CROSS CURRENCY EUROPEAN SWAPTION MODEL

A Cross Currency European Swaption is a European Swaption to enter into a swap to exchange cash flows in two different currencies. The domestic and foreign swap leg cash flows can be fixed or floating. The model outlined here is called a Multi-Currency Terminal Swap Rate Model which generalizes a Terminal Swap Rate Model to incorporate foreign exchange. The main idea behind a Terminal Swap Rate Model is to assume that the discount factors at the option maturity can be written as a function of the underlying swap rates. This assumption reduces the number of stochastic variables that need to be modelled.

A Cross Currency European Swaption gives the holder the option to enter into a swap to exchange cash flows in two different currencies. The domestic and foreign swap leg cash flows can be fixed or floating. The cash flow generation can be referred to as https://finpricing.com/lib/FiBondCoupon.html

The underlying cross-currency swap can be fixed-to-fixed, fixed-to-floating and floating-to-floating types with possible floating spread and principal exchanges which may happen at the beginning of the swap or at the end of the swap or at both the beginning and the end. The floating index interest rate for the CAD is BA rate and the one for USD is the LIBOR rate. In this swaption, the BA-LIBOR basis spread is also considered.

Even for a European cross-currency swaptions, a number of enforced assumptions have to be introduced to reduce the complexity of the problem. Some of the assumptions are purely technical and some of them are supported by historical observations. One of the technical assumptions is that PVBPs for both currencies at a swaption maturity can be approximated by the corresponding forward PVBPs.

Let $m, n \geq 1$ be integers and

$$0 < t_0^d < t_1^d < ... < t_n^d \qquad\qquad 0 < t_0^f < t_1^f < ... < t_m^f$$

Let the domestic and foreign daycount fractions be defined, respectively, as

$$\alpha_j^d = DCF\left(t_{j-1}^d, t_j^d, domesticDaycountBasis\right) \ , \ j = 1,\ldots,n \ , \ \alpha^d = \left(\alpha_1^d,...,\alpha_n^d\right)^T$$

$$\alpha_i^f = DCF\left(t_{i-1}^f, t_i^f, foreignDaycountBasis\right) \ , \ i = 1,\ldots,m \ , \ \alpha^f = \left(\alpha^f{}_1,...,\alpha^f{}_m\right)^T$$

and $f_j^d(t)$, $f_i^f(t)$ be the domestic and foreign forward interest rates seen at time t for the forward accrual periods of $\left(t_{j-1}^d, t_j^d\right), \left(t_{i-1}^f, t_i^f\right)$, $t \leq t_{j-1}^d$, $t \leq t_{i-1}^f$. We define $df_j^d(t)$, $df_i^f(t)$ as the domestic and foreign discount factors at t to the time points t_j^d, t_i^f, respectively.

Let the present values, at time T, of the domestic and foreign fixed leg cash flows be respectively defined as

$$X_T^d(a,b) \equiv \sum_{j=1}^{n} K_d \cdot N_d \cdot \alpha_j^d \cdot df_j^d(T) + a \cdot N_d \cdot df_n^d(T) - b \cdot N_d \cdot df_0^d(T) \qquad (1a)$$

$$X_T^f(a,b) \equiv \left(\sum_{i=1}^{m} K_f \cdot N_f \cdot \alpha_i^f df_i^f(T) + a \cdot N_f \cdot df_m^f(T) - b \cdot N_f \cdot df_0^f(T)\right) \cdot FX_T \qquad (1b)$$

where,

$$T \leq t_0^k, k = d, f$$

N_d, N_f are the domestic and foreign notionals, respectively.

$K_{d,} K_f$ are the domestic and foreign fixed rates, respectively.

FX_T is the FX rate at time T expressed as units of domestic per unit of foreign currency.

$a = 1$ if notional amounts are exchanged at the maturity of the swap, else $a = 0$

$b = 1$ if notional amounts are exchanged at the start of the swap, else $b = 0$

Let the present values, at time T, of the domestic and foreign floating leg cash flows be respectively defined as

$$F_T^d(a,b) \equiv \sum_{j=1}^{n} \left(x + f_j^d(T)\right) \cdot N_d \cdot \alpha_j^d \cdot df_j^d(T) + a \cdot N_d \cdot df_n^d(T) - b \cdot N_d \cdot df_0^d(T) \qquad (1c)$$

$$F_T^f(a,b) \equiv \left(\sum_{i=1}^{m} \left(y + f_i^f(T)\right) \cdot N_f \cdot \alpha_i^f \cdot df_i^f(T) + a \cdot N_f \cdot df_m^f(T) - b \cdot N_f \cdot df_0^f(T) \right) \cdot FX_T \qquad (1d)$$

where,

$T \le t_0^k, k = d, f$

x, y are the domestic and foreign floating rate spreads, respectively.

We define the domestic and foreign PVBP factors, respectively, as.

$$P_t^d \equiv \sum_{j=1}^{n} \alpha_j^d \cdot df_j^d(t) \qquad P_t^f \equiv \sum_{i=1}^{m} \alpha_i^f \cdot df_i^f(t), \quad t \le t_0^k, k = d, f \qquad (2)$$

and the domestic and foreign vanilla swap rates, respectively, as.

$$S_t^d \equiv \frac{df_0^d(t) - df_n^d(t)}{P_t^d} \qquad S_t^f \equiv \frac{df_0^f(t) - df_m^f(t)}{P_t^f}\ ,\ t \le t_0^k, k = d, f \tag{3}$$

With (2) and (3) we can re-express 1(a,b,c,d) at time T , assuming $T = t_0^d = t_o^f$, as:

$$X_T^d(a,b) \equiv K_d \cdot N_d \cdot P_T^d + a \cdot N_d \cdot \left(1 - (\lambda_T + S_T^d) \cdot P_T^d\right) - b \cdot N_d \tag{4a}$$

$$X_T^f(a,b) \equiv \left(K_f \cdot N_f \cdot P_T^f + a \cdot N_f \cdot \left(1 - S_T^f \cdot P_T^f\right) - b \cdot N_f\right) \cdot FX_T \tag{4b}$$

$$F_T^d(a,b) \equiv \left(x + S_T^d\right) \cdot N_d \cdot P_T^d + a \cdot N_d \cdot \left(1 - (\lambda_T + S_T^d) \cdot P_T^d\right) - b \cdot N_d \tag{4c}$$

$$F_T^f(a,b) \equiv \left(\left(y + S_T^f\right) \cdot N_f \cdot P_T^f + a \cdot N_f \cdot \left(1 - S_T^f \cdot P_T^f\right) - b \cdot N_f\right) \cdot FX_T \tag{4d}$$

Note: that we have also added a basis spread, λ_T , to the domestic swap rate.

Further simplifying we have

$$X_T^d(a,b) = c_X^d \cdot S_T^d + q\lambda_T + r_X^d \qquad where, c_X^d = -a \cdot N_d \cdot P_T^d \tag{5a}$$

$$q = -a \cdot N_d \cdot P_T^d$$

$$r_X^d = N_d \cdot \left(K_d \cdot P_T^d + (a - b)\right)$$

$$X_T^f(a,b) = FX_T \cdot \left(c_X^f \cdot S_T^f + r_X^f\right) \qquad where, c_X^f = -a \cdot N_f \cdot P_T^f \tag{5b}$$

$$r_X^f = N_f \cdot \left(K_f \cdot P_T^f + (a - b)\right)$$

$$F_T^d(a,b) = c_F^d \cdot S_T^d + q\lambda_T + r_F^d \quad where, c_F^d = N_d \cdot P_T^d(1 - a) \tag{5c}$$

$$q = -a \cdot N_d \cdot P_T^d$$

$$r_F^d = N_d \cdot \left(x \cdot P_T^d + (a - b)\right)$$

$$F_T^f(a,b) = FX_T \cdot \left(c_F^f \cdot S_T^f + r_F^f\right) \qquad where, c_F^f = N_f \cdot P_T^f(1-a) \tag{5d}$$

$$r_F^f = N_f \cdot \left(y \cdot P_T^f + (a-b)\right)$$

We represent the present values, at time T, of a swap to exchange the domestic and foreign leg cash flows as

$$V\left(T,\beta,X^d,X^f\right) \equiv \beta \cdot \left(X_T^d(a,b) - X_T^f(a,b)\right) \tag{6a}$$

$$V\left(T,\beta,X^d,F^f\right) \equiv \beta \cdot \left(X_T^d(a,b) - F_T^f(a,b)\right) \tag{6b}$$

$$V\left(T,\beta,F^d,X^f\right) \equiv \beta \cdot \left(F_T^d(a,b) - X_T^f(a,b)\right) \tag{6c}$$

$$V\left(T,\beta,F^d,F^f\right) \equiv \beta \cdot \left(F_T^d(a,b) - F_T^f(a,b)\right) \tag{6d}$$

where, $\beta = 1$ indicates a pay-foreign swap and $\beta = -1$ indicates a receive-foreign swap. Let $0 \le T = t_0^d = t_0^f$, then the payoff to the option at maturity can be expressed as:

$$\left[V\left(T,\beta,X^d,X^f\right)\right]^+ \equiv \left[\beta \cdot \left(X_T^d(a,b) - X_T^f(a,b)\right)\right]^+ \tag{7a}$$

$$\left[V\left(T,\beta,X^d,F^f\right)\right]^+ \equiv \left[\beta \cdot \left(X_T^d(a,b) - F_T^f(a,b)\right)\right]^+ \tag{7b}$$

$$\left[V\left(T,\beta,F^d,X^f\right)\right]^+ \equiv \left[\beta \cdot \left(F_T^d(a,b) - X_T^f(a,b)\right)\right]^+ \tag{7c}$$

$$\left[V\left(T,\beta,F^d,F^f\right)\right]^+ \equiv \left[\beta \cdot \left(F_T^d(a,b) - F_T^f(a,b)\right)\right]^+ \tag{7d}$$

We assume the following dynamics

$$d\ln S_t^i = (u_i - {\sigma_i^2}/{2})dt + \sigma_i \cdot dW_t^i \tag{8}$$

$$d\ln FX_t = (u_{FX} - {\sigma_X^2}/{2})dt + \sigma_{FX} \cdot dW_t^{FX} \tag{9}$$

$$d\lambda = \overline{\lambda} + \sigma_\lambda \cdot dW_t^\lambda$$

where,

$$dW_t^j \cdot dW_t^k = \rho_{j,k} dt \qquad (10)$$

$$i = d, f$$

$$j,k = d, f, FX, \lambda$$

$\sigma_d, \sigma_f, \sigma_{FX}, \sigma_\lambda, u_d, u_f, u_{FX}$ are deterministic functions of time.

$W_t^k, k = d, f, M, \lambda$ is a 4-dimensional Brownian motion.

Given the above dynamics the variables $\ln S_T^d, \ln S_T^f, \ln FX_T, \lambda_T$ are joint-normally distributed.

$$\begin{pmatrix} \ln S_T^d \\ \ln S_T^f \\ \ln FX_T \\ \lambda_T \end{pmatrix} \sim N(m, \Sigma) \qquad (11)$$

where,

$$m = \begin{pmatrix} E_t^Q[\ln S_T^d] \\ E_t^Q[\ln S_T^f] \\ E_t^Q[\ln FX_T] \\ E_t^Q[\lambda_T] \end{pmatrix} = \begin{pmatrix} \ln \bar{S}_T^d + (T-t)\cdot\left(-\frac{1}{2}\bar{\sigma}^d(T,t)^2\right) \\ \ln \bar{S}_T^f + (T-t)\cdot\left(-\frac{1}{2}\bar{\sigma}^f(T,t)^2\right) \\ \ln \overline{FX}_T + (T-t)\cdot\left(-\frac{1}{2}\bar{\sigma}^{FX}(T,t)^2\right) \\ \lambda_t + (T-t)\cdot\left(-\frac{1}{2}\bar{\sigma}^\lambda(T,t)^2\right) \end{pmatrix} \qquad (12)$$

where,

$$\bar{\sigma}^i(T,t) = \sqrt{\left(\frac{1}{T-t}\right)\cdot\int_t^T \left(\sigma_\tau^i\right)^2 \cdot d\tau}, \qquad (13)$$

$$i = d, f, FX, \lambda$$

$$\Sigma = \begin{bmatrix} \sigma_{FX}^2 & \sigma_{FX,d} & \sigma_{FX,f} & \sigma_{FX,\lambda} \\ \sigma_{d,FX} & \sigma_d^2 & \sigma_{d,f} & \sigma_{d,\lambda} \\ \sigma_{f,FX} & \sigma_{f,d} & \sigma_f^2 & \sigma_{f,\lambda} \\ \sigma_{\lambda,FX} & \sigma_{\lambda,d} & \sigma_{\lambda,f} & \sigma_\lambda^2 \end{bmatrix} \tag{14}$$

where,

$\sigma_i^2 = (T-t)\cdot\left(\overline{\sigma}^i(T,t)\right)^2$

$i = d, f, FX, \lambda$

$\sigma_{i,j} = \rho_{i,j}\cdot(T-t)\cdot\overline{\sigma}^i(T,t)\cdot\overline{\sigma}^j(T,t)$

$i, j = d, f, FX, \lambda$

$\overline{S}_T^d, \overline{S}_T^f, \overline{FX}_T$ are forward values as seen from time t.

We calulate the time-t value of the options given in 7(a,b,c,d), where $0 \le t \le T$, as

$$\begin{aligned} V_t^{X,X} &= df_T^d(t)\cdot \mathrm{E}_t\left[\left[\beta\cdot\left(X_T^d - X_T^f\right)\right]^+\right] \\ &= df_T^d(t)\cdot E_t\left[\left[\beta\cdot\left(c_X^d\cdot S_T^d + q\cdot\lambda_T + r_X^d - FX_T\cdot\left(c_X^f\cdot S_T^f + r_X^f\right)\right)\right]^+\right] \end{aligned} \tag{15a}$$

$$\begin{aligned} V_t^{X,F} &= df_T^d(t)\cdot \mathrm{E}_t\left[\left[\beta\cdot\left(X_T^d - F_T^f\right)\right]^+\right] \\ &= df_T^d(t)\cdot E_t\left[\left[\beta\cdot\left(c_X^d\cdot S_T^d + q\cdot\lambda_T + r_X^d - FX_T\cdot\left(c_F^f\cdot S_T^f + r_F^f\right)\right)\right]^+\right] \end{aligned} \tag{15b}$$

$$\begin{aligned} V_t^{F,X} &= df_T^d(t)\cdot \mathrm{E}_t\left[\left[\beta\cdot\left(F_T^d - X_T^f\right)\right]^+\right] \\ &= df_T^d(t)\cdot E_t\left[\left[\beta\cdot\left(c_F^d\cdot S_T^d + q\cdot\lambda_T + r_F^d - FX_T\cdot\left(c_X^f\cdot S_T^f + r_X^f\right)\right)\right]^+\right] \end{aligned} \tag{15c}$$

$$V_t^{F,F} = df_T^d(t) \cdot \mathrm{E}_t\left[\left[\beta \cdot \left(F_T^d - F_T^f\right)\right]^+\right]$$

$$= df_T^d(t) \cdot E_t\left[\left[\beta \cdot \left(c_F^d \cdot S_T^d + q \cdot \lambda_T + r_F^d - FX_T \cdot \left(c_F^f \cdot S_T^f + r_F^f\right)\right)\right]^+\right] \quad (15d)$$

Consider the following general form of the conditional expectation in 15(a,b,c,d)

$$E_t\left[V(T)^+\right] = E_t\left[\left[\bar{c}^d \cdot S_T^d + \bar{q} \cdot \lambda_T + \bar{r}^d - FX_T \cdot \left(\bar{c}^f \cdot S_T^f + \bar{r}^f\right)\right]^+\right] \quad (16)$$

where,

$\bar{c}^k = \beta \cdot c^k, k = d, f$

$\bar{r}^k = \beta \cdot r^k, k = d, f$

$\bar{q} = \beta \cdot q$

$$E_t\left[V(T)^+\right] = \int_{-\infty}^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\left(\bar{c}^d \cdot S_T^d + q \cdot \lambda_T + \bar{r}^d - FX_T \cdot \left(\bar{c}^f \cdot S_T^f + \bar{r}^f\right)\right)^+$$

$$\cdot f_{FX_T, S_T^d, S_T^f, \lambda_T}\left(FX_T, S_T^d, S_T^f, \lambda_T\right) \cdot dFX_T dS_T^d dS_T^f d\lambda \quad (17)$$

$f_{FX_T, S_T^d, S_T^f, \lambda_T}\left(FX_T, S_T^d, S_T^f, \lambda_T\right)$ is the corresponding density function.

To solve (23) we condition first on S_T^d, S_T^f and λ_T which yields

$$E_t\left[V(T)^+\right] = \int_{-\infty}^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\left[\int_0^{+\infty}\left(\bar{c}^d \cdot S_T^d + q + \bar{r}^d - FX_T \cdot \left(\bar{c}^f \cdot S_T^f + \bar{r}^f\right)\right)^+ \cdot f_{FX_T|S_T^d, S_T^f, \lambda_T}\left(FX_T\right) \cdot dFX_T\right] \cdot$$

$$\cdot\, f_{S_T^d, S_T^f, \lambda_T}\left(S_T^d, S_T^f, \lambda_T\right) \cdot dS_T^d\, dS_T^f\, d\lambda_T \tag{18}$$

$$E_t\left[V(T)^+\right] = \int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty}\left[\int_0^{+\infty}\left(K\left(S_T^d, \lambda_T\right) - c(S_T^f)FX_T\right)^+ \cdot f_{FX_T|S_T^d, S_T^f, \lambda_T}\left(FX_T\right)\cdot dFX_T\right]\cdot$$

$$\cdot\, f_{S_T^d, S_T^f, \lambda_T}\left(S_T^d, S_T^f, \lambda_T\right) \cdot dS_T^d\, dS_T^f\, d\lambda \tag{19}$$

where,

$$c(S_T^f) = \bar{c}^{\,f} \cdot S_T^f + \bar{r}^{\,f}$$
$$K(S_T^d, \lambda_T) = \bar{c}^{\,d} \cdot S_T^d + \bar{q} \cdot \lambda_T + \bar{r}^{\,d}$$

$f_{FX_T|S_T^d, S_T^f, \lambda_T}\left(FX_T\right)$, $f_{S_T^d, S_T^f, \lambda_T}\left(S_T^d, S_T^f, \lambda_T\right)$ are the corresponding conditional and trivariate densities.

Let

$$BS^*\left(S_T^d, S_T^f, \lambda_T\right) \equiv \int_0^{+\infty}\left(K\left(S_T^d, \lambda_T\right) - c(S_T^f)FX_T\right)^+ \cdot f_{FX_T|S_T^d, S_T^f, \lambda_T}\left(FX_T\right)\cdot dFX_T \tag{20}$$

The BS in $BS^*\left(S_T^d, S_T^f, \lambda_T\right)$ stands for Black-Scholes since depending on the signs of $c(S_T^f)$ and $K\left(S_T^d, \lambda_T\right)$, $BS^*\left(S_T^d, S_T^f, \lambda_T\right)$ reduces to the Black-Scholes equation.

The evaluation of $BS^*\left(S_T^d, S_T^f, \lambda_T\right)$ is as follows:

Dropping the arguments of the functions $c(S_T^f)$ and $K\left(S_T^d, \lambda_T\right)$ we write

$$payoff_T \equiv \left(K - cFX_T\right)^+ \tag{21}$$

Case 1: if $c < 0$ Then $payoff_T = |c| \cdot \left(-\frac{K}{c} + FX_T \right)^+$

Case 1a: if $K < 0$ then $BS^*\left(S_T^d, S_T^f, \lambda_T\right) = |c| \times [Black - Scholes(call)]$

Case 1b: if $K \geq 0$ then $BS^*\left(S_T^d, S_T^f, \lambda_T\right) = |c| \cdot E_t[FX_T] + |K|$

Case 2: if $c > 0$ Then $payoff_T = c \cdot \left(\frac{K}{c} - FX_T \right)^+$

Case 2a: if $K > 0$ then $BS^*\left(S_T^d, S_T^f, \lambda_T\right) = c \times [Black - Scholes(put)]$

Case 2b: if $K \leq 0$ then $BS^*\left(S_T^d, S_T^f, \lambda_T\right) = 0$

Case 3: if $\bar{c}^d = 0$ then $payoff_T = (K)^+$

Case 3a: if $K \geq 0$ then $BS^*\left(S_T^d, S_T^f, \lambda_T\right) = K$

Case 3b: if $K < 0$ $BS^*\left(S_T^d, S_T^f, \lambda_T\right) = 0$

where,

$$Black - Scholes(call) = \exp\left(M^d + \frac{1}{2} V^d \right) \cdot \Theta\left(\frac{M^d + V^d - \ln\left(|K| / |c| \right)}{\sqrt{V^d}} \right) - \frac{|K|}{|c|} \cdot \Theta\left(\frac{M^d - \ln\left(|K| / |c| \right)}{\sqrt{V^d}} \right)$$

$$Black - Scholes(put) = \frac{K}{c} \cdot \Theta\left(-\left(\frac{M^d - \ln\left(K/c \right)}{\sqrt{V^d}} \right) \right) - \exp\left(M^d + \frac{1}{2} V^d \right) \cdot \Theta\left(-\left(\frac{M^d + V^d - \ln\left(K/c \right)}{\sqrt{V^d}} \right) \right)$$

$$M^d \equiv E_t\left[\ln FX_T \mid \ln S_T^d, \ln S_T^f, \lambda_T \right]$$

$$V^d \equiv \mathrm{var}_t\left[\ln FX_T \mid \ln S_T^d, \ln S_T^f, \lambda_T \right]$$

Refer to the Appendix for details on calculating conditional moments of a multivariate normal distribution.

With $BS^*\left(S_T^d, S_T^f, \lambda_T\right)$ well defined we now need to solve

$$E_t\left[V(T)^+\right] = \int_{-\infty}^{+\infty}\int_{0}^{+\infty}\int_{0}^{+\infty} BS^*(S_T^d, S_T^f, \lambda_T) \cdot f_{S_T^d, S_T^f, \lambda_T}(S_T^d, S_T^f, \lambda_T) \cdot dS_T^d dS_T^f d\lambda_T \qquad (22)$$

Let

$$y_1 \equiv \ln S_T^d \qquad y_2 \equiv \ln S_T^f \qquad y_3 = \lambda_T$$

(23a,b,c)

Then

$$E_t\left[V(T)^+\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} BS^*(\exp(y_1), \exp(y_2), y_3) \cdot f_{y_1,y_2,y_3}(y_1, y_2, y_3) \cdot dy_1 dy_2 dy_3 \qquad (24)$$

$f_{y_1,y_2,y_3}(y_1, y_2, y_3)$ is the multivariate normal density function.

We now proceed by conditioning on y_2 and y_3 to integrate with respect to y_1. Then we condition on y_3 to integrate with respect to y_2. Then we integrate with respect to y_3. This allows us to write

$$E_t\left[V(T)^+\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} BS^*(\exp(y_1), \exp(y_2), y_3)$$
$$\cdot f_{y_1|y_2,y_3}(y_1) \cdot f_{y_2|y_3}(y_2) \cdot f_{y_3}(y_3) \cdot dy_1 dy_2 dy \qquad (25)$$

We define the following

$$\bar{u}_{y_1} \equiv E_t[y_1 \mid y_2, y_3] \qquad \bar{u}_{y_2} \equiv E_t[y_2 \mid y_3] \qquad u_{y_3} \equiv E_t[y_3]$$

(26a,b,c)

$$\bar{\sigma}^2_{y_1} \equiv \mathrm{var}_t[y_1 \mid y_2, y_3] \qquad \bar{\sigma}^2_{y_2} \equiv \mathrm{var}_t[y_2 \mid y_3] \qquad \sigma_{y_3} \equiv \mathrm{var}_t[y_3]$$

(27a,b,c)

where, the bars on the variables above indicate that they are conditional moments.

We can now write

$$E_t\left[V(T)^+\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} BS^* \cdot \tfrac{1}{\sqrt{2\pi}\bar{\sigma}_{y_1}} \exp\left(-\tfrac{(y_1-\bar{u}_{y_1})^2}{2\bar{\sigma}^2_{y_1}}\right) \cdot \tfrac{1}{\sqrt{2\pi}\bar{\sigma}_{y_2}} \exp\left(-\tfrac{(y_2-\bar{u}_{y_2})^2}{2\bar{\sigma}^2_{y_2}}\right)$$

$$\cdot \tfrac{1}{\sqrt{2\pi}\sigma_{y_3}} \exp\left(-\tfrac{(y_3-u_{y_3})^2}{2\sigma^2_{y_3}}\right) \cdot dy_1 dy_2 dy_3 \qquad (28)$$

We make the following change of variables

$$y_1 = z_1\bar{\sigma}_{y_1} + \bar{u}_{y_1} \qquad y_2 = z_2\bar{\sigma}_{y_2} + \bar{u}_{y_2} \qquad y_3 = z_3\sigma_{y_3} + u_{y_3}$$

(29a,b,c)

$$z_1 = \sqrt{2} \cdot x_1 \qquad z_2 = \sqrt{2} \cdot x_2 \qquad z_3 = \sqrt{2} \cdot x_3$$

(30a,b,c)

(29a,b,c) & (30a,b,c) imply

$$y_1 = \sqrt{2}x_1\overline{\sigma}_{y_1}^2 + \overline{u}_{y_1} \qquad y_2 = \sqrt{2}x_2\overline{\sigma}_{y_2}^2 + \overline{u}_{y_2} \qquad y_3 = \sqrt{2}x_3\sigma_{y_3} + u_{y_3}$$

(31a,b,c)

Which allows us to express

$$E_t\left[V(T)^+\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} BS^*\left(\exp(\sqrt{2}x_1\overline{\sigma}_{y_1} + \overline{u}_{y_1}), \exp(\sqrt{2}x_2\overline{\sigma}_{y_2} + \overline{u}_{y_2}), \exp(\sqrt{2}x_3\sigma_{y_3} + u_{y_3})\right)$$

$$\cdot \pi^{-3/2} \cdot \exp\left(-x_1^2\right) \cdot \exp\left(-x_2^2\right) \cdot \exp\left(-x_3^2\right) \cdot dx_1 dx_2 dx_3 \qquad (32)$$