{aligned}$$

We know that

$$\begin{aligned}\log\left(Y_k \cdot (1 - \frac{0.01}{\log^3(q_k)})\right) &= \log\left(\frac{e^{\frac{0.2}{\log^2(q_k)}}}{(1 - \frac{0.01}{\log^3(q_k)})} \cdot (1 - \frac{0.01}{\log^3(q_k)})\right)\\ &= \log\left(e^{\frac{0.2}{\log^2(q_k)}}\right)\\ &= \frac{0.2}{\log^2(q_k)}.\end{aligned}$$

Consequently, we obtain that

$$\log q_k + \log\left(Y_k \cdot (1 - \frac{0.01}{\log^3(q_k)})\right) \geq (\log q_k + \frac{0.2}{\log^2(q_k)}).$$

By Proposition 10, we prove that

$$\prod_{i=1}^{k} \frac{q_i}{q_i - 1} \leq e^{\gamma} \cdot (\log q_k + \frac{0.2}{\log^2(q_k)}) \leq e^{\gamma} \cdot \log(Y_k \cdot \theta(q_k))$$

when $q_k > 7232121212$. □

We use the following Propositions:

Proposition 11. *[4, Lemma 1 pp. 2]. Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of n as a product of prime numbers $q_1 < \ldots < q_k$ with natural numbers $a_1, \ldots, a_k$ as exponents. Then,*

$$f(n) = \prod_{i=1}^{k} \frac{q_i^{a_i+1} - 1}{q_i^{a_i} \cdot (q_i - 1)} = \left(\prod_{i=1}^{k} \frac{q_i}{q_i - 1}\right) \cdot \left(\prod_{i=1}^{k} (1 - \frac{1}{q_i^{a_i+1}})\right).$$

Proposition 12. *[5, Lemma 3.3 pp. 8]. Let $x \geq 11$. For $y > x$ we have*

$$\frac{\log\log y}{\log\log x} < \sqrt{\frac{y}{x}}.$$

This is the main insight.

Theorem 1. *Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of a superabundant number $n > 5040$ as the product of the first k consecutive primes $q_1 < \ldots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ as exponents. Suppose that* $\mathsf{Robin}(n)$ *fails. Then,*

$$n < \alpha^2 \cdot (N_k)^{Y_k},$$

where $N_k = \prod_{i=1}^{k} q_i$ is the primorial number of order k and $\alpha = \prod_{i=1}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$.

Proof. When $\mathsf{Robin}(n)$ fails, then $q_k > e^{31.018189471}$ by Proposition 2. By the Proposition 11, we notice that

$$f(n) = \left(\prod_{i=1}^{k} \frac{q_i}{q_i - 1}\right) \cdot \prod_{i=1}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

However, we know that

$$\prod_{i=1}^{k} \frac{q_i}{q_i - 1} \leq e^{\gamma} \cdot \log\left(Y_k \cdot \theta(q_k)\right)$$

by Lemma 2, when $q_k > e^{31.018189471} > 7232121212$. If we multiply both sides by the value of $\alpha = \prod_{i=1}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$, then we obtain that

$$f(n) \leq e^{\gamma} \cdot \log\left(Y_k \cdot \theta(q_k)\right) \cdot \alpha.$$

Since $\mathsf{Robin}(n)$ fails, then

$$e^{\gamma} \cdot \log\log n \leq e^{\gamma} \cdot \log\left(Y_k \cdot \theta(q_k)\right) \cdot \alpha$$

due to

$$e^{\gamma} \cdot \log\log n \leq f(n).$$

That's the same as

$$\log\log n \leq \log\left(Y_k \cdot \theta(q_k)\right) \cdot \alpha$$

which is equivalent to

$$\frac{\log\log n}{\log\left(Y_k \cdot \theta(q_k)\right)} \leq \alpha.$$

We know that

$$\log\left(Y_k \cdot \theta(q_k)\right) = \log\log(N_k)^{Y_k}.$$

We assume that $(N_k)^{Y_k} > n > 5040 > 11$ since $0 < \alpha < 1$. Consequently,

$$\sqrt{\frac{n}{(N_k)^{Y_k}}} < \frac{\log\log n}{\log\log(N_k)^{Y_k}}$$

by Proposition 12. In this way, we obtain that

$$n < \alpha^2 \cdot (N_k)^{Y_k}$$

and therefore, the proof is done. □

Corollary 1. *Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of a superabundant number $n > 5040$ as the product of the first k consecutive primes $q_1 < \ldots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ as exponents. If $\mathsf{Robin}(n)$ fails, then*

$$n < \alpha^2 \cdot (N_k)^{1.000208229291},$$

where $N_k = \prod_{i=1}^{k} q_i$ is the primorial number of order k and $\alpha = \prod_{i=1}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$.

Proof. For $q_k > e^{31.018189471}$, we know that $Y_k < 1.000208229291$ after of evaluating in the value of q_k due to Y_k is strictly decreasing. □

This is the main theorem.

Theorem 2. *The Riemann Hypothesis is true.*

Proof. There are infinitely many superabundant numbers by Proposition 5. Let $n > 5040$ be a large enough superabundant number. Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of this superabundant number n as the product of the first k consecutive primes $q_1 < \ldots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ as exponents, since n must be a Hardy-Ramanujan integer by Proposition 3. Suppose that $\mathsf{Robin}(n)$ fails. By Theorem 1, we have

$$n < \alpha^2 \cdot (N_k)^{Y_k}$$

which is the same as

$$\frac{n}{N_k} < \alpha^2 \cdot (N_k)^{Y_k - 1}.$$

For every prime r, $\nu_r(n)$ goes to infinity as long as n goes to infinity when n is superabundant by Propositions 4 and 8. In this way, the fraction $\frac{n}{N_k}$ is strictly increasing for large enough k. Moreover, the value of $(N_k)^{Y_k - 1}$ is strictly decreasing for large enough k. Certainly, we have $(N_k)^{Y_k - 1} > (N_{k+1})^{Y_{k+1} - 1}$ which implies that $(N_k)^{Y_k - Y_{k+1}} > (q_{k+1})^{Y_{k+1} - 1}$ for every large enough k. This is is possible since $\lim_{k \to \infty} Y_k = 1$. Since the superabundant number n is large enough, then we deduce that

$$\frac{n}{N_k} \geq \alpha^2 \cdot (N_k)^{Y_k - 1}$$

can be always satisfied because of Proposition 4. In this way, we obtain a contradiction under the assumption that $\mathsf{Robin}(n)$ fails. To sum up, the study of this arbitrary large enough superabundant number n reveals that $\mathsf{Robin}(n)$ holds on anyway. Accordingly, $\mathsf{Robin}(n)$ holds for all large enough superabundant numbers n. This contradicts the fact that there are infinitely many superabundant numbers n, such that $\mathsf{Robin}(n)$ fails when the Riemann Hypothesis is false according to Lemma 1. By reductio ad absurdum, we prove that the Riemann Hypothesis is true. □

3 Conclusions

Practical uses of the Riemann Hypothesis include many propositions that are known to be true under the Riemann Hypothesis and some that can be shown to be equivalent to the Riemann Hypothesis. Indeed, the Riemann Hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf Hypothesis, the Large Prime Gap Conjecture, etc. Certainly, a proof of the Riemann Hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

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