Binomial Coefficients and Identities in Combinatorial Geometric Series

Chinnaraji Annamalai
School of Management, Indian Institute of Technology, Kharagpur, India
Email: anna@iitkgp.ac.in
https://orcid.org/0000-0002-0992-2584

Abstract: This paper presents binomial theorems on combinatorial identities that are derived from the binomial coefficients in combinatorial geometric series. The coefficient of each term in combinatorial geometric series refers to a binomial coefficient. These ideas can enable the scientific researchers to solve the real life problems.

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1. Introduction

When the author of this article was trying to develop the multiple summations of geometric series, a new idea was stimulated his mind to create a combinatorial geometric series [1-7]. The combinatorial geometric series is a geometric series whose coefficient of each term of the geometric series denotes the binomial coefficient V_n^r. In this article, binomial identities and binomial theorems are provided using the binomial coefficients in combinatorial geometric series.

2. Combinatorial Geometric Series

The combinatorial geometric series [1-7] is derived from the multiple summations of geometric series. The coefficient of each term in the combinatorial refers to the binomial coefficient V_n^r.

$$\sum_{i_1=0}^{n} \sum_{i_2=i_1}^{n} \sum_{i_3=i_2}^{n} \cdots \sum_{i_r=i_{r-1}}^{n} x^{i_r} = \sum_{i=0}^{n} V_i^r x^i \;\; \& \;\; V_n^r = \frac{(n+1)(n+2)(n+3)\cdots(n+r-1)(n+r)}{r!},$$

where $n \geq 0, r \geq 1 \;\; and \;\; n, r \in N = \{0, 1, 2, 3, \cdots\}$.

Here, $\sum_{i=0}^{n} V_i^r x^i$ denotes the combinatorial geometric series and V_n^r the binomial coefficient.

Binomial identities of V_r^n are given below:

$(i). V_n^n = 2V_n^{n-1}$.

$(ii). V_{kn}^{kn} = 2V_{kn}^{kn-1}, (n, k \geq 1 \;\; \& \;\; n, k \in N)$.

$(iii). V_{n-d}^{n-d} = 2V_{n-d-1}^{n-d}, (n > d \geq 0 \;\; \& \;\; d, n \in N)$.

$Proof$ for Binomial Identity (i):

$$V_n^n = \frac{(n+1)(n+2)(n+3)\cdots(n+n-1)(n+n)}{n!} = \frac{2n(n+1)(n+2)\cdots(n+n-1)}{n!}$$

$$= \frac{2n(n+1)(n+2)\cdots(n+n-1)}{(n-1)!\,n} = \frac{2(n+1)(n+2)\cdots(n+n-1)}{(n-1)!}$$

Here, $V_n^{n-1} = \dfrac{(n+1)(n+2)\cdots(n+n-1)}{(n-1)!}$.

$\therefore 2V_n^{n-1} = V_n^n$.

Proof for Binomial Identity (ii):

$$V_{kn}^{kn} = \frac{(kn+1)(kn+2)(kn+3)\cdots(kn+kn-1)(kn+kn)}{n!}$$
$$= \frac{2(kn)(kn+1)(kn+2)(kn+3)\cdots(kn+kn-1)}{(kn-1)!\,(kn)}$$
$$= \frac{2(kn+1)(kn+2)(kn+3)\cdots(kn+kn-1)}{(kn-1)!} = 2V_{kn}^{kn-1}.$$

$\therefore\ 2V_{kn}^{kn-1} = V_{kn}^{kn}$.

For example, $2V_n^{n-1} = V_n^n$; $2V_{2n}^{2n-1} = 2V_{2n}^{2n}$; $2V_{3n}^{3n-1} = 2V_{3n}^{3n}$; for $k = 1, 2, 3, \cdots$

Proof for Binomial Identity (iii):

$$V_{n-d}^{n-d} = \frac{(n-d+1)(n-d+2)(n-d+3)\cdots(n-d+n-d-1)(n-d+n-d)}{(n-d)!}$$
$$= \frac{2(n-d)(n-d+1)(n-d+2)(n-d+3)\cdots(n-d+n-d-1)}{(n-d-1)!\,(n-d)} = 2V_{n-d-1}^{n-d}$$
$$= \frac{2(n-d+1)(n-d+2)(n-d+3)\cdots(n-d+n-d-1)}{(n-d-1)!} = 2V_{n-d}^{n-d-1}$$

$\therefore\ 2V_{n-d}^{n-d-1} = V_{n-d}^{n-d}$

For examples, $2V_{n-1}^{n-2} = V_{n-1}^{n-1}$; $2V_{n-2}^{n-3} = V_{n-2}^{n-2}$; $2V_{n-3}^{n-4} = V_{n-3}^{n-3}$; for $d = 0, 1, 2, 3, \cdots$

For examples for $2V_{n+d}^{n+d-1} = V_{n+d}^{n+d}$, $2V_{n+2}^{n+1} = V_{n+2}^{n+2}$; $2V_{n+3}^{n+2} = V_{n+3}^{n+3}$; $\cdots$

Theorem 2.1: $\dfrac{(n+r)!}{n!\,r!} = \prod_{i=1}^{r} \dfrac{n+i}{r!} = V_n^r$, where $n, r \geq 0$ & $n, r \in N$.

Proof. $\binom{n+r}{n} = \dfrac{(n+r)!}{n!\,(n+r-n)!} = \dfrac{(n+r)!}{n!\,r!} = \dfrac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!} = V_n^r$.

That is, $V_n^r = \dfrac{(n+1)(n+2)(n+3)\cdots(n+r)}{r!} = \prod_{i=1}^{r} \dfrac{n+i}{r!}$.

From the above expressions, we conclude that

$$\frac{(n+r)!}{n!\,r!} = \prod_{i=1}^{r} \frac{n+i}{r!} = V_n^r, \text{where } n, r \geq 0 \ \&\ n, r \in N.$$

Theorem 2.2: $2\dfrac{(2n-1)!}{n!\,(n-1)!} = \dfrac{(2n)!}{(n!)^2}$, where $n \geq 1$ & $n \in N$.

$Proof$. By applying Theorem 2.1 to $2V_n^{n-1} = V_n^n$, *we get that*

$$2\frac{(n+n-1)!}{n!\,(n-1)!} = \frac{(n+n)!}{n!\,n!} \Longrightarrow 2\frac{(2n-1)!}{n!\,(n-1)!} = \frac{(2n)!}{(n!)^2} \left(\because V_n^r = \frac{(n+r)!}{n!\,r!}\right).$$

Corollary 2.1: $2\frac{(2kn-1)!}{(kn)!\,(kn-1)!} = \frac{(2kn)!}{(kn!)^2}$, where $k, n \geq 1$ & $k, n \in N$.

Similarly, we can constitute the corollaries for the other binomial identities such that $V_{n-d}^{n-d} = 2V_{n-d}^{n-d-1}$ and $V_{n+d}^{n+d} = 2V_{n+d}^{n+d-1}$

3. Conclusion

In this article, the combinatorial geometric series and its binomial coefficients and combinatorial identities were introduced and theorems on binomial and multinomial coefficients and factorials discussed with detailed proofs for research and development further.

References

[1] Annamalai, C. (2020) Optimized Computing Technique for Combination in Combinatorics. *hal*-0286583. https://doi.org/10.31219/osf.io/9p4ek.

[2] Annamalai, C. (2020) Novel Computing Technique in Combinatorics. *hal*-02862222. https://doi.org/10.31219/osf.io/m9re5.

[3] Annamalai, C. (2022) Annamalai's Binomial Identity and Theorem, *SSRN Electronic Journal*. http://dx.doi.org/10.2139/ssrn.4097907.

[4] Annamalai, C. (2022) Computation Method for Combinatorial Geometric Series and its Applications. *COE, Cambridge University Press.* https://doi.org/10.33774/coe-2022-pnx53-v22.

[5] Annamalai, C. (2022) Computing Method for Combinatorial Geometric Series and Binomial Expansion. *SSRN Electronic Journal*. http://dx.doi.org/10.2139/ssrn.4168016.

[6] Annamalai, C. (2022) Factorials and Integers for Applications in Computing and Cryptography. *COE, Cambridge University Press.* https://doi.org/10.33774/coe-2022-b6mks.

[7] Annamalai, C. (2022) Application of Factorial and Binomial identities in Computing and Cybersecurit. *Research Square.* https://doi.org/10.21203/rs.3.rs-1666072/v2.