NON-ARCHIMEDEAN WELCH BOUNDS AND NON-ARCHIMEDEAN ZAUNER CONJECTURE

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Date: September 2, 2022

Abstract: Let $\mathbb{K}$ be a non-Archimedean (complete) valued field satisfying

$$\left|\sum_{j=1}^{n}\lambda_j^2\right| = \max_{1\leq j\leq n}|\lambda_j|^2, \quad \forall \lambda_j\in\mathbb{K}, 1\leq j\leq n, \forall n\in\mathbb{N}.$$

For $d\in\mathbb{N}$, let $\mathbb{K}^d$ be the standard d-dimensional non-Archimedean Hilbert space. Let $m\in\mathbb{N}$ and $\text{Sym}^m(\mathbb{K}^d)$ be the non-Archimedean Hilbert space of symmetric m-tensors. We prove the following result. If $\{\tau_j\}_{j=1}^n$ is a collection in $\mathbb{K}^d$ satisfying $\langle \tau_j, \tau_j\rangle = 1$ for all $1\leq j\leq n$ and the operator $\text{Sym}^m(\mathbb{K}^d)\ni x\mapsto \sum_{j=1}^n\langle x, \tau_j^{\otimes m}\rangle\tau_j^{\otimes m}\in \text{Sym}^m(\mathbb{K}^d)$ is diagonalizable, then

$$\max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^{2m}\} \geq \frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}. \tag{1}$$

We call Inequality (1) as the non-Archimedean version of Welch bounds obtained by Welch [*IEEE Transactions on Information Theory, 1974*]. We formulate non-Archimedean Zauner conjecture.

Keywords: Non-Archimedean valued field, non-Archimedean Hilbert space, Welch bound, Zauner conjecture.

Mathematics Subject Classification (2020): 12J25, 46S10, 47S10.

1. Introduction

Forty-eight years ago, Prof. L. Welch proved the following result [80].

Theorem 1.1. *[80] (**Welch Bounds**) Let $n > d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{C}^d$, then*

$$\sum_{1\leq j,k\leq n}|\langle \tau_j, \tau_k\rangle|^{2m} = \sum_{j=1}^{n}\sum_{k=1}^{n}|\langle \tau_j, \tau_k\rangle|^{2m} \geq \frac{n^2}{\binom{d+m-1}{m}}, \quad \forall m\in\mathbb{N}.$$

In particular,

$$\sum_{1\leq j,k\leq n}|\langle \tau_j, \tau_k\rangle|^{2} = \sum_{j=1}^{n}\sum_{k=1}^{n}|\langle \tau_j, \tau_k\rangle|^{2} \geq \frac{n^2}{d}.$$

Further,

$$\textbf{(Higher order Welch bounds)} \quad \max_{1\leq j,k\leq n, j\neq k}|\langle \tau_j, \tau_k\rangle|^{2m} \geq \frac{1}{n-1}\left[\frac{n}{\binom{d+m-1}{m}}-1\right], \quad \forall m\in\mathbb{N}.$$

In particular,

$$\textbf{\textit{(First order Welch bound)}} \quad \max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle|^2 \geq \frac{n-d}{d(n-1)}.$$

There are infinitely many applications of Theorem 1.1 such as in the study of root-mean-square (RMS) absolute cross relation of unit vectors [67], frame potential [9, 14, 18], correlations [66], codebooks [27], numerical search algorithms [81, 82], quantum measurements [69], coding and communications [72, 76], code division multiple access (CDMA) systems [49, 50], wireless systems [64], compressed/compressive sensing [1, 6, 29, 32, 68, 74, 75, 77], 'game of Sloanes' [45], equiangular tight frames [73], equiangular lines [23, 31, 44, 57], digital fingerprinting [56] etc.

Different proofs/improvements of Theorem 1.1 have been done in [19, 24, 25, 28, 42, 65, 72, 78, 79]. In 2021 M. Krishna derived continuous version of Theorem 1.1 [51]. In 2022 M. Krishna obtained Theorem 1.1 for Hilbert C*-modules [53] and Banach spaces [52].

In this paper we derive non-Archimedean Welch bounds (Theorem 2.3). We formulate non-Archimedean Zauner conjecture (Conjecture 3.2).

2. Non-Archimedean Welch bounds

Let $\mathbb{K}$ be a non-Archimedean (complete) valued field satisfying

$$\left|\sum_{j=1}^{n}\lambda_j^2\right| = \max_{1\leq j\leq n}|\lambda_j|^2, \quad \forall \lambda_j\in \mathbb{K}, 1\leq j\leq n, \forall n\in \mathbb{N}. \tag{2}$$

Such non-Archimedean fields exist, see [61]. Throughout the paper, we assume that our non-Archimedean field satisfies Equation (2). For $d\in\mathbb{N}$, let $\mathbb{K}^d$ be the standard non-Archimedean Hilbert space equipped with the inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d\rangle \coloneqq \sum_{j=1}^{d} a_j b_j, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{K}^d.$$

Theorem 2.1. ***(First Order Non-Archimedean Welch Bound)*** *If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{K}^d$ such that the operator $S_\tau : \mathbb{K}^d \ni x \mapsto \sum_{j=1}^{n}\langle x, \tau_j\rangle \tau_j \in \mathbb{K}^d$ is diagonalizable, then*

$$\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^2\right|, |\langle \tau_j, \tau_k\rangle|^2\right\} \geq \frac{1}{|d|}\left|\sum_{j=1}^{n}\langle \tau_j, \tau_j\rangle\right|^2.$$

In particular, if $\langle \tau_j, \tau_j\rangle = 1$ for all $1\leq j\leq n$, then

$$\textbf{\textit{(First order non-Archimedean Welch bound)}} \quad \max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^2\} \geq \frac{|n|^2}{|d|}.$$

Proof. We first note that

$$\operatorname{Tra}(S_\tau) = \sum_{j=1}^{n}\langle \tau_j, \tau_j\rangle,$$

$$\operatorname{Tra}(S_\tau^2) = \sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j, \tau_k\rangle\langle \tau_k, \tau_j\rangle.$$

Let $\lambda_1, \dots, \lambda_d$ be the diagonal entries in the diagonalization of S_τ. Then using the diagonalizability of S_τ and the non-Archimedean Cauchy-Schwarz inequality (Theorem 2.4.2 [61]), we get

$$\begin{aligned}\left|\sum_{j=1}^{n}\langle\tau_j,\tau_j\rangle\right|^2 &= |\operatorname{Tra}(S_\tau)|^2 = \left|\sum_{k=1}^{d}\lambda_k\right|^2 \leq |d|\left|\sum_{k=1}^{d}\lambda_k^2\right| = |d||\operatorname{Tra}(S_\tau^2)| \\ &= |d|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\tau_j,\tau_k\rangle\langle\tau_k,\tau_j\rangle\right| = |d|\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^2 + \sum_{j,k=1,j\neq k}^{n}\langle\tau_j,\tau_k\rangle\langle\tau_k,\tau_j\rangle\right| \\ &\leq |d|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^2\right|, |\langle\tau_j,\tau_k\rangle\langle\tau_k,\tau_j\rangle|\right\} \\ &= |d|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^2\right|, |\langle\tau_j,\tau_k\rangle|^2\right\}.\end{aligned}$$

Whenever $\langle\tau_j,\tau_j\rangle = 1$ for all $1 \leq j \leq n$,

$$|n|^2 \leq |d|\max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle\tau_j,\tau_k\rangle|^2\}.$$

□

We next obtain higher order non-Archimedean Welch bounds. We use the following vector space result.

Theorem 2.2. *[13, 21] If $\mathcal{V}$ is a vector space of dimension d and $Sym^m(\mathcal{V})$ denotes the vector space of symmetric m-tensors, then*

$$dim(Sym^m(\mathcal{V})) = \binom{d+m-1}{m}, \quad \forall m \in \mathbb{N}.$$

Theorem 2.3. ***(Higher Order Non-Archimedean Welch Bounds)*** *Let $m \in \mathbb{N}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{K}^d$ such that the operator $S_\tau : Sym^m(\mathbb{K}^d) \ni x \mapsto \sum_{j=1}^{n}\langle x, \tau_j^{\otimes m}\rangle\tau_j^{\otimes m} \in Sym^m(\mathbb{K}^d)$ is diagonalizable, then*

$$\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}\right|, |\langle\tau_j,\tau_k\rangle|^{2m}\right\} \geq \frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\sum_{j=1}^{n}\langle\tau_j,\tau_j\rangle^m\right|^2.$$

In particular, if $\langle\tau_j,\tau_j\rangle = 1$ for all $1 \leq j \leq n$, then

$$\textbf{\textit{(Higher order non-Archimedean Welch bound)}} \quad \max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle\tau_j,\tau_k\rangle|^{2m}\} \geq \frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}.$$

Proof. Let $\lambda_1, \dots, \lambda_{\dim(\mathrm{Sym}^m(\mathbb{K}^d))}$ be the diagonal entries in the diagonalization of S_τ. Then

$$
\begin{aligned}
\left|\sum_{j=1}^{n}\langle \tau_j, \tau_j\rangle^m\right|^2 &= \left|\sum_{j=1}^{n}\langle \tau_j^{\otimes m}, \tau_j^{\otimes m}\rangle\right|^2 = |\operatorname{Tra}(S_\tau)|^2 = \left|\sum_{k=1}^{\dim(\operatorname{Sym}^m(\mathbb{K}^d))}\lambda_k\right|^2\\
&\leq |\dim(\operatorname{Sym}^m(\mathbb{K}^d))|\left|\sum_{k=1}^{\dim(\operatorname{Sym}^m(\mathbb{K}^d))}\lambda_k^2\right| = |\dim(\operatorname{Sym}^m(\mathbb{K}^d))||\operatorname{Tra}(S_\tau^2)|\\
&= \left|\binom{d+m-1}{m}\right||\operatorname{Tra}(S_\tau^2)| = \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j^{\otimes m}, \tau_k^{\otimes m}\rangle\langle \tau_k^{\otimes m}, \tau_j^{\otimes m}\rangle\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j, \tau_k\rangle^m\langle \tau_k, \tau_j\rangle^m\right|\\
&= \left|\binom{d+m-1}{m}\right|\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^{2m} + \sum_{j,k=1, j\neq k}^{n}\langle \tau_j, \tau_k\rangle^m\langle \tau_k, \tau_j\rangle^m\right|\\
&\leq \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^{2m}\right|, |\langle \tau_j, \tau_k\rangle^m\langle \tau_k, \tau_j\rangle^m|\right\}\\
&= \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^{2m}\right|, |\langle \tau_j, \tau_k\rangle|^{2m}\right\}.
\end{aligned}
$$

Whenever $\langle \tau_j, \tau_j\rangle = 1$ for all $1 \leq j \leq n$,

$$
|n|^2 \leq \left|\binom{d+m-1}{m}\right|\max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^{2m}\}.
$$

□

Remark 2.4. *Theorems 2.1 and 2.3 hold by replacing $\mathbb{K}^d$ by a d-dimensional non-Archimedean Hilbert space over $\mathbb{K}$.*

3. Non-Archimedean Zauner Conjecture and open problems

Theorem 2.1 straight away brings the following question.

Question 3.1. ***Given non-Archimedean field $\mathbb{K}$ satisfying Equation (2), for which $(d, n) \in \mathbb{N}\times\mathbb{N}$, there exist vectors $\tau_1, \dots, \tau_n \in \mathbb{K}^d$ satisfying the following.***

- **(i)** $\langle \tau_j, \tau_j\rangle = 1$ ***for all*** $1 \leq j \leq n$.
- **(ii)** ***The operator*** $S_\tau : \mathbb{K}^d \ni x \mapsto \sum_{j=1}^{n}\langle x, \tau_j\rangle\tau_j \in \mathbb{K}^d$ ***is diagonalizable.***
- **(iii)**

$$
\max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^2\} = \frac{|n|^2}{|d|}.
$$

A particular case of Question 3.1 is the following non-Archimedean version of Zauner conjecture (see [2–5,10–12,33,36,43,48,51,55,63,70,84] for Zauner conjecture in Hilbert spaces, [53] for Zauner conjecture in Hilbert C*-modules and [52] for Zauner conjecture in Banach spaces).

Conjecture 3.2. ***(Non-Archimedean Zauner Conjecture) Let $\mathbb{K}$ be a non-Archimedean field satisfying Equation (2). For each $d \in \mathbb{N}$, there exist vectors $\tau_1, \dots, \tau_{d^2} \in \mathbb{K}^d$ satisfying the following.***

(i) $\langle \tau_j, \tau_j\rangle = 1$ ***for all*** $1 \leq j \leq d^2$***.***
(ii) ***The operator*** $S_\tau : \mathbb{K}^d \ni x \mapsto \sum_{j=1}^{d^2}\langle x, \tau_j\rangle \tau_j \in \mathbb{K}^d$ ***is diagonalizable.***
(iii)

$$|\langle \tau_j, \tau_k\rangle|^2 = |d|, \quad \forall 1 \leq j, k \leq d^2, j \neq k.$$

We remember the definition of Gerzon's bound which allows us to recall the bounds which are in the same way to Welch bounds in Hilbert spaces.

Definition 3.3. *[45] Given* $d \in \mathbb{N}$*, define* ***Gerzon's bound***

$$\mathcal{Z}(d, \mathbb{K}) := \begin{cases} d^2 & \text{if} \quad \mathbb{K} = \mathbb{C} \\ \frac{d(d+1)}{2} & \text{if} \quad \mathbb{K} = \mathbb{R}. \end{cases}$$

Theorem 3.4. *[16, 22, 41, 45, 58, 62, 71, 81] Define* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$ *and* $m := \dim_{\mathbb{R}}(\mathbb{K})/2$*. If* $\{\tau_j\}_{j=1}^n$ *is any collection of unit vectors in* $\mathbb{K}^d$*, then*

(i) ***(Bukh-Cox bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq \frac{\mathcal{Z}(n-d, \mathbb{K})}{n(1+m(n-d-1)\sqrt{m^{-1}+n-d}) - \mathcal{Z}(n-d, \mathbb{K})} \quad \text{if} \quad n > d.$$

(ii) ***(Orthoplex/Rankin bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq \frac{1}{\sqrt{d}} \quad \text{if} \quad n > \mathcal{Z}(d, \mathbb{K}).$$

(iii) ***(Levenstein bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq \sqrt{\frac{n(m+1) - d(md+1)}{(n-d)(md+1)}} \quad \text{if} \quad n > \mathcal{Z}(d, \mathbb{K}).$$

(iv) ***(Exponential bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq 1 - 2n^{\frac{-1}{d-1}}.$$

Theorem 3.4 and Theorem 2.1 give the following problem.

Question 3.5. ***Whether there is a non-Archimedean version of Theorem 3.4? In particular, does there exists a version of***

(i) ***non-Archimedean Bukh-Cox bound?***
(ii) ***non-Archimedean Orthoplex/Rankin bound?***
(iii) ***non-Archimedean Levenstein bound?***
(iv) ***non-Archimedean Exponential bound?***

As written already, Welch bounds have applications in study of equiangular lines. Therefore we wish to formulate equiangular line problem for non-Archimedean Hilbert spaces. For the study of equiangular lines in Hilbert spaces we refer [7, 8, 15, 17, 26, 34, 35, 37, 40, 46, 47, 54, 59, 60, 83], quaternion Hilbert space we refer [30], octonion Hilbert space we refer [20], finite dimensional vector spaces over finite fields we refer [38, 39] and for Banach spaces we refer [52] (there equiangular line problem for Banach spaces is not mentioned explicitly but Zauner conjecture for Banach spaces is formulated. One can easily formulate equiangular line problem using that).

Question 3.6. ***(Non-Archimedean Equiangular Line Problem) Let*** $\mathbb{K}$ ***be a non-Archimedean field. Given*** $a \in \mathbb{K}$***,*** $d \in \mathbb{N}$ ***and*** $\gamma > 0$***, what is the maximum*** $n = n(\mathbb{K}, a, d, \gamma) \in \mathbb{N}$ ***such that there exist vectors*** $\tau_1, \dots, \tau_n \in \mathbb{K}^d$ ***satisfying the following.***

(i) $\langle \tau_j, \tau_j \rangle = a$ ***for all*** $1 \leq j \leq n$***.***
(ii) $|\langle \tau_j, \tau_k \rangle|^2 = \gamma$ ***for all*** $1 \leq j, k \leq n, j \neq k$***.***

In particular, whether there is a non-Archimedean Gerzon bound?

Question 3.6 can be easily modified to formulate question of non-Archimedean regular s-distance sets.

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