

Commutative Ring and Field on the Binomial Coefficients of Combinatorial Geometric Series

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Abstract: This paper discusses an abelian group, ring, and field under addition and multiplication of the binomial coefficients in combinatorial geometric series. The coefficient for each term in combinatorial geometric series refers to a binomial coefficient. This idea can enable the scientific researchers to solve the real life problems.

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1. Introduction

When the author of this article was trying to develop the multiple summations of geometric series, a new idea was stimulated his mind to create a combinatorial geometric series [1-9]. The combinatorial geometric series is a geometric series whose coefficient of each term of the geometric series denotes the binomial coefficient V_n^r . In this article, an abelian group, ring, and field are discussed on the binomial coefficients of combinatorial geometric series.

2. Combinatorial Geometric Series

The combinatorial geometric series [1-9] is derived from the multiple summations of geometric series. The coefficient of each term in the combinatorial refers to the binomial coefficient V_n^r .

$$\sum_{i_1=0}^n \sum_{i_2=i_1}^n \sum_{i_3=i_2}^n \cdots \sum_{i_r=i_{r-1}}^n x^{i_r} = \sum_{i=0}^n V_i^r x^i \quad \& \quad V_n^r = \frac{(n+1)(n+2)(n+3) \cdots (n+r-1)(n+r)}{r!},$$

where $n \geq 0, r \geq 1$ and $n, r \in N = \{1, 2, 3, \dots\}$.

Here, $\sum_{i=0}^n V_i^r x^i$ refers to the combinatorial geometric series and

V_n^r is the binomial coefficient for combinatorial geometric series.

$V_0^1 = 1; V_1^1 = 2; V_2^1 = 3; V_3^1 = 4; V_4^1 = 5; V_5^1 = 6; \dots$

$N = \{V_0^1, V_1^1, V_2^1, V_3^1, V_4^1, V_5^1, \dots\}$ is a set of natural numbers.

$W = \{0, V_0^1, V_1^1, V_2^1, V_3^1, V_4^1, V_5^1, \dots\}$ is a set of whole numbers.

$Z = \{\dots, -V_2^1, -V_1^1, -V_0^1, 0, V_0^1, V_1^1, V_2^1, \dots\}$ is a set of integers.

$\{+, -, \times, \div, \dots\}$ is a set of binary operators, where $+$ is used for addition,
–for subtraction, \times for multiplication, \div for division, etc.

Note that $V_0^r + V_1^r + V_2^r + V_3^r + \dots + V_{n-1}^r + V_n^r = V_n^{r+1} \Rightarrow V_{n-1}^{r+1} + V_n^r = V_n^{r+1}$,

where $V_n^r = V_0^{r-1} + V_1^{r-1} + V_2^{r-1} + \dots + V_n^{r-1}$ and $V_0^r + V_1^r + V_2^r + \dots + V_{n-1}^r = V_{n-1}^{r+1}$.

Proof for $V_0^r + V_1^r + V_2^r + V_3^r + \dots + V_{n-1}^r + V_n^r = V_n^{r+1}$.

$$V_0^r + V_1^r + V_2^r + V_3^r + \dots + V_{n-1}^r + V_n^r = V_n^{r+1} \Rightarrow V_{n-1}^{r+1} + V_n^r = V_n^{r+1}.$$

$$\text{Note that } V_n^r = \frac{(n+r)!}{n!r!}.$$

$$V_n^r + V_{n-1}^{r+1} = \frac{(n+r)!}{n!r!} + \frac{(n-1+r+1)!}{(n-1)!(r+1)!} = (n+r)! \left(\frac{r+1}{n!(r+1)!} + \frac{n}{n!(r+1)!} \right).$$

$$V_n^r + V_{n-1}^{r+1} = (n+r)! \left(\frac{n+r+1}{n!(r+1)!} \right) = \frac{(n+r+1)!}{n!(n+r)!} = V_n^{r+1}.$$

$$\therefore V_n^r + V_{n-1}^{r+1} = V_n^{r+1}.$$

Hence, it is proved.

Also, $V_n^r = V_n^{r-1} + V_r^{n-1}$, which is the sum of partitions of V_n^r .

In general, addition or multiplication of any two binomial coefficients is an integer as all binomial coefficients are integers such that $V_n^r \in \mathbb{N}$.

3. Abelian Group, Ring and Field

$Z = \{-V_r^n, 0, V_r^n \mid n \geq 1, r \geq 0 \text{ \& } n, r \in \mathbb{N}\}$ is a set of integers.

Closure property: Addition of any two binomial coefficients is also a binomial coefficient.

$$(V_m^n + V_p^q) \in Z \text{ for all } V_m^n, V_p^q \in Z.$$

Associativity: For all $V_m^n, V_p^q, V_r^s \in Z$, $V_m^n + (V_p^q + V_r^s) = (V_m^n + V_p^q) + V_r^s$.

Identity element: $0 + V_r^n = V_r^n + 0 = V_r^n$, where 0 is an identity element.

Inverse element: $V_r^n + (-V_r^n) = (-V_r^n) + V_r^n = 0$, where $-V_r^n$ is an additive inverse.

Commutativity: $V_m^n + V_p^q = V_p^q + V_m^n$ for all $V_m^n, V_p^q \in Z$.

$(Z, +)$ is an abelian group under addition [1].

A **RING** is a non-empty set R which is CLOSED under two binary operators $+$ and \times and satisfying the following properties:

(1) R is an abelian group under $+$.

(2) R is an associativity of \times . For $a, b, c \in R$, $a \times (b \times c) = (a \times b) \times c$.

(3) R has distributive properties, i.e. for all $a, b, c \in R$ the following identities hold:

$$a \times (b + c) = (a \times b) + (a \times c) \text{ and } (b + c) \times a = (b \times a) + (c \times a).$$

$\therefore (Z, +, \times)$ is a **RING** (Note that $(Z, +, \times)$ is a **Ring with Unity** which has 1 as identity element such that $1 \times V_r^n = V_r^n \times 1 = V_r^n$ and also Commutative Ring : $V_m^n \times V_p^q = V_p^q \times V_m^n$).

A **FIELD** is a non-empty set F which is CLOSED under two binary operators $+$ and \times and satisfying the following properties:

- (1) F is an abelian group under $+$.
- (2) $F - \{0\}$ is an abelian group under \times .

$\therefore (Z, +, \times)$ is a **FIELD**.

4. Conclusion

In this article, an abelian group, ring, and field were formed on the binomial coefficients of combinatorial geometric series under addition and multiplication. This new idea can enable the scientific researchers for research and development further.

References

- [1] Annamalai, C. (2022) Abelian Group on the Binomial Coefficients of Combinatorial Geometric Series. *COE, Cambridge University Press*. <https://doi.org/10.33774/coe-2022-mzwxk>.
- [2] Annamalai, C. (2022) Computation of Combinatorial Geometric Series and its Combinatorial Identities for Machine Learning and Cybersecurity. *COE, Cambridge University Press*. <https://doi.org/10.33774/coe-2022-b6mks>.
- [3] Annamalai, C. (2022) Novel Binomial Series and its Summations. *SSRN Electronic Journal*. <http://dx.doi.org/10.2139/ssrn.4078523>.
- [4] Annamalai, C. (2022) Successive Partition Method for Binomial Coefficient in Combinatorial Geometric Series. *Zenodo*. <https://doi.org/10.5281/zenodo.7060159>.
- [5] Annamalai, C. (2022) Partition of Multinomial Coefficient. *Zenodo*. <https://doi.org/10.5281/zenodo.7063004>.
- [6] Annamalai, C. (2022) A Generalized Method for proving the Theorem derived from the Binomial Coefficients in Combinatorial Geometric Series. *Zenodo*. <https://doi.org/10.5281/zenodo.7047548>.
- [7] Annamalai, C. (2022) Computation Method for Combinatorial Geometric Series and its Applications. *COE, Cambridge University Press*. <https://doi.org/10.33774/coe-2022-pnx53-v22>.
- [8] Annamalai, C. (2022) Computing Method for Combinatorial Geometric Series and Binomial Expansion. *SSRN Electronic Journal*. <http://dx.doi.org/10.2139/ssrn.4168016>.
- [9] Annamalai, C. (2022) Factorials and Integers for Applications in Computing and Cryptography. *COE, Cambridge University Press*. <https://doi.org/10.33774/coe-2022-b6mks>.