Goldbach's Conjecture — A Route to the Inconsistency of Arithmetic

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Abstract. This paper proves an inconsistency in Peano arithmetic (PA) by showing that a strengthened form of the strong Goldbach conjecture as well as its negation can be derived. This contradiction is the consequence of two properties of a specific set which we use to reformulate the conjecture.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *Both SSGB and the negation ¬SSGB hold*.

Proof. We define the set S_g := { (pk, mk, qk) | k, m $\in \mathbb{N}$; p, q $\in \mathbb{P}_3$, p < q; m = (p + q) / 2 }.

SSGB is equivalent to saying that every integer x ≥ 4 is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers x ≥ 4 appear as m in a middle component mk of S_g. So, by the definition of S_g we have

$$\text{SSGB} \iff \forall x \in \mathbb{N}_4 \quad \exists (pk, mk, qk) \in S_g \quad x = m.$$

$$\neg\text{SSGB} \iff \exists x \in \mathbb{N}_4 \quad \forall (pk, mk, qk) \in S_g \quad x \neq m.$$

The set S_g has the following two properties.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g ("*covering*"), because every integer x ≥ 3 can be written as some pk with k = 1 when x is prime, as some pk with k ≠ 1 when x is composite and not a power of 2, or as (3 + 5)k / 2 when x is a power of 2; p $\in \mathbb{P}_3$, k $\in \mathbb{N}$. So we have

(C) $\forall x \in \mathbb{N}_3 \quad \exists (pk, mk, qk) \in S_g \quad x = pk \ \vee \ x = mk = 4k.$

A few examples of the covering:

x = 19: (**19·1**, 21·1, 23·1), (**19·1**, 60·1, 101·1)

x = 36: (**3·12**, 7·12, 11·12)

x = 38: (**19·2**, 21·2, 23·2)

x = 42: (**3·14**, 5·14, 7·14), (**7·6**, 9·6, 11·6)

x = 64: (3·16, **4·16**, 5·16)

x = 10000: (**5·2000**, 6·2000, 7·2000).

Second, according to the statement SSGB, all pairs (p, q) of distinct odd primes are used in the definition of the set S_g ("*maximality*"). So we have

(M) $\forall$ p, q $\in \mathbb{P}_3$, p < q $\forall$ k $\in \mathbb{N}$ (pk, mk, qk) $\in S_g$, where m = (p + q) / 2.

The proof is motivated by the following view.

There are two possibilities for S_g, exactly one of which must occur: Either there is an n $\in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to ¬SSGB.

Since, due to (C), every n given by ¬SSGB as well as every multiple nk, k $\in \mathbb{N}$, equals a component of some S_g triple that exists by definition, the covering of $\mathbb{N}_3$ by the S_g triples if n exists (¬SSGB) is equal to that if n does not exist (SSGB). This causes a contradiction because in the case SSGB the numbers m defined in S_g take all integer values x ≥ 4 whereas in the case ¬SSGB they don't.

First of all, we note that each of the two properties (C) and (M) is a condition sine qua non for the proof, for the following reasons.

¬(C) immediately implies ¬SSGB, since an n ≥ 4 different from all S_g triple components pk, mk, qk is in particular different from all m in S_g.
The proof would no longer be possible if, for example, we omitted the factor k in the definition of S_g, because then the corresponding (C) could no longer be guaranteed.

Similarly, the property (M) rules out the possibility that there is an $n \geq 4$ different from all m (i.e. ¬SSGB) and n is the arithmetic mean of a pair of primes not used in S_g. Thus (M) excludes the possibility that ¬SSGB applies due to a missing prime number pair. This means that the proof would no longer be possible here either if we left out any prime number pair in the formulation of SSGB and S_g.

We will now show that ((C) $\wedge$ (M)) leads to a contradiction.

We split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, with

$S_g+(y) := \{ (pk, mk, qk) \in S_g \mid \exists\, k' \in \mathbb{N} \quad pk = yk' \;\vee\; mk = yk' \;\vee\; qk = yk' \}$ and

$S_g-(y) := \{ (pk, mk, qk) \in S_g \mid \forall\, k' \in \mathbb{N} \quad pk \neq yk' \;\wedge\; mk \neq yk' \;\wedge\; qk \neq yk' \}$.

Let $n \in \mathbb{N}_4$ be given by ¬SSGB as described above. Then, we have

(*) ¬SSGB => $S_g = S_g+(n) \cup S_g-(n)$.

More precisely, under the assumption ¬SSGB with the associated n the set S_g can be written as the disjoint union of the following triples.

(i) S_g triples of the form (pk = nk', mk, qk) with k = k' in case n is prime, due to (C)

(ii) S_g triples of the form (pk = nk', mk, qk) with k ≠ k' in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form (3k, 4k = nk', 5k) in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form (pk = nk', mk, qk), (pk, mk = nk', qk) or (pk, mk, qk = nk')

and

(v) S_g triples of the form (pk ≠ nk', mk ≠ nk', qk ≠ nk'), i.e. those S_g triples where none of the nk' equals a component.

So, $S_g+(n)$ is the union of the triples of the above types (i) to (iv) and $S_g-(n)$ is the union of the triples of type (v).

Now, we define

$S_1 := \{ (pk, mk, qk) \in S_g \mid \neg SSGB \text{ holds} \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid \ \ SSGB \text{ holds} \}$.

So, by (*) we obtain

(1) $\neg SSGB \Rightarrow S_1 = S_g = S_g+(n) \cup S_g-(n)$.

Since $S_g+(n) \cup S_g-(n)$ is independent of n, we can write

(1') $\forall\, y \in \mathbb{N}_3 \quad \neg SSGB \Rightarrow S_1 = S_g = S_g+(y) \cup S_g-(y)$.

Under the assumption SSGB there is no n as above. Therefore, under this assumption, we can choose an arbitrary $y \in \mathbb{N}_3$ such that $S_g = S_g+(y) \cup S_g-(y)$. So, we obtain

(2) $\forall\, y \in \mathbb{N}_3 \quad SSGB \Rightarrow S_2 = S_g = S_g+(y) \cup S_g-(y)$.

We will make use of the following trivial principle.

If two sets of (possibly infinitely many) x-tuples are equal, then the sets of their corresponding i-th components are equal; $1 \le i \le x$.

To this end, for each $k \in \mathbb{N}$ we define

$M(k) := \{ mk \mid (pk, mk, qk) \in S_g \}$

$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$

$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}$.

Then, applying the principle above to the middle component of the triples (pk, mk, qk), (1') and (2) imply

$\forall\, k \in \mathbb{N} \quad \forall\, y \in \mathbb{N}_3 \quad \neg SSGB \Rightarrow M_1(k) = M(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \}$

$\wedge$

$\forall\, k \in \mathbb{N} \quad \forall\, y \in \mathbb{N}_3 \quad SSGB \Rightarrow M_2(k) = M(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \}$.

We set $M := M(1)$, $M_1 := M_1(1)$ and $M_2 := M_2(1)$. So we get

$\forall y \in \mathbb{N}_3 \quad \neg SSGB \Rightarrow M_1 = M = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$

(3) $\wedge$

$\forall y \in \mathbb{N}_3 \quad SSGB \Rightarrow M_2 = M = \{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$.

On the other hand, under the assumption SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas under $\neg$SSGB they don't. Therefore, we have

(4.1) $SSGB \Rightarrow M_2 = M = \mathbb{N}_4$

and

(4.2) $\neg SSGB \Rightarrow M_1 = M \neq \mathbb{N}_4$.

Since for every $y \in \mathbb{N}_3$ $S_g+(y) \cup S_g-(y)$ equals S_g, we have that for every $y \in \mathbb{N}_3$ the set $\{ m \mid (p, m, q) \in S_g+(y) \cup S_g-(y) \}$ is the same in both implications of (3) and thus the set M is equal to that set.

Therefore, from (3) and (4.1) we obtain

$(\neg SSGB \Rightarrow M_1 = M = \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 = M = \mathbb{N}_4)$,

and from (3) and (4.2) we obtain

$(\neg SSGB \Rightarrow M_1 = M \neq \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 = M \neq \mathbb{N}_4)$.

This implies

$(\neg SSGB \Rightarrow M_1 = \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 = \mathbb{N}_4)$

(5) $\wedge$

$(\neg SSGB \Rightarrow M_1 \neq \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 \neq \mathbb{N}_4)$.

Also, we have SSGB => M_1 = { } and ¬SSGB => M_2 = { }. Together with (4.1) and (4.2), this yields

(6) SSGB <=> $M_2 = \mathbb{N}_4$

and

(7) $M_1 \neq \mathbb{N}_4$.

Then, due to (6) and (7), (5) becomes

(¬SSGB => FALSE $\wedge$ TRUE)

$\wedge$

(TRUE $\wedge$ SSGB => ¬SSGB).

And this yields (SSGB $\wedge$ ¬SSGB).

□