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ON TIME-HOMOGENEOUS PIECEWISE DETERMINISTIC MARKOV PROCESSES

NIKITA RATANOV

Abstract. In this note, we present some observations related to piecewise deterministic Markov processes. We also give some explicit representations for invariant and first-crossing time distributions.

1. Introduction

This paper is motivated by the simple idea of studying known examples of stochastic equations when a Markov process with finite variation and with a finite number of patterns switching at random times is used instead of white noise. The solution of the stochastic equation modified in this way can be considered as a Markov-modulated piecewise deterministic process.

The simplest example of a piecewise deterministic Markov process is the telegraph process, $\Xi(t) = (\Gamma(t), \varepsilon(t)),\ t \geq 0$, with values in $(-\infty, \infty) \times \{0, 1\}$ and with infinitesimal generator

$$\mathcal{L}^{\Gamma} = \begin{pmatrix} -\lambda_0 + \gamma_0 \dfrac{\mathrm{d}}{\mathrm{d}x} & \lambda_0 \\ & \\ \lambda_1 & -\lambda_1 + \gamma_1 \dfrac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix}$$

introduces by M.Kac [6]. Here, $\lambda_0,\ \lambda_1 > 0$ denote the intensities of switching between two states $\{0, 1\}$, and γ_0, γ_1 are the alternating velocities of a particle moving on a line,

$$\Gamma(t) = \int_0^t \gamma_{\varepsilon(s)} \mathrm{d}s.$$

The marginal distributions of the integrated telegraph process $\Gamma = \Gamma(t)$ can be described by the (generalised) probability density function

$$\mathbf{p}^{\Gamma}(t, x) = (p_0(t, x), p_1(t, x)), \qquad p_i(t, x)\mathrm{d}x = \mathbb{P}\{\Gamma(t) \in \mathrm{d}x \mid \varepsilon(0) = i\}, \quad i \in \{0, 1\}.$$

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In what follows, we will need explicit formulae for $\mathbf{p}^{\Gamma}(t,x)$,

$$\begin{aligned} p_0(t,x)\mathrm{d}x = {} & \mathrm{e}^{-\lambda_0 t}\delta_{\gamma_0 t}(\mathrm{d}x) \\ & +\theta(t,x)\Big[\lambda_0 I_0\left(2\sqrt{\lambda_0\lambda_1\xi_0(t,x)\xi_1(t,x)}\right) \\ & + \sqrt{\lambda_0\lambda_1\frac{\xi_0(t,x)}{\xi_1(t,x)}}I_1\left(2\sqrt{\lambda_0\lambda_1\xi_0(t,x)\xi_1(t,x)}\right)\Big]\mathrm{d}x, \end{aligned} \tag{1.1}$$

$$\begin{aligned} p_1(t,x)\mathrm{d}x = {} & \mathrm{e}^{-\lambda_1 t}\delta_{\gamma_1 t}(\mathrm{d}x) \\ & +\theta(t,x)\Big[\lambda_1 I_0\left(2\sqrt{\lambda_0\lambda_1\xi_0(t,x)\xi_1(t,x)}\right) \\ & + \sqrt{\lambda_0\lambda_1\frac{\xi_1(t,x)}{\xi_0(t,x)}}I_1\left(2\sqrt{\lambda_0\lambda_1\xi_0(t,x)\xi_1(t,x)}\right)\Big]\mathrm{d}x, \end{aligned} \tag{1.2}$$

where $\xi_0(t,x) = (x-\gamma_1 t)/(\gamma_0-\gamma_1)$, $\xi_1(t,x) = 1-\xi_0(t,x) = (\gamma_0 t - x)/(\gamma_0-\gamma_1)$, $\theta(t,x) = \exp(-\lambda_0\xi_0(t,x)-\lambda_1\xi_1(t,x))/(\gamma_0-\gamma_1)$ and I_0, I_1 are the modified Bessel functions, see, e.g., [8]; $\delta_a(\mathrm{d}x)$ denotes the Dirac δ-function.

The class of non-diffusion stochastic models based on piecewise deterministic Markov processes was introduced into scientific use by Davis [2]. Many authors, guided by various fields of application, have studied these processes from the point of view of martingale and ergodicity. Since a continuous telegraph process is not a martingale, for the purposes of financial modelling, various versions of a telegraph process equipped with jumps, [11, 8], or Markov modulated diffusion, [12, 13] are considered. The study of some ergodic properties and invariant distributions for these processes was stimulated by the modelling of neuronal activity, [10, 15], models of bacterial chemotaxis, [4]. See also an extensive and detailed review on this subject in [3].

Recently, the so-called *Poisson Stochastic Index processes* have also been introduced, studied, and fruitfully used for similar purposes, [19].

In this project, we will focus on a special case of piecewise deterministic processes defined by a stochastic equation of the form

$$X(t) = x + \int_0^t G_{\varepsilon(s)}(s, X(s))\mathrm{d}\Gamma(s), \qquad t \ge 0. \tag{1.3}$$

An important example of such a process, defined by the linear equation

$$X(t) = x + \int_0^t \left[a_{\varepsilon(s)} + \gamma_{\varepsilon(s)}X(s)\right]\mathrm{d}s, \qquad t \ge 0, \tag{1.4}$$

has recently been studied, [14, 15, 17] Since, with an appropriate scaling, the telegraph process converges to the diffusion process W, and equation (1.4) takes the form

$$\bar{X}(t) = x + at + bW(t) + \gamma\int_0^t \bar{X}(s)\mathrm{d}s,$$

which is the Langevin equation in the classic form, then this example (1.4) can be thought of as a generalised Ornstein-Uhlenbeck process.

In the case of the process X defined by equation (1.4), the distributions of the first crossing times with applications to neural modelling was analysed in

[14, 15]. This analysis was continued in [17] by studying invariant distributions and distributions of exponential functionals.

In this paper, we study the time-homogeneous case of equation (1.3):

$$X(t) = x + \int_0^t G(X(s))\mathrm{d}\Gamma(s), \qquad t \geq 0, \tag{1.5}$$

where the profile $G = G(x)$ is specified by a continuous positive function.

The Markov process $(X(t),\ \varepsilon(t)),\ t \geq 0$, (1.5), is determined by an infinitesimal generator

$$\mathcal{L} = \begin{pmatrix} -\lambda_0 + \gamma_0 G(x)\dfrac{\mathrm{d}}{\mathrm{d}x} & \lambda_0 \\ & \\ \lambda_0 & -\lambda_1 + \gamma_1 G(x)\dfrac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix}. \tag{1.6}$$

An important examples with $G(x) = \rho + |x|$ and $G(x) = \rho + |x|^\beta/\beta,\ \rho > 0, \beta > 0$, are examined explicitly below, see Example 3.7.

2. Time-homogeneous piecewise deterministic processes: invariant distributions and first crossing times

Let $G = G(x)$ be a positive smooth function and $\varepsilon = \varepsilon(t) \in \{0, 1\}$ be a two-state Markov process switching at random times $\tau_n,\ n \geq 1,\ \tau_0 = 0$.

Let us define two deterministic flows $\phi_0 = \phi_0(t, x)$ and $\phi_1 = \phi_1(t, x)$, $t \geq 0$, on the line as solutions of the Cauchy problems for first-order differential equations,

$$\frac{\partial \phi_i(t, x)}{\partial t} - \gamma_i G(x)\frac{\partial \phi_i(t, x)}{\partial x} = 0, \qquad t > 0, \qquad i \in \{0, 1\}, \tag{2.1}$$

with the initial condition $\phi_i(0, x) = x,\ x \in (-\infty, \infty)$.

By definition, the continuous piecewise deterministic process $X = X(t)$ defined by (1.5) sequentially follows the patterns ϕ_0 and ϕ_1, alternating at switching times τ_n, that is,

$$X(t) = \sum_{n\geq 0} \phi_{\varepsilon_n}(t - \tau_n, X(\tau_n))\mathbf{1}_{\{\tau_n \leq t < \tau_{n+1}\}}, \qquad \varepsilon_n = \varepsilon(\tau_n), \qquad t \geq 0. \tag{2.2}$$

The distribution $\mathbf{P}^X(t,\ \mathrm{d}y \mid x) = (P_0(t, \mathrm{d}y \mid x),\ P_1(t, \mathrm{d}y \mid x))$ of $X(t)$,

$$P_i(t, \mathrm{d}y \mid x) = \mathbb{P}\{X(t) \in \mathrm{d}y \mid \varepsilon(0) = i,\ X(0) = x\}, \qquad i \in \{0, 1\},$$

follows the coupled integral equations, $t \geq 0$,

$$\begin{cases} P_0(t, \mathrm{d}y \mid x) = \mathrm{e}^{-\lambda_0 t}\delta_{\phi_0(t,x)}(\mathrm{d}y) + \displaystyle\int_0^t \lambda_0 \mathrm{e}^{-\lambda_0 \tau} P_1(t - \tau, \mathrm{d}y \mid \phi_0(\tau, x))\mathrm{d}\tau, \\ P_1(t, \mathrm{d}y \mid x) = \mathrm{e}^{-\lambda_1 t}\delta_{\phi_1(t,x)}(\mathrm{d}y) + \displaystyle\int_0^t \lambda_1 \mathrm{e}^{-\lambda_1 \tau} P_0(t - \tau, \mathrm{d}y \mid \phi_1(\tau, x))\mathrm{d}\tau, \end{cases} \tag{2.3}$$

where $\lambda_0, \lambda_1 > 0$ are switching intensities.

To describe an invariant distribution, we need the $L_2(K)$-adjoint operator $\mathcal{L}^*$ to the generator $\mathcal{L}$, (1.6). For any measurable set $K \subset \mathbb{R}$, we have

$$\begin{aligned}\Big(\mathcal{L}\vec{f}, \vec{\varphi}\Big)_{L_2(K)} &= \int_K \Big(- \lambda_0 f_0(x) + \gamma_0 G(x) f_0'(x) + \lambda_0 f_1(x)\Big)\varphi_0(x)\mathrm{d}x \\ &\quad + \int \Big(\lambda_1 f_0(x) - \lambda_1 f_1(x) + \gamma_1 G(x) f_1'(x)\Big)\varphi_1(x)\mathrm{d}x \\ = \Big[\gamma_0 G(x) f_0(x)\varphi_0(x) &+ \gamma_1 G(x) f_1(x)\varphi_1(x)\Big]|_{x\in\partial K} \\ &+ \int_K f_0(x)\Big[- \lambda_0\varphi_0(x) + \lambda_1\varphi_1(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\gamma_0 G(x)\varphi_0(x)\right)\Big]\mathrm{d}x \\ &+ \int_K f_1(x)\Big[\lambda_0\varphi_0(x) - \lambda_1\varphi_1(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\gamma_1 G(x)\varphi_1(x)\right)\Big]\mathrm{d}x.\end{aligned}$$

Therefore, the (generalised) probability density function $\vec{\pi} = (\pi_0(x),\ \pi_1(x))$, $\pi_i(x) := \mathbb{P}\{X(t) \in \mathrm{d}x,\ \varepsilon(0) = i\}/\mathrm{d}x,\ i \in \{0, 1\}$, of an invariant distribution of $\Xi = \Xi(t)$ supported on K satisfies the equation

$$\mathcal{L}^*[\vec{\pi}](x) = 0, \qquad x \in K, \tag{2.4}$$

supplied with the boundary conditions $\pi_0(x)|_{x\in\partial K} = \pi_1(x)|_{x\in\partial K} = 0$, where

$$\mathcal{L}^*[\vec{\varphi}](x) = \begin{pmatrix} -\lambda_0\varphi_0(x) - \gamma_0\dfrac{\mathrm{d}}{\mathrm{d}x}\left(G(x)\varphi_0(x)\right) + \lambda_1\varphi_1(x) \\ \\ \lambda_0\varphi_0(x) - \lambda_1\varphi_1(x) - \gamma_1\dfrac{\mathrm{d}}{\mathrm{d}x}\left(G(x)\varphi_1(x)\right)\end{pmatrix}. \tag{2.5}$$

Rewriting (2.4)-(2.5) in scalar form, we obtain the equivalent system

$$\begin{cases}\gamma_0\dfrac{\mathrm{d}}{\mathrm{d}x}\left[G(x)\pi_0(x)\right] = -\lambda_0\pi_0(x) + \lambda_1\pi_1(x), \\ \gamma_1\dfrac{\mathrm{d}}{\mathrm{d}x}\left[G(x)\pi_1(x)\right] = \lambda_0\pi_0(x) - \lambda_1\pi_1(x),\end{cases} \qquad x \in K. \tag{2.6}$$

Our next task is to describe the first crossing probabilities.

Let $T(x, y)$ be the moment when the process $X(t)$ reaches the threshold y for the first time, starting from the point x:

$$T(x, y) = \inf\{t > 0 \mid X(t) = y, X(0) = x\}.$$

We set $T(x, y) = +\infty$ if the threshold y is never reached. Similarly to (2.3) one can find that the distribution $\mathbf{F}^X(\mathrm{d}t, y \mid x) = (F_0(\mathrm{d}t, y \mid x), F_1(\mathrm{d}t, y \mid x))$,

$$F_i(\mathrm{d}t, y \mid x) = \mathbb{P}\{T(x, y) \in \mathrm{d}t \mid \varepsilon(0) = i, X(0) = x\}, \qquad i \in \{0, 1\},$$

satisfies the system

$$\begin{cases} F_0(\mathrm{d}t, y \mid x) = \mathrm{e}^{-\lambda_0 t_0(x,y)}\delta_{t_0(x,y)}(\mathrm{d}t) \\ \qquad\qquad + \displaystyle\int_0^{t_0(x,y)\wedge t} \lambda_0 \mathrm{e}^{-\lambda_0\tau} F_1(-\tau+\mathrm{d}t, y \mid \phi_0(\tau, x))\mathrm{d}\tau, \\ F_1(\mathrm{d}t, y \mid x) = \mathrm{e}^{-\lambda_1 t_1(x,y)}\delta_{t_1(x,y)}(\mathrm{d}t) \\ \qquad\qquad + \displaystyle\int_0^{t_1(x,y)\wedge t} \lambda_1 \mathrm{e}^{-\lambda_1\tau} F_0(-\tau+\mathrm{d}t, y \mid \phi_1(\tau, x))\mathrm{d}\tau. \end{cases}$$

Here, $t_0(x, y)$ and $t_1(x, y)$ denote the time to reach the threshold y without switching states.

For the simplest case of the telegraph process, the distributions $\mathbf{F}^\Gamma(\mathrm{d}t, y \mid x) = (F_0(\mathrm{d}t, y \mid x), F_1(\mathrm{d}t, y \mid x))$, are well studied, see e.g. [8, 9]. For the sake of completeness, we write down the explicit formulae below.

For the case of $\gamma_0 > 0 > \gamma_1$ and $x < y$,

$$\begin{aligned} F_0(\mathrm{d}t, y \mid x) =& \mathrm{e}^{-\lambda_0 t}\delta_{(y-x)/\gamma_0}(\mathrm{d}t) \\ &+ \frac{\lambda_0\lambda_1(y-x)\theta(t,x)}{\sqrt{\lambda_0\lambda_1\xi_0(t,x)\xi_1(t,x)}} I_1(2\sqrt{\lambda_0\lambda_1\xi_0(t,x)\xi_1(t,x)})\mathrm{d}t, \\ F_1(\mathrm{d}t, y \mid x) =& \frac{\lambda_1\theta(t,x)}{\xi_0(t,x)}\Big[xI_0(2\sqrt{\lambda_0\lambda_1\xi_0(t,x)\xi_1(t,x)}) \\ &- \frac{\gamma_1}{\sqrt{\lambda_0\lambda_1}}\sqrt{\frac{\xi_1(t,x)}{\xi_0(t,x)}} I_1(2\sqrt{\lambda_0\lambda_1\xi_0(t,x)\xi_1(t,x)})\Big]\mathrm{d}t. \end{aligned} \tag{2.7}$$

3. Time-homogeneous dynamics: marginal and invariant distributions. Blow-up probabilities

Consider the piecewise deterministic process X, defined by (2.2) with a positive continuous profile $G = G(x)$. In differential form, equation (1.5) is equivalent to the initial value problem for the autonomous stochastic equation

$$\mathrm{d}X(t) = G(X(t))\mathrm{d}\Gamma(t), \qquad t > 0, \tag{3.1}$$

with initial condition $X(0) = x$.

Let

$$\Phi(x) = \int_0^x \frac{\mathrm{d}y}{G(y)}. \tag{3.2}$$

The mapping $x \to \Phi(x)$ can be considered as a rectifying diffeomorphism to the initial value problem (3.1), see [1].

Therefore, the solution $X = X(t)$ of equation (3.1) can be explicitly expressed by means of the underlying telegraph process Γ :

$$X(t) = \Phi^{-1}(\Phi(x) + \Gamma(t)), \qquad t \geq 0. \tag{3.3}$$

Note that the representation (3.3) is invariant under the transform $\Phi \to k\Phi + a$, $k \neq 0$.

Theorem 3.1. *Let $X = X(t)$ be defined by (1.5) or, equivalently, by (3.1). Let the velocities γ_0, γ_1 of the underlying telegraph process $\Gamma(t)$ have opposite signs.*

a) *If at least one of the integrals converges,*

$$\int_{-\infty}^{0}\frac{\mathrm{d}y}{G(y)}<\infty \ \textit{or} \int_{0}^{\infty}\frac{\mathrm{d}y}{G(y)}<\infty, \tag{3.4}$$

then process X goes to infinity a.s. in a finite time (blow-up);

b) *If both integrals diverge,*

$$\int_{-\infty}^{0}\frac{\mathrm{d}y}{G(y)}=\infty \ \textit{and} \ \int_{0}^{\infty}\frac{\mathrm{d}y}{G(y)}=\infty, \tag{3.5}$$

then process X is well-defined for all $t\geq 0$. In this case, the distribution of the random variable $X(t)$, $t\geq 0$, is supported on the interval $I_{t,x}$ ending in the points $\phi_0(t,x)$ and $\phi_1(t,x)$ and determined by the probability density function $\mathbf{p}^X(t,\cdot\mid x)=(p_0(t,\cdot\mid x),\ p_1(t,\cdot\mid x))$ of the form:

$$\mathbf{p}^X(t,y\mid x)=\left|\frac{\mathrm{d}\Phi(y)}{\mathrm{d}y}\right|\cdot\mathbf{p}^\Gamma(t,\Phi(y)-\Phi(x)), \tag{3.6}$$

$$y \text{ is between } \phi_0(t,x) \text{ and } \phi_1(t,x),$$

where Φ is defined by (3.2), and $\mathbf{p}^\Gamma(t,z)$ is the probability density function of the telegraph process $\Gamma(t)$, (1.1)-(1.2).

Proof. In case a), (3.4), the function $\Phi(x)$ is bounded (at least on one side), and the inverse function Φ^{-1} is defined on a bounded (at least on one side) interval $\Delta=(A_-,\ A_+)$, where $A_-:=-\int_{-\infty}^{0}\frac{\mathrm{d}y}{G(y)}$, $A_+:=\int_0^{+\infty}\frac{\mathrm{d}y}{G(y)}$. Therefore, $X(t)$ goes to infinity when the telegraph process $\Phi(x)+\Gamma(t)$ leaves the interval Δ, which happens a.s. within a finite time $\mathcal{T}_x$, see, for example, [8].

Otherwise, (3.5), both functions, $\Phi(x)=\int_0^x \mathrm{d}y/G(y)$ and Φ^{-1}, are monotonic and are defined on the whole line. Therefore (3.6) follows from the representation (3.3). Detailed comments on this issue can be found in [18].

Formulae (3.6) follow from the representation (3.3). □

The case of both positive velocities is similar to Theorem 3.1.

Corollary 3.2. *Let $\gamma_0,\gamma_1>0$.*

a) *If $A=\int_0^\infty G(y)^{-1}\mathrm{d}y<\infty$, then the process $X=X(t)$ goes to $+\infty$ a.s. in finite time.*

b) *If $\int_0^\infty G(y)^{-1}\mathrm{d}y=\infty$, then the process $X=X(t)$ is a subordinator, and its distribution is given by (3.6).*

The case of both negative velocities is symmetric.

The blow-up scenario requires more detail. For example, we need to know the probability of a process going to $-\infty$ versus to $+\infty$. Let $\mathcal{T}_x$ be the time instant when the process X leaving the interval Δ.

By virtue of representation (3.3), the following auxiliary result will be very useful.

Lemma 3.3. *Let $a<0<b$ and a stopping time $\mathcal{T}^\Gamma(a;x)$ (respectively, $\mathcal{T}^\Gamma(b;x)$) be the time when the process $x+\Gamma(t)$ first crosses the threshold a (respectively, b). Let $u(x)=\mathbb{P}\{\mathcal{T}^\Gamma(b;x)<\mathcal{T}^\Gamma(a;x)\mid\varepsilon(0)=0\}, a<b, a\leq x\leq b$.*

Function $u(x)$ has the form:

$$u(x) = \begin{cases} \dfrac{\alpha_1 - \alpha_0 \exp(\Delta\alpha \cdot (x-a))}{\alpha_1 - \alpha_0 \exp(\Delta\alpha \cdot (b-a))}, & \alpha_0 \neq \alpha_1, \\ \\ \dfrac{1+\alpha(x-a)}{1+\alpha(b-a)}, & \alpha_0 = \alpha_1 = \alpha, \end{cases} \tag{3.7}$$

where $\alpha_0 = \lambda_0/\gamma_0,\ \alpha_1 = \lambda_1/\gamma_1$, and $\Delta\alpha = \alpha_0 - \alpha_1$.

Proof. Let $u_0(x) = u(x) = \mathbb{P}\{\mathcal{T}^\Gamma(b;x) < \mathcal{T}^\Gamma(a;x) \mid \varepsilon(0) = 0\}$ and $u_1(x) = \mathbb{P}\{\mathcal{T}^\Gamma(b;x) < \mathcal{T}^\Gamma(a;x) \mid \varepsilon(0) = 1\}$. These functions follow the system of coupled equations

$$\begin{cases} u_0'(x) = \alpha_0(u_0(x) - u_1(x)), \\ u_1'(x) = \alpha_1(u_0(x) - u_1(x)), \end{cases} \quad a < x < b, \tag{3.8}$$

supplied with the boundary conditions $u_0(b) = 1,\ u_1(a) = 0$, cf [7, pp.192-193].

By virtue of (3.8), $\alpha_1 u_0'(x) = \alpha_0 u_1'(x)$ and $(u_0 - u_1)'(x) = (\alpha_0 - \alpha_1)(u_0 - u_1)(x)$. Therefore,

$$u_0(x) = A_0 \exp\left((\alpha_0 - \alpha_1)(x-a)\right) + B, \qquad u_1(x) = A_1 \exp\left((\alpha_0 - \alpha_1)(x-a)\right) + B,$$

such that

$$\alpha_0 A_1 = \alpha_1 A_0, \qquad A_1 + B = 0, \qquad A_0 \mathrm{e}^{(\alpha_0 - \alpha_1)(b-a)} + B = 1,$$

which give (3.7) for $\alpha_0 \neq \alpha_1$. The case $\alpha_0 = \alpha_1$ is analysed similarly. □

Theorem 3.4. *Let $\gamma_0 > 0 > \gamma_1$ and both integrals in (3.4) converge.*

Therefore, if the process starts at point x, $A_- < \Phi(x) < A_+$, with positive velocity γ_0, then

$$\begin{aligned} \mathbb{P}\{X(\mathcal{T}_x) = +\infty \mid \varepsilon(0) = 0\} &= 1 - \mathbb{P}\{X(\mathcal{T}_x) = -\infty \mid \varepsilon(0) = 0\} \\ &= \begin{cases} \dfrac{\alpha_1 - \alpha_0 \exp(\Delta\alpha \cdot (\Phi(x) - A_-))}{\alpha_1 - \alpha_0 \exp(\Delta\alpha \cdot (A_+ - A_-))}, & \text{if } \alpha_0 \neq \alpha_1, \\ \\ \dfrac{1 + \alpha \cdot (\Phi(x) - A_-)}{1 + \alpha \cdot (A_+ - A_-)}, & \text{if } \alpha_0 = \alpha_1 = \alpha. \end{cases} \end{aligned} \tag{3.9}$$

Here $A_- = -\int_{-\infty}^0 G(y)^{-1}\mathrm{d}y,\ A_+ = \int_0^\infty G(y)^{-1}\mathrm{d}y$.

Proof. First, note that $X(\mathcal{T}_x) = +\infty$ occurs if and only if the underlying telegraph process $\Phi(x) + \Gamma(t)$ leaves the interval $\Delta = [A_-,\ A_+]$ on the right (through the point A_+). Therefore, formula (3.9) follows from (3.7), Lemma 3.3. □

In view of applications, the description of invariant distributions is of special interest.

Since the mapping Φ is monotonically increasing, the process $X = X(t)$ is a subordinator when both velocities of Γ are positive (the same is true for $-X$ if both velocities are negative). Thus, there is no invariant measure in this case.

The following theorem shows how to describe the invariant distribution when telegraphic velocities are of opposite signs.

Theorem 3.5. *Let function $G = G(x)$ be continuous, positive,$G(x) > 0\ \forall x$, and even, $G(-x) = G(x)$. Let $X = X(t)$ be defined by* (3.1) *with the telegraph process having velocities of opposite signs, $\gamma_0 \cdot \gamma_1 < 0$, such that $\alpha_0 + \alpha_1 > 0$, where no blow-up occurs,* (3.5).

The invariant measure for X exists and is defined by the probability density functions

$$\begin{aligned} \pi_0(x) &= \frac{1}{2}\frac{\alpha_0+\alpha_1}{\gamma_0-\gamma_1}c_0(x)G(x)^{-1}\exp\left(-(\alpha_0+\alpha_1)\left|\int_0^x \frac{\mathrm{d}y}{G(y)}\right|\right), \\ \pi_1(x) &= \frac{1}{2}\frac{\alpha_0+\alpha_1}{\gamma_0-\gamma_1}c_1(x)G(x)^{-1}\exp\left(-(\alpha_0+\alpha_1)\left|\int_0^x \frac{\mathrm{d}y}{G(y)}\right|\right), \end{aligned} \tag{3.10}$$

where

$$c_0(x) = \begin{cases} -\gamma_1, & x > 0, \\ \gamma_0, & x < 0, \end{cases} \qquad c_1(x) = \begin{cases} \gamma_0, & x > 0, \\ -\gamma_1 & x < 0. \end{cases}$$

Proof. The linear system (2.6) considered on the whole line $(-\infty, \infty)$ has the following solution: for $x \in (0, +\infty)$ by direct substitution into (2.6) one can verify that

$$\begin{aligned} \pi_0(x) &= -C\gamma_0^{-1}G(x)^{-1}\exp\left(-(\alpha_0+\alpha_1)\int_0^x G(y)^{-1}\mathrm{d}y\right), \\ \pi_1(x) &= C\gamma_1^{-1}G(x)^{-1}\exp\left(-(\alpha_0+\alpha_1)\int_0^x G(y)^{-1}\mathrm{d}y\right), \end{aligned}$$

which can be extended to $(-\infty, 0)$ by symmetry. The indefinite constant C can be found from the normalisation condition,

$$\int_{-\infty}^{\infty} (\pi_0(x) + \pi_1(x))\,\mathrm{d}x = 1.$$

□

Corollary 3.6. *In the symmetric case, that is, if $\gamma_0 = -\gamma_1 = \gamma$, then*

$$\pi_0(x) = \pi_1(x) = \frac{1}{4}(\alpha_0+\alpha_1)G(x)^{-1}\exp\left(-(\alpha_0+\alpha_1)\left|\int_0^x G(y)^{-1}\mathrm{d}y\right|\right),$$
$$-\infty < x < \infty.$$

Let us take a few more examples.

Example 3.7. Let $G(x) = \rho + |x|,\ \rho > 0$. In this case, the rectifying diffeomorphism Φ for the stochastic equation

$$\mathrm{d}X(t) = (\rho + |X(t)|)\mathrm{d}\Gamma(t), \tag{3.11}$$

is given by $\Phi(x) = \mathrm{sign}(x) \cdot \ln(1 + |x|/\rho)$ which is odd and strictly increasing function on the line $(-\infty, \infty)$, so that

$$\Phi^{-1}(y) = \rho(\mathrm{e}^{|y|} - 1)\mathrm{sign}(y)$$

is also strictly increasing.

Notice that the alternating patterns $\phi_0(t,x)$, $\phi_1(t,x)$ that form the process $X = X(t)$ depend on the value of the corresponding velocity γ. In this case, if $\gamma > 0$, then $\phi(t,x) = (\rho + x)\mathrm{e}^{\gamma t} - \rho$ with $x > 0$, and for $x < 0$,

$$\phi(t,x) = \begin{cases} \rho - (\rho - x)\mathrm{e}^{-\gamma t}, & \text{for small } t, \text{such that } t < t_* = \frac{1}{\gamma}\ln\left(1 - x/\rho\right), \\ \frac{\rho}{1-x/\rho}\mathrm{e}^{\gamma t} - \rho, & \text{for large } t,\ t > t_*. \end{cases}$$

If $\gamma < 0$, then the formulae are symmetric.

Let $\gamma_0 \cdot \gamma_1 < 0$. By virtue of (3.10), the invariant probability density functions for $X(t)$ defined by (3.11) have the form:

$$\pi_0(x) = \frac{1}{2}\frac{\alpha_0 + \alpha_1}{\gamma_0 - \gamma_1} c_0(x)\rho^{\alpha_0+\alpha_1}(\rho + |x|)^{-1-(\alpha_0+\alpha_1)},$$
$$\pi_1(x) = \frac{1}{2}\frac{\alpha_0 + \alpha_1}{\gamma_0 - \gamma_1} c_1(x)\rho^{\alpha_0+\alpha_1}(\rho + |x|)^{-1-(\alpha_0+\alpha_1)},$$

This example can be easily generalised. Instead of (3.11), consider the initial value problem,

$$\begin{aligned} \mathrm{d}X(t) =& \left(\rho + \frac{|X(t)|^\beta}{\beta}\right)\mathrm{d}\Gamma(t), \qquad t > 0, \\ X(0) =& x, \end{aligned}$$

where $\rho > 0$ and $\beta > 0$.

Let $\Phi = \Phi(x)$, $x \in (-\infty, \infty)$ be an odd and strictly increasing function which is defined so that for $x \geq 0$

$$\Phi(x) = \int_0^x \left(\rho + u^\beta/\beta\right)^{-1}\mathrm{d}u = \int_0^{x^\beta} \frac{v^{-1+1/\beta}}{\beta\rho + v}\mathrm{d}v.$$

By virtue of [5, 3.194.5],

$$\Phi(x) = x\rho^{-1} \cdot {}_2F_1(1, 1/\beta; 1 + 1/\beta; -x^\beta/(\beta\rho)), \qquad x > 0, \tag{3.12}$$

with odd extension to $x < 0$. Here ${}_2F_1$ is the Gauss hypergeometric function. If $0 < \beta \leq 1$, there is no blow-up, (3.5), and one can obtain an invariant distribution, as in the previous case.

If $\beta > 1$, then

$$2A = \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{1 + |y|^\beta/\beta} < \infty,$$

that is, the blow-up condition (3.4) holds. Therefore, the inverse function Φ^{-1} is only defined on the interval $(-A, A)$, and the process $X = \Phi^{-1}(\Phi(x) + \Gamma(t))$ goes to infinity in a.s. finite time $\mathcal{T}_x$, where

$$\mathcal{T}_x = \sup\{t > 0 \mid -A - \Phi(x) < \Gamma(t) < A - \Phi(x)\},$$

where $\Phi(x)$ is defined by (3.12). The form of distribution of $\mathcal{T}_x$ follows from (2.7) and Theorem 3.4 . See also [16].

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NIKITA RATANOV: LABORATORY OF FINANCIAL MODELLING, CHELYABINSK STATE UNIVERSITY, CHELYABINSK, 454001, RUSSIAN FEDERATION

E-mail address: rtnnkt@gmail.com