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ON A CERTAIN SYSTEM OF STOCHASTIC EQUATIONS WITH MEAN DERIVATIVES, CONNECTED WITH HYDRODYNAMICS

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ABSTRACT. We study a system of two special equations with mean derivatives on the group of Sobolev diffeomorphisms of the flat n-dimensional torus that leads to a flow on the torus, described by a system of two equations, one of which is the Burgers equation, and the second one is a continuity-type equation. We prove the existence of solution theorem and interpret this flow as a flow of a special viscous fluid.

Introduction

The paper is devoted to the Lagrangiaan approach to hydrodynamics initiated by the well-known works by V.I. Arnold [1] and then by D. Ebin and J. Marsden [2]. In [2], in the language of infinite-dimensional Riemanninan geometry on the Sobolev group of volume-preserving diffeomorphisms and the group of all the Sobolev group of diffeomorphisms of the compact manifolds there was given the description of ideal fluids. In particular, it was shown that the flow of ideal incompressible fluid is described by the second order ordinary differenetial equation with covariant derivatives of weak Riemannian metric on the group of volume preserving diffeomorphisms. In the case of zero external force the flow is described by the equation of geodesic. For the flows generated by the Hops equation, it was shown the same on the group of all Sobolev diffeomorphisms.

We show that the flows of viscous fluids are described by stochastic analogues of the Ebin and Marsden equations, in which the covariant derivatives are replaced by the second order backward mean derivatives. The concept of mean derivatives was introduced by E. Nelson (see, e.g., [3]). We use the extended version of this machinery, see, e.g., [4].

In spite of the fact that the construction is based on the Stochastic Analysis, the results are obtained for deterministic (not random) fluids. Unlike Ebin and Marsden, we consider the hydrodynamics only on the flat n-dimensional torus and we essentially use the properties of the torus in our constructions. Note that

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the investigation of fluid motion on the torus is a well-known problem in the hydrodynamics.

We study a system of two special equations with mean derivatives on the group of Sobolev diffeomorphisms of the flat n-dimensional torus that leads to a flow on the torus, described by a system of two equations, one of which is the Burgers equation, and the second one is a continuity-type equation. We prove the existence of solution theorem and interpret this flow as a flow of a special viscous fluid.

1. Mean derivatives

For simplicity of presentation, we describe the theory of mean derivatives for processes in $\mathbb{R}^n$. However, due to the fact that the geometry on the torus is inherited from the Euclidean geometry on $\mathbb{R}^n$, this presentation is unchanged applied to the torus.

Consider a random process $\xi(t)$ in $\mathbb{R}^n$ (where we specify the σ-algebra of Borel sets), $t \in [0,T]$, defined on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t)$ belongs to the space $L_1(\Omega, \mathbb{R}^n)$ for every t.

Denote by $\mathcal{N}_t^\xi$ the σ-subalgebra "presence" in $\mathcal{F}$ generated by preimages of Borel sets from $\mathbb{R}^n$ under the mapping $\xi(t) : \Omega \to \mathbb{R}^n$. $\mathcal{N}_t^\xi$ is assumed to be complete, i.e. containing all zero probability sets.

For convenience, we denote by E_t^ξ the conditional mathematical expectation of $E(\cdot|\mathcal{N}_t^\xi)$ relative to the "presence" $\mathcal{N}_t^\xi$ of $\xi(t)$.

Following E. Nelson, we introduce the concepts of forward backward and symmetric mean derivative.

The forward mean derivative $D\xi(t)$ of the process $\xi(t)$ at time t is an L_1 random element of the form

$$D\xi(t) = \lim_{\triangle t \to +0} E_t^\xi\left(\frac{\xi(t + \triangle t) - \xi(t)}{\triangle t}\right), \tag{1.1}$$

where the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$, and the symbol $\triangle t \to +0$ means that $\triangle t$ tends to zero 0 and $\triangle t > 0$.

The backward mean derivative $D_*\xi(t)$ of the process $\xi(t)$ at the time instant t is an L_1-random element

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^\xi\left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t}\right) \tag{1.2}$$

where (as in (i)) the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P})$, and the symbol $\Delta t \to +0$ means that $\Delta t \to 0$ and $\Delta t > 0$.

The symmetric mean derivative D_S is given by the formula $\frac{1}{2}(D + D_*)$. The derivative $D_S\xi(t)$ is called the current velocity of $\xi(t)$.

We mainly need to work with D_* and D_S. That's why let's take a closer look at their properties.

It follows from the properties of the conditional expectation that $D_*\xi(t)$ can be represented as a superposition of $\xi(t)$ and a measurable Borel vector field (regression)

$$a(t,x) = \lim_{\Delta t \to +0} E\left(\frac{\xi(t) - \xi(t - \Delta t)}{\Delta t}\Big|\xi(t) = x\right) \tag{1.3}$$

on $\mathbb{R}^n$. This means that $D_*\xi(t) = a(t, \xi(t))$.

We introduce the symbol v^ξ for the regression of current velocity.

Let $Z(t, x)$ be a C^2-smooth vector field on $\mathbb{R}^n$, and $\xi(t)$ be a stochastic process in $\mathbb{R}^n$.

The L_1-limit of the form

$$D_*Z(t, \xi(t)) = \lim_{\Delta t \to +0} E_t^\xi (L \frac{Z(t, \xi(t)) - Z(t - \Delta t, \xi(t - \Delta t))}{\Delta t}) \tag{1.4}$$

is called backward mean derivative of Z along $\xi(\cdot)$ at the time instant t.

Of course, $D_*Z(t, \xi(t))$ can be represented as the superposition of $\xi(t)$ with a certain Borel measurable vector field (regression). This vector field (if this does not lead to a confusion) we will denote by the same symbol D_*Z.

For a process with diffusion coefficient $\sigma^2 I$ in $\mathbb{R}^n$ the following formula holds

$$D_*Z = \frac{\partial}{\partial t} Z + (Y_*^0 \cdot \nabla)Z - \frac{\sigma^2}{2} \Delta Z, \tag{1.5}$$

where $\nabla = (\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n})$, Δ is Laplacian, dot denotes scalar product in $\mathbb{R}^n$ and the vector field $Y_*^0(t, x)$ is the regression of backward mean derivative.

Second order derivative $D_*D_*\xi(t)$ we describe as the first derivative D_* of the regression (vector field) $D_*\xi$.

In the case when ξ has a diffusion coefficient $\sigma^2 I$, denote the regression $D_*\xi$ by the symbol Y. Then according to the formula (1.5) we get

$$D_*D_*\xi = \left(\frac{-\sigma^2}{2} \Delta + Y \cdot \nabla + \frac{\partial}{\partial t} \right) Y, \tag{1.6}$$

where the right-hand side of the formula (1.6) is the same as the left-hand side of the Burgers equations with viscosity $\frac{\sigma^2}{2}$.

Let the process $\xi(t)$ have a diffusion coefficient Q, which is a smooth symmetric $(2, 0)$ non-degenerate tensor field. This means that a smooth non-degenerate symmetric $(0, 2)$ tensor field Q^{-1} is well defined, and it can be considered a new Riemannian metric.

Note that in the case of the diffusion coefficient Q in the formula (1.6), the expression $Y \cdot \nabla$ is replaced by the covariant derivative of the Riemannian metric Q^{-1} with respect to Y.

Denote by the symbol $\rho^\xi(t, x)$ the probabilistic density of element $\xi(t)$ with respect to the volume form $dt \wedge \Lambda$ on $\mathbb{R} \times \mathbb{R}^n$, where Λ is the Euclidean form volume on the torus, i.e., for any bounded continuous function $f \colon [0, T] \times \mathbb{R}^n \to \mathbb{R}$ the formula

$$\begin{aligned} \int_0^T E(f(t, \xi(t)))\, dt &= \int_0^T \left(\int_\Omega f(t, \xi(t)) d\mathsf{P} \right) \mathsf{dt} \\ &= \int_{[0,T] \times \mathbb{R}^n} f(t, x) \rho^\xi(t, x) dt \wedge \Lambda_\alpha \end{aligned}$$

takes place.

Let $\xi(t)$ be a diffusion process with diffusion coefficient Q as above and density ρ^ξ. For $v^\xi(t, x)$ and $\rho^\xi(t, x)$ of this process, the following relation of the type of continuity equation is satisfied the following kind

$$\frac{\partial \rho^{\xi}(t,x)}{\partial t} = -Div(\rho^{\xi}(t,x)v^{\xi}(t,x)), \tag{1.7}$$

where Div denotes the divergence with respect to Riemannian metric Q^{-1}. If the diffusion coefficient is not autonomous, then for each t we consider the Riemannian metric and its divergence on the corresponding level surface. For a process with diffusion coefficient I Div is the usual divergence div.

2. The Sobolev groups of diffeomorphisms

Let $\mathcal{T}^n$ be a flat n-dimensional torus and $\mathcal{D}^s(\mathcal{T}^n)$ – its Sobolev group of diffeomorphisms of the class $H^s(s > n/2+1)$. Recall that for $s > n/2+1$ the mappings from H^s are C^1 smooth.

$\mathcal{D}^s(\mathcal{T}^n)$ is a Hilbert manifold and group with respect to superposition with unity $e = id$. Tangent space $T_e\mathcal{D}^s(\mathcal{T}^n)$ is the space of all H^s-vector fields on $\mathcal{T}^n$.

In any tangent space $T_f\mathcal{D}^s(\mathcal{T}^n)$ one can define L^2-scalar product by the formula

$$(X,Y) = \int_{\mathcal{T}^n} \langle X(m), Y(m)\rangle_{f(m)}\mu(dm) \tag{2.1}$$

The family of these scalar products forms the so-called weak Riemannian metric on $\mathcal{D}^s(\mathcal{T}^n)$. In particular, in $T_e\mathcal{D}^s(\mathcal{T}^n)$ (2.1) becomes

$$(X,Y)_e = \int\limits_{\mathcal{T}^n} \langle X(m), Y(m)\rangle_m \mu(dm). \tag{2.2}$$

Right shift $R_f : \mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$, where $R_f(\Theta) = \Theta \circ f$ for $\Theta, f \in \mathcal{D}^s(T^n)$ is a C^∞-smooth mapping. The tangent mapping to the right shift is $TR_f(X) = X \circ f$ for $X \in T\mathcal{D}^s(T^n)$.

On the other hand, left shift $L_f : \mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$, where $L_f(\Theta) = f \circ \Theta$ for $\Theta, f \in \mathcal{D}^s(T^n)$, is only continuous. Let us specify the vector $x \in \mathbb{R}^n$ and denote by $l_x : T^n \to T^n$ mapping$l_x(m) = m + x$ modulo factorization with respect to the integer lattices of the space $\mathbb{R}^n$. Note that the left shift Ll_x C^∞-smooth.

Recall that $T\mathcal{T}^n = \mathcal{T}^n \times \mathbb{R}^n$. We introduce the operators

$$B : T\mathcal{T}^n \to \mathbb{R}^n,$$

projections to the second factor in $\mathcal{T}^n \times \mathbb{R}^n$, and

$$A(m) : \mathbb{R}^n \to \mathcal{T}^n_m,$$

the linear isomorphism of $\mathbb{R}^n$ inverse to B onto the tangent space to $\mathcal{T}^n$ for $m \in \mathcal{T}^n$.

Let's introduce

$$Q_{g(m)} = A(g(m)) \circ B,$$

where $g \in \mathcal{D}^s(\mathcal{T}^n)$, $m \in \mathcal{T}^n$, For every $Y \in T_f\mathcal{D}^s(\mathcal{T}^n)$ we get that $Q_gY = A(g(m)) \circ B(Y(m)) \in T_g\mathcal{D}^s(\mathcal{T}^n)$ for all $f \in \mathcal{D}^s(\mathcal{T}^n)$.

In complete analogy to finite-dimensional case, a geodesic is a smooth curve $g(t)$ in $\mathcal{D}^s(\mathcal{T}^n)$ such that

$$\frac{\bar{D}}{dt}\dot{g}(t) = 0. \tag{2.3}$$

For such a curve $g(t)$ we construct the vector $v(t) \in T_e\mathcal{D}^s(\mathcal{T}^n)$ by the formula $v(t) = \dot{g}(t) \circ g^{-1}(t)$.

If $g(t)$ is a geodesic, the curve $R_f g(t)$ is also geodetic.

Let $g(t)$ be a geodesic and $x \in \mathbb{R}^n$ be some vector. then $l_x g(t)$ is a geodesic.

Consider the operator $\bar{A} : \mathcal{D}^s(\mathcal{T}^n) \times \mathbb{R}^n \to T\mathcal{D}^s(\mathcal{T}^n)$ such that $\bar{A}_e$ is the same as A introduced above and for each $g \in \mathcal{D}^s(\mathcal{T}^n)$ the mapping $\bar{A}_g : \mathbb{R}^n \to T_g\mathcal{D}^s(\mathcal{T}^n)$ is constructed from $\bar{A}_e$ by right shift, i.e. for $X \in \mathbb{R}^n$:

$$\bar{A}_g(X) = TR_g \circ A_e(X) = (A \circ g)(X).$$

Every right-invariant vector field $\bar{A}(X)$ is C^∞-smooth on $\mathcal{D}^s(\mathcal{T}^n)$ for every $X \in \mathbb{R}^n$.

For any point $m \in T^n$ denote by $exp_m : T_mT^n \to T^n$ the mapping that sends the vector $X \in T_mT^n$ to the point $m+X$ modulo the factorization with respect to the integer lattice on T^n. The family of such mappings generates a mapping $\overline{exp} : T_e\mathcal{D}^s(T^n) \to \mathcal{D}^s(T^n)$ that sends the vector $X \in T_e\mathcal{D}^s(T^n)$ into $e + X \in \mathcal{D}^s(T^n)$, where $e + X$ is the diffeomorphism T^n of the form: $(e + X)(m) = m + X(m)$.

Consider the superposition $\overline{exp} \circ \bar{A}_e : \mathbb{R}^n \to \mathcal{D}^s(\mathcal{T}^n)$. By construction for arbitrary $X \in \mathbb{R}^n$ we get that $\overline{exp} \circ \bar{A}_e(X)(m) = m + X$, i.e., the same the vector X is added to each point m.

Let $w(t)$ be a Wiener process in $\mathbb{R}^n$ given on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Construct the random process

$$W^{(\sigma)}(t) = \overline{exp} \circ \bar{A}_e(\sigma w(t)) \tag{2.4}$$

on $\mathcal{D}^s(\mathcal{T}^n)$. By construction, for $\omega \in \Omega$ the corresponding sample the trajectory $W^{(\sigma)}_\omega(t)$ is a diffeomorphism of the form $W^{(\sigma)}_\omega(t)(m) = m + \sigma w_\omega(t)$. Note that for given $\omega \in \Omega$ and given $t \in \mathbb{R}$ we get that $w(t)_\omega$ is a constant vector in $\mathbb{R}^n$. This means that for a given ω and t the action $W^{(\sigma)}_\omega(t)$ is the same as $l_{w(s)_\omega}$.

In terms of the Wiener process $W^{(\sigma)}(t)$, one can introduce analogues of ordinary stochastic differential equations in the Ito form. In what follows, we consider equations for which the coefficient of $W^{(\sigma)}(t)$ is equal to I. Recall that σ is already included in the construction of $W^{(\sigma)}(t)$.

Mean derivatives are also introduced in complete analogy to the usual definition. We are interested in the operator of the second backward derivative D_*D_*. Recall that we understand this expression as the application operator D_* to the regression of the backward derivative of the random process (i.e. to a vector field).

3. The main result

Here and below we assume that $s > n/2 + 2$. So the diffeomorphisms from $\mathcal{D}^s(\mathcal{T}^n)$ are C^2-smooth as well as vector fields from $T_e\mathcal{D}^s(\mathcal{T}^n)$ on the torus. Everywhere below we use the same stochastic process $W^{(\sigma)}(t)$ constructed from the Wiener process $w(t)$ in $\mathbb{R}^n$ by the formula (2.4).

Consider the equation

$$D_*D_*\xi(t) = 0 \tag{3.1}$$

We interpret the solution of this equation as a random flow, whose mathematical expectation is the flow of a viscous fluid with viscosity coefficient $\frac{\sigma^2}{2}$. To make sure With this, we turn to the Euler description, i.e. describe the corresponding equations in $T_e\mathcal{D}^s(\mathcal{T}^n)$.

Transferring both sides of the (3.1) equation to e, we get the finite-dimensional equation on the torus of the form $D_*D_*\xi(t) = F$. Note that in this case, when defining D_*, the conditional mathematical expectation is replaced by the usual mathematical expectation, but nevertheless less (1.6) remains true. Let's denote the regression $D_*\xi$ by v. Then from equality (1.6) we get in $T_e\mathcal{D}^s(\mathcal{T}^n)$ the Burgers equation

$$\frac{\partial}{\partial t}v + (v\cdot\nabla)v - \frac{\sigma^2}{2}\Delta v = 0$$

Consider on $\mathcal{D}^s(\mathcal{T}^n)$ system of equations

$$D_*D_*\xi(t) = 0 \tag{3.2}$$

and

$$D_S\xi(t) = \frac{1}{2}D_*\xi(t). \tag{3.3}$$

From (3.3) it follows that for the solution of this system its forward mean derivative $D\xi(t) = 0$. This means that $\xi(t)$ is a martingale. In other words, one can consider any martingale e.g. $W^{(\sigma)}(t)$. And then at the right-hand side corresponding to this martingale (we emphasize that not for every right-hand side) this system has a solution.

Introduce the notation $D_*\xi(t) = u(t)_{\xi(t)}$ and transfer by right shifts on the group all $u(t)_{\xi(t)}$ to the tangebt space at the unit e. As well as above, when defining D_*, the conditional mathematical expectation is replaced by the usual mathematical expectation. Thus in the tangent space at e we obtain the deterministic curve $u_e(t)$. It is a non-autonomus vector field on the torus.

Theorem 3.1. *The curve $u_e(t)$ is a solution of the system*

$$\frac{\partial}{\partial t}u + (u\cdot\nabla)u - \frac{\sigma^2}{2}\Delta u = 0, \tag{3.4}$$

$$\frac{\partial\rho^\xi(t,x)}{\partial t} = -\frac{1}{2}Div(\rho^\xi(t,x)u^\xi(t,x)). \tag{3.5}$$

To prove the Theorem, we transfer both equations (3.2) and (3.3) by right shifts to e. As stated above, equation (3.2) becomes the Burgers equation (3.4) on the torus, i.e. it describes a viscous fluid. Since $W^{(s)}$ is a martingale, one can easily derive from formula (1.7) that equation (3.3) becomes the equation of the continuity type (3.5). We interpret the ρ obtained from this equation as the density of the fluid.

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