Multinomial Theorem on the Binomial Coefficients for Combinatorial Geometric Series

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Abstract: This paper presents a multinomial theorem on the binomial coefficients for combinatorial geometric series. The coefficient for each term in combinatorial geometric series denotes a binomial coefficient. These ideas can enable the scientific researchers to solve the real life problems.

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1. Introduction

When the author of this article was trying to develop the multiple summations of geometric series, a new idea was stimulated his mind to create a combinatorial geometric series [1-9]. The combinatorial geometric series is a geometric series whose coefficient of each term of the geometric series denotes the binomial coefficient V_n^r. In this article, binomial identities and multinomial theorem is provided using the binomial coefficients for combinatorial geometric series.

2. Combinatorial Geometric Series

The combinatorial geometric series [1-9] is derived from the multiple summations of geometric series. The coefficient of each term in the combinatorial refers to the binomial coefficient V_n^r.

$$\sum_{i_1=0}^{n}\sum_{i_2=i_1}^{n}\sum_{i_3=i_2}^{n}\cdots\sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \ \& \ V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r-1)(n+r)}{r!},$$

where $n \geq 0, r \geq 1$ and $n, r \in N = \{0, 1, 2, 3, \cdots\}$.

Here, $\sum_{i=0}^{n} V_i^r x^i$ refers to the combinatorial geometric series and

V_n^r is the binomial coefficient for combinatorial geometric series.

Binomial identities [1-5] on the binomial coefficients V_r^n for combinatorial geometric series are given below:

(i) $V_n^0 = V_0^n = 1$ for $n = 0, 1, 2, 3, \cdots$

(ii) $V_r^m = V_m^r$, $(m, r \geq 1 \ \& \ m, r \in N)$.

$(iii). V_n^n = 2V_n^{n-1}, (n \geq 1 \ \& \ n \in N)$.

$(iv). V_{kn}^{kn} = 2V_{kn}^{kn-1}, (n, k \geq 1 \ \& \ n, k \in N)$.

$(v).\ V_{n-d}^{n-d} = 2V_{n-d}^{n-d-1}, (n > d \geq 0\ \ \&\ \ d, n \in N).$

$(vi).\ V_{n+d}^{n+d} = 2V_{n+d}^{n+d-1}, (n > d \geq 0\ \ \&\ \ d, n \in N).$

$(vii).\ V_0^r + V_1^r + V_2^r + \cdots + V_n^r = V_n^{r+1}$ (or) $V_n^0 + V_n^1 + V_n^2 + \cdots + V_n^r = V_{n+1}^r.$

Theorem 2.1: $\dfrac{(n+r)!}{n!\,r!} = V_n^r$, where $n, r \geq 0\ \ \&\ \ n, r \in N$.

$$Proof.\binom{n+r}{n} = \frac{(n+r)!}{n!\,(n+r-n)!} = \frac{(n+r)!}{n!\,r!} = \frac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!} = V_n^r.$$

$$\text{That is, } V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!} = \prod_{i=1}^{r}\frac{n+i}{r!}.$$

From the above expressions, we conclude that

$$\frac{(n+r)!}{n!\,r!} = V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!}, \text{where } n, r \geq 0\ \ \&\ \ n, r \in N.$$

Theorem 2.2 : For any k nonnegative integers $n_1, n_2, n_3, \cdots and\ n_k$,

$$V_{n_1}^{n_2+n_3+n_4+\cdots+n_k} \times V_{n_2}^{n_3+n_4+\cdots+n_k} \times V_{n_3}^{n_4+n_5+\cdots+n_k} \times \cdots \times V_{n_{k-1}}^{n_k} = \frac{(n_1+n_2+n_3+\cdots+n_k)!}{(n_1!\,n_2!\,n_3!\cdots n_k!)}.$$

$Proof.$ Let us apply Theorem 2.1 to the following expression:

$$V_{n_1}^{n_2+n_3+n_4+\cdots+n_k} \times V_{n_2}^{n_3+n_4+\cdots+n_k} \times V_{n_3}^{n_4+n_5+\cdots+n_k} \times \cdots \times V_{n_{k-1}}^{n_k}$$

$$= \frac{(n_1+n_2+n_3+\cdots+n_k)!}{n_1!\,(n_2+n_3+\cdots+n_k)!} \times \frac{(n_2+n_3+\cdots+n_k)!}{n_2!\,(n_3+n_4+\cdots+n_k)!} \times \frac{(n_3+n_4+\cdots+n_k)!}{n_3!\,(n_4+n_5+\cdots+n_k)!} \times \cdots \times \frac{(n_{k-1}+n_k)!}{n_{k-1}!\,n_k!} = \frac{(n_1+n_2+n_3+\cdots+n_k)!}{n_1!\,n_2!\,n_3!\cdots n_k!}.$$

Thus,

$$\frac{(n_1+n_2+n_3+\cdots+n_k)!}{n_1!\,n_2!\,n_3!\cdots n_k!} = V_{n_1}^{n_2+n_3+n_4+\cdots+n_k} \times V_{n_2}^{n_3+n_4+\cdots+n_k} \times V_{n_3}^{n_4+n_5+\cdots+n_k} \times \cdots \times V_{n_{k-1}}^{n_k}.$$

Conclusion

In this article, the combinatorial geometric series and binomial identities on the binomial coefficients for combinatorial geometric series were given and a multinomial theorem discussed with detailed proofs for research and development further.

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