Note on the Riemann Hypothesis

International Conference on Recent Developments in Mathematics (ICRDM 2022)

Frank Vega, CopSonic France

vega.frank@gmail.com

August 24-26, 2022

Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 1984, Guy Robin stated a new criterion for the Riemann Hypothesis. We prove that the Riemann Hypothesis is true using the Robin's criterion.

Note

This result is an extension of my article "Robin's criterion on divisibility" published by **The Ramanujan Journal** (03 May 2022).

References

This presentation and references can be found at my *ResearchGate Project*: `https://www.researchgate.net/project/The-Riemann-Hypothesis`.

The Riemann Hypothesis is considered by many to be the most important unsolved problem in pure mathematics. It was proposed by Bernhard Riemann (1859).

The Riemann Hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of twenty-three unsolved problems.

This is one of the Clay Mathematics Institute's Millennium Prize Problems.

Objectives

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer d divides n. Define $f(n)$ as $\frac{\sigma(n)}{n}$.

We provide a proof for the Riemann Hypothesis using the properties of the f function.

We say that Robin(n) holds provided that

$$f(n) < e^{\gamma} \cdot \log \log n,$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm.

Proposition 1

Robin(n) *holds for all natural numbers* $n > 5040$ *if and only if the Riemann Hypothesis is true (Robin, 1984, Theorem 1 pp. 188).*

Superabundant Numbers

A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer (Choie et al., 2007, pp. 367).

If n is superabundant, then n is a Hardy-Ramanujan integer (Alaoglu and Erdős, 1944, Theorem 1 pp. 450).

The following is a key Lemma.

Lemma 2

If the Riemann Hypothesis is false, then there are infinitely many superabundant numbers n such that $\mathrm{Robin}(n)$ *fails (i.e.* $\mathrm{Robin}(n)$ *does not hold).*

If the Riemann Hypothesis is false, then there are infinitely many colossally abundant numbers $n > 5040$ such that Robin(n) fails (Robin, 1984, Proposition pp. 204).

Every colossally abundant number is superabundant (Alaoglu and Erdős, 1944, pp. 455).

In this way, the proof is complete. ■

A Strictly Decreasing Sequence

For every prime number $q_k > 2$, we define the sequence:

$$Y_k = \frac{e^{\frac{0.2}{\log^2(q_k)}}}{(1 - \frac{1}{\log(q_k)})}.$$

As the prime number q_k increases, the sequence Y_k is strictly decreasing (Vega, 2022, Lemma 6.1 pp. 6).

We use the following Propositions:

Proposition 3

(Vega, 2022, Theorem 6.6 pp. 8). Let $\prod_{i=1}^{k} q_i^{a_i}$ *be the representation of a superabundant number* $n > 5040$ *as the product of the first* k *consecutive primes* $q_1 < \ldots < q_k$ *with the natural numbers* $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ *as exponents. Suppose that* Robin(n) *fails. Then,*

$$\alpha_n < \frac{\log \log (N_k)^{Y_k}}{\log \log n},$$

where $N_k = \prod_{i=1}^{k} q_i$ *is the primorial number of order* k *and* $\alpha_n = \prod_{i=1}^{k} \left(\frac{q_i^{a_i+1}}{q_i^{a_i+1} - 1} \right)$.

Proposition 4

(Nazardonyavi and Yakubovich, 2013, Lemma 3.3 pp. 8). Let $x \geq 11$*. For* $y > x$*, we have*

$$\frac{\log \log y}{\log \log x} < \sqrt{\frac{y}{x}}.$$

This is the main insight.

Lemma 5

Let $\prod_{i=1}^{k} q_i^{a_i}$ be the representation of a superabundant number $n > 5040$ as the product of the first k consecutive primes $q_1 < \ldots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ as exponents. Suppose that Robin(n) *fails. Then,*

$$\alpha_n < \sqrt{\frac{(N_k)^{Y_k}}{n}},$$

where $N_k = \prod_{i=1}^{k} q_i$ is the primorial number of order k and $\alpha_n = \prod_{i=1}^{k} \left(\frac{q_i^{a_i+1}}{q_i^{a_i+1}-1} \right)$.

When $n > 5040$ is a superabundant number and Robin(n) fails, then we have

$$\alpha_n < \frac{\log\log(N_k)^{Y_k}}{\log\log n}.$$

We assume that $(N_k)^{Y_k} > n > 5040 > 11$ since $\alpha_n > 1$. Consequently,

$$\frac{\log\log(N_k)^{Y_k}}{\log\log n} < \sqrt{\frac{(N_k)^{Y_k}}{n}}.$$

As result, we obtain that

$$\alpha_n < \sqrt{\frac{(N_k)^{Y_k}}{n}}$$

and thus, the proof is done. ■

Main Theorem

This is the main theorem.

Theorem 6

The Riemann Hypothesis is true.

We know there are infinitely many superabundant numbers (Alaoglu and Erdős, 1944, Theorem 9 pp. 454).

In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$.

For every prime q, $\nu_q(n)$ goes to infinity as long as n goes to infinity when n is superabundant (Nazardonyavi and Yakubovich, 2013, Theorem 4.4 pp. 12), (Alaoglu and Erdős, 1944, Theorem 7 pp. 454).

Let $n_k > 5040$ be a large enough superabundant number such that q_k is the largest prime factor of n_k. Suppose that Robin(n_k) fails. In the same way, let $n_{k'}$ be another superabundant number much greater than n_k such that Robin($n_{k'}$) fails too. Under our assumption, we have

$$\alpha_{n_k} < \sqrt{\frac{(N_k)^{Y_k}}{n_k}}$$

and

$$\alpha_{n_{k'}} < \sqrt{\frac{(N_{k'})^{Y_{k'}}}{n_{k'}}}.$$

Hence,

$$\alpha_{n_{k'}} \cdot \alpha_{n_k} < \sqrt{\frac{(N_{k'})^{Y_{k'}}}{n_{k'}}} \cdot \alpha_{n_k}.$$

Consequently,

$$\alpha_{n_{k'}} \cdot \alpha_{n_k} < \sqrt{\frac{(N_{k'})^{Y_{k'}}}{n_{k'}}} \cdot \sqrt{\frac{(N_k)^{Y_k}}{n_k}}.$$

So,

$$(\alpha_{n_{k'}} \cdot \alpha_{n_k})^2 < \frac{(N_{k'})^{Y_{k'}}}{n_{k'}} \cdot \frac{(N_k)^{Y_k}}{n_k}.$$

However, we know that

$$(\alpha_{n_{k'}} \cdot \alpha_{n_k})^2 > 1.$$

Moreover, we can see that

$$\frac{(N_{k'})^{Y_{k'}}}{n_{k'}} \cdot \frac{(N_k)^{Y_k}}{n_k} \leq 1$$

since the following inequality

$$Y_k \leq \frac{\log(n_{k'} \cdot n_k)}{\log((N_{k'})^{\frac{Y_{k'}}{Y_k}} \cdot N_k)}$$

is satisfied for $n_{k'}$ much greater than n_k, because of $\frac{Y_{k'}}{Y_k} < 1$ and $\lim_{k \to \infty} Y_k = 1$. In this way, we obtain the contradiction $1 < 1$ under the assumption that Robin(n_k) fails.

To sum up, the study of this arbitrary large enough superabundant number $n_k > 5040$ reveals that Robin(n_k) holds on anyway.

Accordingly, Robin(n) holds for all large enough superabundant numbers n.

This contradicts the fact that there are infinite superabundant numbers n, such that Robin(n) fails when the Riemann Hypothesis is false.

By reductio ad absurdum, we prove that the Riemann Hypothesis is true. ■

The Riemann Hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Large Prime Gap Conjecture, etc.

Certainly, a proof of the Riemann Hypothesis could spur considerable advances in many mathematical areas, such as number theory and pure mathematics in general.

References

Leonidas Alaoglu and Paul Erdős. On highly composite and similar numbers. *Transactions of the American Mathematical Society*, 56(3):448–469, 1944. URL https://doi.org/10.2307/1990319.

YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, and Patrick Solé. On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux*, 19 (2):357–372, 2007. URL https://doi.org/10.5802/jtnb.591.

Sadegh Nazardonyavi and Semyon Yakubovich. Superabundant numbers, their subsequences and the Riemann hypothesis. *arXiv preprint arXiv:1211.2147v3*, 2013. version 3 (Submitted on 26 Feb 2013). arXiv URL https://arxiv.org/pdf/1211.2147v3.pdf.

Guy Robin. Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. *J. Math. pures appl*, 63(2):187–213, 1984. URL http://zakuski.utsa.edu/~jagy/Robin_1984.pdf.

Frank Vega. Robin's criterion on divisibility. *The Ramanujan Journal*, pages 1–11, 2022. URL https://doi.org/10.1007/s11139-022-00574-4.