# p-ADIC WELCH BOUNDS AND p-ADIC ZAUNER CONJECTURE

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**Abstract**: Let $p$ be a prime. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ be the standard $d$-dimensional p-adic Hilbert space. Let $m \in \mathbb{N}$ and $\text{Sym}^m(\mathbb{Q}_p^d)$ be the p-adic Hilbert space of symmetric m-tensors. We prove the following result. Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying (i) $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$ and (ii) there exists $b \in \mathbb{Q}_p$ satisfying $\sum_{j=1}^n \langle x, \tau_j \rangle \tau_j = bx$ for all $x \in \mathbb{Q}_p^d$. Then

$$\max_{1\leq j,k \leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^{2m}\} \geq \frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}. \quad (1)$$

We call Inequality (1) as the p-adic version of Welch bounds obtained by Welch [*IEEE Transactions on Information Theory, 1974*]. Inequality (1) differs from the non-Archimedean Welch bound obtained recently by M. Krishna as one can not derive one from another. We formulate p-adic Zauner conjecture.
**Keywords**: p-adic number field, p-adic Hilbert space, Welch bound, Zauner conjecture.
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## 1. Introduction

In 1974 Prof. L. Welch proved the following result [86].

**Theorem 1.1.** *[86] (**Welch Bounds**) Let $n > d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{C}^d$, then*

$$\sum_{1\leq j,k\leq n} |\langle \tau_j, \tau_k\rangle|^{2m} = \sum_{j=1}^n \sum_{k=1}^n |\langle \tau_j, \tau_k\rangle|^{2m} \geq \frac{n^2}{\binom{d+m-1}{m}}, \quad \forall m \in \mathbb{N}.$$

*In particular,*

$$\sum_{1\leq j,k\leq n} |\langle \tau_j, \tau_k\rangle|^{2} = \sum_{j=1}^n \sum_{k=1}^n |\langle \tau_j, \tau_k\rangle|^{2} \geq \frac{n^2}{d}.$$

*Further,*

$$\textbf{(Higher order Welch bounds)} \quad \max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle|^{2m} \geq \frac{1}{n-1}\left[\frac{n}{\binom{d+m-1}{m}} - 1\right], \quad \forall m \in \mathbb{N}.$$

*In particular,*

$$\textbf{(First order Welch bound)} \quad \max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle|^{2} \geq \frac{n-d}{d(n-1)}.$$

It is impossible to list all applications of Theorem 1.1. A few are: in the study of root-mean-square (RMS) absolute cross relation of unit vectors [73], frame potential [10, 15, 19], correlations [72], codebooks [30],

numerical search algorithms [87, 88], quantum measurements [75], coding and communications [78, 82], code division multiple access (CDMA) systems [54, 55], wireless systems [70], compressed/compressive sensing [2, 7, 32, 35, 74, 80, 81, 83], 'game of Sloanes' [48], equiangular tight frames [79], equiangular lines [24, 34, 47, 63], digital fingerprinting [62] etc.
Theorem 1.1 has been improved/different proofs were given in [20, 25, 26, 31, 45, 71, 78, 84, 85]. In 2021 M. Krishna derived continuous version of Theorem 1.1 [56]. In 2022 M. Krishna obtained Theorem 1.1 for Hilbert C*-modules [58], Banach spaces [57] and non-Archimedean Hilbert spaces [59].
In this paper we derive p-adic Welch bounds (Theorem 2.3). We formulate p-adic Zauner conjecture (Conjecture 3.3).

## 2. p-adic Welch bounds

Let $p$ be a prime. For $d \in \mathbb{N}$, let $\mathbb{Q}_p^d$ be the standard p-adic Hilbert space equipped with the inner product

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d \rangle \coloneqq \sum_{j=1}^d a_j b_j, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{Q}_p^d.$$

Let $I_{\mathbb{Q}_p^d}$ be the identity operator on $\mathbb{Q}_p^d$. Note that $\mathbb{Q}_p^d$ is not a non-Archimedean Hilbert space as it does not satisfies Equation (2) in [59] (see Page 40, [67]). We refer [1, 28, 29, 51, 52] for more on p-adic Hilbert spaces.

**Theorem 2.1.** ***(First Order p-adic Welch Bound)*** *Let $p$ be a prime and $n, d \in \mathbb{N}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{Q}_p^d$ such that there exists $b \in \mathbb{Q}_p$ satisfying*

$$\sum_{j=1}^n \langle x, \tau_j \rangle \tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d,$$

*then*

$$\max_{1 \leq j,k \leq n, j \neq k} \left\{ \left| \sum_{l=1}^n \langle \tau_l, \tau_l \rangle^2 \right|, |\langle \tau_j, \tau_k \rangle|^2 \right\} \geq \frac{1}{|d|} \left| \sum_{j=1}^n \langle \tau_j, \tau_j \rangle \right|^2.$$

*In particular, if $\langle \tau_j, \tau_j \rangle = 1$ for all $1 \leq j \leq n$, then*

$$\textbf{\textit{(First order p-adic Welch bound)}} \quad \max_{1 \leq j,k \leq n, j \neq k} \{|n|, |\langle \tau_j, \tau_k \rangle|^2\} \geq \frac{|n|^2}{|d|}.$$

*Proof.* Define $S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^n \langle x, \tau_j \rangle \tau_j \in \mathbb{Q}_p^d$. Then

$$bd = \operatorname{Tra}(bI_{\mathbb{Q}_p^d}) = \operatorname{Tra}(S_\tau) = \sum_{j=1}^n \langle \tau_j, \tau_j \rangle,$$

$$b^2 d = \operatorname{Tra}(b^2 I_{\mathbb{Q}_p^d}) = \operatorname{Tra}(S_\tau^2) = \sum_{j=1}^n \sum_{k=1}^n \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle.$$

Therefore

$$\begin{aligned}\left|\sum_{j=1}^{n}\langle\tau_j,\tau_j\rangle\right|^2&=|\operatorname{Tra}(S_\tau)|^2=|bd|^2=|d||b^2d|=|d|\left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\tau_j,\tau_k\rangle\langle\tau_k,\tau_j\rangle\right|\\&=|d|\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^2+\sum_{j,k=1,j\neq k}^{n}\langle\tau_j,\tau_k\rangle\langle\tau_k,\tau_j\rangle\right|\\&\leq|d|\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^2\right|,|\langle\tau_j,\tau_k\rangle\langle\tau_k,\tau_j\rangle|\right\}\\&=|d|\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^2\right|,|\langle\tau_j,\tau_k\rangle|^2\right\}.\end{aligned}$$

Whenever $\langle\tau_j,\tau_j\rangle=1$ for all $1\leq j\leq n$,

$$|n|^2\leq|d|\max_{1\leq j,k\leq n,j\neq k}\{|n|,|\langle\tau_j,\tau_k\rangle|^2\}.$$

□

We next derive higher order p-adic Welch bounds. For this, we need the following result.

**Theorem 2.2.** *[14, 22] If $\mathcal{V}$ is a vector space of dimension $d$ and $Sym^m(\mathcal{V})$ denotes the vector space of symmetric $m$-tensors, then*

$$dim(Sym^m(\mathcal{V}))=\binom{d+m-1}{m},\quad\forall m\in\mathbb{N}.$$

**Theorem 2.3.** ***(Higher Order p-adic Welch Bounds)*** *Let $p$ be a prime and $n,d,m\in\mathbb{N}$. If $\{\tau_j\}_{j=1}^n$ is any collection in $\mathbb{Q}_p^d$ such that there exists $b\in\mathbb{Q}_p$ satisfying*

$$\sum_{j=1}^{n}\langle x,\tau_j^{\otimes m}\rangle\tau_j^{\otimes m}=bx,\quad\forall x\in Sym^m(\mathbb{Q}_p^d),$$

*then*

$$\max_{1\leq j,k\leq n,j\neq k}\left\{\left|\sum_{l=1}^{n}\langle\tau_l,\tau_l\rangle^{2m}\right|,|\langle\tau_j,\tau_k\rangle|^{2m}\right\}\geq\frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\sum_{j=1}^{n}\langle\tau_j,\tau_j\rangle^m\right|^2.$$

*In particular, if $\langle\tau_j,\tau_j\rangle=1$ for all $1\leq j\leq n$, then*

$$\textbf{\textit{(Higher order p-adic Welch bound)}}\quad\max_{1\leq j,k\leq n,j\neq k}\{|n|,|\langle\tau_j,\tau_k\rangle|^{2m}\}\geq\frac{|n|^2}{\left|\binom{d+m-1}{m}\right|}.$$

*Proof.* Define $S_\tau:\operatorname{Sym}^m(\mathbb{Q}_p^d)\ni x\mapsto\sum_{j=1}^n\langle x,\tau_j^{\otimes m}\rangle\tau_j^{\otimes m}\in\operatorname{Sym}^m(\mathbb{Q}_p^d)$. Then Then

$$b\dim(\mathrm{Sym^m}(\mathbb{Q}_\mathrm{p}^\mathrm{d}))=\operatorname{Tra}(bI_{\operatorname{Sym}^m(\mathbb{Q}_p^d)})=\operatorname{Tra}(S_\tau)=\sum_{j=1}^{n}\langle\tau_j^{\otimes m},\tau_j^{\otimes m}\rangle,$$

$$b^2\dim(\mathrm{Sym^m}(\mathbb{Q}_\mathrm{p}^\mathrm{d}))=\operatorname{Tra}(b^2I_{\operatorname{Sym}^m(\mathbb{Q}_p^d)})=\operatorname{Tra}(S_\tau^2)=\sum_{j=1}^{n}\sum_{k=1}^{n}\langle\tau_j^{\otimes m},\tau_k^{\otimes m}\rangle\langle\tau_k^{\otimes m},\tau_j^{\otimes m}\rangle.$$

Therefore

$$\begin{aligned}
\left|\sum_{j=1}^{n}\langle \tau_j, \tau_j\rangle^m\right|^2 &= \left|\sum_{j=1}^{n}\langle \tau_j^{\otimes m}, \tau_j^{\otimes m}\rangle\right|^2 = |\operatorname{Tra}(S_\tau)|^2 = \left|b\dim(\operatorname{Sym}^{\mathrm{m}}(\mathbb{Q}_{\mathrm{p}}^{\mathrm{d}}))\right|^2\\
&= \left|\dim(\operatorname{Sym}^{\mathrm{m}}(\mathbb{Q}_{\mathrm{p}}^{\mathrm{d}}))\right| \left|b^2\dim(\operatorname{Sym}^{\mathrm{m}}(\mathbb{Q}_{\mathrm{p}}^{\mathrm{d}}))\right|\\
&= \left|\dim(\operatorname{Sym}^{\mathrm{m}}(\mathbb{Q}_{\mathrm{p}}^{\mathrm{d}}))\right| \left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j^{\otimes m}, \tau_k^{\otimes m}\rangle\langle \tau_k^{\otimes m}, \tau_j^{\otimes m}\rangle\right|\\
&= \left|\binom{d+m-1}{m}\right| \left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j^{\otimes m}, \tau_k^{\otimes m}\rangle\langle \tau_k^{\otimes m}, \tau_j^{\otimes m}\rangle\right|\\
&= \left|\binom{d+m-1}{m}\right| \left|\sum_{j=1}^{n}\sum_{k=1}^{n}\langle \tau_j, \tau_k\rangle^m\langle \tau_k, \tau_j\rangle^m\right|\\
&= \left|\binom{d+m-1}{m}\right| \left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^{2m}+\sum_{j,k=1, j\neq k}^{n}\langle \tau_j, \tau_k\rangle^m\langle \tau_k, \tau_j\rangle^m\right|\\
&\leq \left|\binom{d+m-1}{m}\right| \max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^{2m}\right|, |\langle \tau_j, \tau_k\rangle^m\langle \tau_k, \tau_j\rangle^m|\right\}\\
&= \left|\binom{d+m-1}{m}\right| \max_{1\leq j,k\leq n, j\neq k}\left\{\left|\sum_{l=1}^{n}\langle \tau_l, \tau_l\rangle^{2m}\right|, |\langle \tau_j, \tau_k\rangle|^{2m}\right\}.
\end{aligned}$$

Whenever $\langle \tau_j, \tau_j\rangle = 1$ for all $1 \leq j \leq n$,

$$|n|^2 \leq \left|\binom{d+m-1}{m}\right| \max_{1\leq j,k\leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^{2m}\}.$$

□

**Remark 2.4.** *Conditions given in the Theorem 2.3 says that the operator $S_\tau$ in the proof of Theorem 2.3 is diagonalizable. Thus Theorem 2.3 is restrictive as the hypothesis is stronger than that of Theorem 2.3 in [59]. However, note that the field $\mathbb{Q}_p$ does not satisfies the Equation (2) in [59] (see [67]) and hence neither the results in this paper can be derived from the results in [59] nor the results in [59] can be derived from the results in this paper.*

**Remark 2.5.** *Theorems 2.1 and 2.3 hold by replacing $\mathbb{Q}_p^d$ by a d-dimensional p-adic Hilbert space over any non-Archimedean (complete) valued field (such as $\mathbb{C}_p$).*

## 3. p-adic Zauner Conjecture and open problems

Using Theorem 2.1 we ask the following question.

**Question 3.1.** ***Given a prime $p$, for which $(d,n) \in \mathbb{N}\times\mathbb{N}$, there exist vectors $\tau_1, \dots, \tau_n \in \mathbb{Q}_p^d$ satisfying the following.***

- **(i)** $\langle \tau_j, \tau_j\rangle = 1$ ***for all*** $1 \leq j \leq n$.
- **(ii)** ***There exists*** $b \in \mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{n}\langle x, \tau_j\rangle\tau_j = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

**(iii)**

$$\max_{1\leq j,k \leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^2\}=\frac{|n|^2}{|d|}.$$

We can formulate a strong form of Question 3.1 as follows.

**Question 3.2.** ***Given a prime*** $p$***, for which*** $(d,n)\in \mathbb{N}\times\mathbb{N}$***, there exist vectors*** $\tau_1, \dots, \tau_n \in \mathbb{Q}_p^d$ ***satisfying the following.***

- **(i)** $\langle \tau_j, \tau_j\rangle =1$ ***for all*** $1\leq j \leq n$***.***
- **(ii)** ***There exists*** $b \in \mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{n}\langle x, \tau_j\rangle \tau_j=bx, \quad \forall x \in \mathbb{Q}_p^d.$$

- **(iii)**

$$\max_{1\leq j,k \leq n, j\neq k}\{|n|, |\langle \tau_j, \tau_k\rangle|^2\}=\frac{|n|^2}{|d|}.$$

- **(iv)** $\|\tau_j\|=1$ ***for all*** $1\leq j \leq n$***.***

Why Question 3.2 is different than Question 3.1? Reason is that unlike non-Archimedean Hilbert spaces, in p-adic Hilbert spaces, norm is not defined as $\sqrt{|\langle \cdot, \cdot \rangle|}$. A particular case of Question 3.1 is the following p-adic version of Zauner conjecture (see [3–6, 11–13, 36, 39, 46, 53, 56, 61, 69, 76, 90] for Zauner conjecture in Hilbert spaces, [58] for Zauner conjecture in Hilbert C*-modules, [57] for Zauner conjecture in Banach spaces and [59] for Zauner conjecture in non-Archimedean Hilbert spaces).

**Conjecture 3.3.** ***(p-adic Zauner Conjecture) Let*** $p$ ***be a prime. For each*** $d\in \mathbb{N}$***, there exist vectors*** $\tau_1, \dots, \tau_{d^2} \in \mathbb{Q}_p^d$ ***satisfying the following.***

- **(i)** $\langle \tau_j, \tau_j\rangle =1$ ***for all*** $1\leq j \leq d^2$***.***
- **(ii)** ***There exists*** $b \in \mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{d^2}\langle x, \tau_j\rangle \tau_j=bx, \quad \forall x \in \mathbb{Q}_p^d.$$

- **(iii)**

$$|\langle \tau_j, \tau_k\rangle|^2=|n|, \quad \forall 1\leq j,k \leq d^2, j\neq k.$$

Question 3.2 gives the following Zauner conjecture.

**Conjecture 3.4.** ***(p-adic Zauner Conjecture - strong form) Let*** $p$ ***be a prime. For each*** $d\in \mathbb{N}$***, there exist vectors*** $\tau_1, \dots, \tau_{d^2} \in \mathbb{Q}_p^d$ ***satisfying the following.***

- **(i)** $\langle \tau_j, \tau_j\rangle =1$ ***for all*** $1\leq j \leq d^2$***.***
- **(ii)** ***There exists*** $b \in \mathbb{Q}_p$ ***satisfying***

$$\sum_{j=1}^{d^2}\langle x, \tau_j\rangle \tau_j=bx, \quad \forall x \in \mathbb{Q}_p^d.$$

- **(iii)**

$$|\langle \tau_j, \tau_k\rangle|^2=|n|, \quad \forall 1\leq j,k \leq d^2, j\neq k.$$

- **(iv)** $\|\tau_j\|=1$ ***for all*** $1\leq j \leq d^2$***.***

We recall the definition of Gerzon's bound which allows us to remember companions to Welch bounds in Hilbert spaces.

**Definition 3.5.** *[48] Given $d \in \mathbb{N}$, define* ***Gerzon's bound***

$$\mathcal{Z}(d,\mathbb{K}) := \begin{cases} d^2 & \text{if} \quad \mathbb{K}=\mathbb{C} \\ \frac{d(d+1)}{2} & \text{if} \quad \mathbb{K}=\mathbb{R}. \end{cases}$$

**Theorem 3.6.** *[17,23,44,48,64,68,77,87] Define $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and $m := \dim_{\mathbb{R}}(\mathbb{K})/2$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{K}^d$, then*

(i) ***(Bukh-Cox bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq \frac{\mathcal{Z}(n-d,\mathbb{K})}{n(1+m(n-d-1)\sqrt{m^{-1}+n-d})-\mathcal{Z}(n-d,\mathbb{K})} \quad \text{if} \quad n>d.$$

(ii) ***(Orthoplex/Rankin bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq \frac{1}{\sqrt{d}} \quad \text{if} \quad n>\mathcal{Z}(d,\mathbb{K}).$$

(iii) ***(Levenstein bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq \sqrt{\frac{n(m+1)-d(md+1)}{(n-d)(md+1)}} \quad \text{if} \quad n>\mathcal{Z}(d,\mathbb{K}).$$

(iv) ***(Exponential bound)***

$$\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle| \geq 1-2n^{\frac{-1}{d-1}}.$$

Theorem 3.6 and Theorem 2.1 give the following problem.

**Question 3.7.** ***Whether there is a p-adic version of Theorem 3.6? In particular, does there exists a version of***

(i) ***p-adic Bukh-Cox bound?***
(ii) ***p-adic Orthoplex/Rankin bound?***
(iii) ***p-adic Levenstein bound?***
(iv) ***p-adic Exponential bound?***

We already wrote that Welch bounds have applications in study of equiangular lines. We wish to formulate equiangular line problem for p-adic Hilbert spaces. For the study of equiangular lines in Hilbert spaces we refer [8,9,16,18,27,37,38,40,43,49,50,60,65,66,89], quaternion Hilbert spaces we refer [33], octonion Hilbert spaces we refer [21], finite dimensional vector spaces over finite fields we refer [41,42], for Banach spaces we refer [57] and for non-Archimedean Hilbert spaces we refer [59].

**Question 3.8.** ***(p-adic Equiangular Line Problem) Let $p$ be a prime. Given $a \in \mathbb{Q}_p$, $d \in \mathbb{N}$ and $\gamma > 0$, what is the maximum $n = n(p,a,d,\gamma) \in \mathbb{N}$ such that there exist vectors $\tau_1, \dots, \tau_n \in \mathbb{Q}_p^d$ satisfying the following.***

**(i)** $\langle \tau_j, \tau_j\rangle = a$ ***for all*** $1 \leq j \leq n$.
**(ii)** $|\langle \tau_j, \tau_k\rangle|^2 = \gamma$ ***for all*** $1 \leq j,k \leq n, j \neq k$.

***In particular, whether there is a p-adic Gerzon bound?***

Question 3.8 can be easily lifted to formulate question of p-adic regular $s$-distance sets.

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