MODULAR CUBIC SEQUENCE PATTERN AND ITS APPLICATION TO THE DIOPHANTINE EQUATION $X^3 + Y^3 + Z^3 = K$

A PREPRINT

Eduardo J. Acuña T.
eacuna1@uc.edu.ve

Samuel J. Flores
sflores4@uc.edu.ve

Paul F. Marrero. R.
paulqed0@protonmail.com

August 29, 2022

ABSTRACT

We propose a new sequence-based algorithm to track and find solutions to the Diophantine equation $X^3 + Y^3 + Z^3 = k$, for a fixed integer $k > 0$.

1 Introduction

The following Diophantine equation, known as the sum of the 3 cubes, represents one of the most complex problems one can attempt to solve, in 2019 Booker. A [1] proposes an algorithm to find the number 33, later the solution for the number 42 will be discovered, but there are still numbers to be discovered.
However, although there are the following numbers in the list, it does not answer the question of how many numbers are solutions to K? for this reason by studying the patterns generated with Primordial Algebra (2021)[4] it was possible to determine the sequence in which all the cubed numbers are found once classified by the module (9).

1.1 Notation

- Let n be any integer number
- Let Φ^Z be any of the class of any $\mathbb{Z}$, we replace "Z" to denote the value of the class.
- Let Z_ϕ be any Primal Number, which is the representative value of the class and can be operated with.

2 Set of cubes congruent to primordial numbers

A primordial number 2021 [4] is the representative element of a primordial class, these elements form a finite group, which when operated can generate members of the same group, the sub-group of the primordial numbers are the set of integers modulo 9, whose remainder is congruent to that primordial number, this denotes a direct relationship of behavior of a primordial number and its integer component.

Let the set **C=**$\{\pm\, 1_\phi, \pm\, 8_\phi, \pm\, 9_\phi\}$ where **C** $\subseteq \mathbb{Z}$, be the set formed by the only 6 values that a primordial number raised to the cube can take.

3 Sequential Functions $3n + m$

3.1 Definition 0

Let f be the sequential function such that $f(n) = (3n+m)^3$ with n, m integers and $m \in \{-3, -2, -1, 1, 2, 3\}$ from where:

$$f(n) \equiv c \ \ (mod \ \ 9), \ \text{for any } c \in \mathbf{C}.$$

3.2 Definition 1: specific cases

$$\text{Let } f : \mathbb{Z} \to \mathbb{Z}$$

$$f(n) = \begin{cases} m^3, \ \ if \ \ n = 0, \\ (3n+m)^3 \ / \ 1 \leq m \leq 3 \ \ \ if \ \ n > 0, \\ (3n+m)^3 \ / -3 \leq m \leq -1 \ \ \ if \ \ n < 0. \end{cases}$$

3.3 Definition 2

For a $n > 0$ we have that $f : \mathbb{N} \to \mathbb{N}$. Therefore the sequential functions $3n + m$ have the form:

$$\begin{array}{lll} f(n) = & (3n+1)^3 & \equiv 1 \ (mod \ \ 9) \\ f(n) = & (3n+2)^3 & \equiv 8 \ (mod \ \ 9) \\ f(n) = & (3n+3)^3 & \equiv 9 \ (mod \ \ 9) \end{array}$$

In the cases of functions $3n + m$ where $n < 0$, shall be characterized by $f : \mathbb{Z}^- \to \mathbb{Z}^-$ and:

$$\begin{array}{lll} f(n) = & (3n-1)^3 & \equiv -1 \ (mod \ \ 9) \\ f(n) = & (3n-2)^3 & \equiv -8 \ (mod \ \ 9) \\ f(n) = & (3n-3)^3 & \equiv -9 \ (mod \ \ 9) \end{array}$$

4 Algorithm Modular Cubic Sequences (A.M.C.S.)

From the definition 0. 3.1, we have that $f : \mathbb{Z} \to \mathbb{Z}$ from where:

$$f(n) = (3n+m)^3.$$

Input: $k > 0$.
Output: There exists a solution (x, y, z) for $x^3 + y^3 + z^3 = k$ or the message "there is no solution because you entered a k ≡ 4,5 (mod 9)" if there is not.

Step 1: Calculate $k \ mod \ \ 9$.

Step 2: Let $\{X\} = 3n + a$; $\{Y\} = 3n + b$; $\{Z\} = 3n + d$. Continuously take n as $n = 0$. Then apply n in $(\{X\}, \{Y\}, \{Z\})$ such that $\{X\} = 3(0) + a = a; \{Y\} = 3(0) + b = b; \{Z\} = 3(0) + d = d$ with integers $a, b, d \in \{\pm 1, \pm 2, \pm 3\}$.

Step 3: Raise to the third power to $(\{X\}, \{Y\}, \{Z\})$ to get $(\{X^3\}, \{Y^3\}, \{Z^3\})$ = (a^3, b^3, d^3). **Then** $a^3, b^3, d^3 \in \{\pm 1_\phi, \pm 8_\phi, \pm 9_\phi\}$

Step 4: Call to Step 1 and apply the path combinations from the Sam algorithm so that: $a^3 + b^3 + d^3 \equiv k \ \ (mod \ \ 9)$, and write "There exist a congruent solution for the Diophantine equation".

Step 5: Selecting the tern (a^3, b^3, d^3) from Step 4, apply cube root on the previous tern i.e., $(\sqrt[3]{a^3}, \sqrt[3]{b^3}, \sqrt[3]{d^3}) = (a, b, d)$. **Then** do $\{X\} = a \ \wedge \ \{Y\} = b \ \wedge \ \{Z\} = d$.

Step 6: Add $3n$ to each element of $(\{X\}, \{Y\}, \{Z\})$ to get $\{X\} = 3n+a; \{Y\} = 3n+b; \{Z\} = 3n+d, \ with \ (a, b, d)$ selected from (Step 4).

Step 7:

Case 1: If $a > 0$, **then** apply any integer $n > 0$ in $\{X\}$ and develop the sequence as follows
$\{X\} = x_1, x_2, x_3, \cdots, x_{n-1}, x_n$. **Else** you have an $a < 0$, now apply all integer $n < 0$ in $\{X\}$ to get:
$\{X\} = x_1, x_2, x_3, \cdots, x_{n-1}, x_n$.

Case 2: If $b > 0$, **then** apply any integer $n > 0$ in $\{Y\}$ and develop the sequence as follows
$\{Y\} = y_1, y_2, y_3, \cdots, y_{n-1}, y_n$. **Else** you have an $b < 0$, now apply all integer $n < 0$ in $\{Y\}$ to get:
$\{Y\} = y_1, y_2, y_3, \cdots, y_{n-1}, y_n$.

Case 3: If $d > 0$, **then** apply any integer $n > 0$ in $\{Z\}$ and develop the sequence as follows
$\{Z\} = z_1, z_2, z_3, \cdots, z_{n-1}, z_n$. **Else** you have an $d < 0$ in $\{Y\}$ to get:
$\{Z\} = z_1, z_2, z_3, \cdots, z_{n-1}, z_n$.

Step 8: Raise to the third power to $(\{X\}, \{Y\}, \{Z\})$ such that:
$\{X\}^3 = x_1^3, x_2^3, x_3^3, \cdots, x_{n-1}^3, x_n^3; \ \{Y\}^3 = y_1^3, y_2^3, y_3^3, \cdots, y_{n-1}^3, y_n^3; \ \{Z\}^3 = z_1^3, z_2^3, z_3^3, \cdots, z_{n-1}^3, z_n^3$.
Of the above successions **If** we found what for values (x_i^3, y_i^3, z_i^3) with an integer i such that $1 \leq i \leq n$ we have that $x_i^3 + y_i^3 + z_i^3 \equiv k \ (mod \ \ 9)$, **then** write "This is a congruent solution to k modulo 9 for the Diophantine equation". **If not** write "nonexistence".

Step 9: **If** when $(\{X\}^3, \{Y\}^3, \{Z\}^3)$ is evaluated, we found values (x_i^3, y_i^3, z_i^3) such that $x_i^3 + y_i^3 + z_i^3 = k$, **Then** write "There is a fixed solution for the Diophantine equation".

5 Apply the AMSC

ZONA EN REVISIÓN...

Let $k = 42$ apply the **AMSC** we have that:

1) $42 \ \ mod \ \ 9 \ = \ 6 \ \equiv \ 6_\phi$.

2) Now let $n = 0$. Then we designate to: a,b,d the following values:

$$a = -1$$
$$b = 2$$
$$d = 2$$

Therefore:

$$\{X\} = (3(0) - 1); \ \{Y\} = (3(0) + 2); \ \{Z\} = (3(0) + 2).$$

3) $X = -1;\ Y = 2;\ Z = 2$. Now raising to the third power to (X, Y, Z) we verify that $(-1)^3+(2)^3+(2)^3 = 15$. from the last we check that

$$15 \equiv 6\ (mod\ \ 9)$$

"Exist a solution congruent with 6 modulo 9."

4) According to the "specific definitions" as $a = -1$, then we apply $n < 0$ on $\{X\}$, consequently we obtain the following numerical sequence:

$$\{X\} = -1, -7, -10, -13, \cdots, -80538738812075974, \cdots, -x_n.$$

The same happens with b, d, to which, when applying the "specific definitions" on $\{Y\}, \{Z\}$, we are left with:

$$\{Y\} = 5, 8, 11, 14, \cdots, 80435758145817515, \cdots, y_n.$$

$$\{Z\} = 5, 8, 11, 14, \cdots, 12602123297335631, \cdots, z_n.$$

5) Now raising to the third power to $(\{X\}, \{Y\}, \{Z\})$:

$$\{X\}^3 = -64, -343, -1000, -2197, \cdots, (-80538738812075974)^3, \cdots, (-x_n)^3.$$

$$\{Y\}^3 = 125, 512, 1331, 2744, \cdots, (80435758145817515)^3, \cdots, (y_n)^3.$$

$$\{Z\}^3 = 125, 512, 1331, 2744, \cdots, (12602123297335631)^3, \cdots, (z_n)^3.$$

6) Finally, if we add the following terms of the sequence:

$$X_i^3 = (-80538738812075974)^3.$$

$$Y_i^3 = (80435758145817515)^3.$$

$$Z_i^3 = (12602123297335631)^3.$$

we can verify that $X_i^3 + Y_i^3 + Z_i^3 = 42$. Ergo *there is a fixed solution for this diophantine equation.*

5.1 Relation to the S.A.M. algorithm.

The AMSC algorithm and the SAM existence algorithm (2021) [2] are complementary, since with SAM for a given K we can know the primal class paths of that number, and with AMSC we can run the sequence based on the paths that SAM gives us.

5.2 Example

Lest take the number 33 from Booker [1].

Input: K=33
Step 1: SAM Evaluating 33 for SAM, we find that 33 is a class 6 number.

$$33 \equiv \Phi^6 = 6_\phi$$

Step 2: SAM will give us the following combinations of paths for the sum of the 3 cubes with class 6 results:

a $(8_\phi) + (-1_\phi) + (-1_\phi)$
b $(-1_\phi) + (8_\phi) + (8_\phi)$
c $(8_\phi) + (8_\phi) + (8_\phi)$

Step 3: AMCS Turning roads into sequences.

a.s $(3 * n_x + 2)^3 + (3 * -n_y - 1)^3 + (3 * -n_z - 1)^3$
b.s $(3 * -n_x - 1)^3 + (3 * n_y + 2)^3 + (3 * n_z + 2)^3$
c.s $(3 * n_x + 2)^3 + (3 * n_y + 2)^3 + (3 * n_z + 2)^3$

As we know 33 has only one known solution so far, SAM indicates that the path to that solution is $(8_\phi)+(-1_\phi)+(-1_\phi)$, which means that the sequence is the $(3 * n_x + 2)^3 + (3 * -n_y - 1)^3 + (3 * -n_z - 1)^3$.

$$(8_\phi) + (-1_\phi) + (-1_\phi) = (3 * n_x + 2)^3 + (3 * -n_y - 1)^3 + (3 * -n_z - 1)^3 = 33$$

Output: 33 is a valid solution:

$$33 = (3 * 2955376325095842 + 2)^3 + (3 * -912037156269013 - 1)^3 + (3 * -2926135147620746 - 1)^3$$

6 Some Remarks.

1 The AMCS algorithm can find all the data reported by Huisman S. (2017). [3].

2 The sequence, $(3n+m)$ without being cubed, generates the whole set $\mathbb{Z}-\{0\}$.

3 From the previous point, since the matrixE_9^ϕ is composed of all integers less zero, $\mathbb{Z}-\{0\}$, we can confirm that the sequence generates all the elements of the matrixE_9^ϕ.

4 the inverse function of the sequence $(3n+m)$ for core solutions:

$$n_c = \frac{n-m}{3}$$

5 Moving in the sequence $(3n+m)$ is the same as adding 1 to n as it is to adding 3 to m.

$$(3(n+1)+m)^3 = (3n+(m+3))^3$$

6 Although the sequence itself is not a solution to the problem, it is an approximation to it, since any valid (x^3, y^3, z^3) triad for a given K will belong to the sequence.

Bibliography

[1] A. R. Booker. Cracking the problem with 33. *Research in Number Theory*, 5(3):1–6, 2019.

[2] S. Flores, E. Acuña, and P. Marrero. Existential refinement on the search of integer solutions for the diophantine equation $x^3+y^3+z^3=n$, 2021. URL https://arxiv.org/abs/2103.17037.

[3] S. G. Huisman. Newer sums of three cubes, 2016. URL https://arxiv.org/abs/1604.07746.

[4] E. J. A. Tarazona, S. Flores, and P. F. Marrero. Matrize_ϕˆ 9. clasificación de números enteros. algebra primordial. *Revista MATUA ISSN: 2389-7422*, 8(1):10–45, 2021.