Black Option Model

Black's vanilla option pricing model can be applied to a wide range of vanilla European options such as caps/floors, European swaptions, bond options, bond futures options and interest rate (IR) futures options.

Black's option pricing model, which is in a closed-form formula, can be applied to vanilla European type options under the Black-Scholes framework. Black's option pricing formula has been widely applied in fixed income derivative market for years. A vanilla European **call** option (see https://finpricing.com/lib/EqCallable.html) can be defined by its payoff at maturity as

$$[X_T - K]^+$$

where X is an underlying rate, T is the payoff reset time and K is the strike price. Accordingly, the payoff for a vanilla European **put** option is

$$[K - X_T]^+$$

The matured payoffs is paid at a settlement time T' which is greater than or equal to T. Under the Black-Scholes framework, the key assumption is that, in the risk-neutral measure with respect to the zero bond price matured at T', T is log-normally distributed with a single parameter X σ, the volatility of the underlying rate X.

Black's vanilla option pricing model can be applied to pricing a variety of instruments including caps/floors, European swaptions, bond options, bond futures options and IR futures options. In the case of caps/floors and European swaptions1, *X* is the forward term rate and forward swap rate, respectively. For European bond options, the rate *X* represents the bond price. For European bond futures options and European IR futures options, *X* stands for bond futures price and Euro-Dollar futures price, respectively.

In the Black's model, the fair price for the vanilla European **call** option as above is

$$C = e^{-rT'}[F_0 \cdot N(d_1) - K \cdot N(d_2)]$$

where *F*0 is a current expectation of the underlying rate *X* at maturity, *r* is the risk-free interest rate, *N*(·) denotes the cumulative distribution function (cdf) for a standard normal random variable and

$$d_1 = \frac{\log(F_0 / K) + \sigma^2 T / 2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\log(F_0 / K) - \sigma^2 T / 2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

A bond **put** option with the same specifications has a fair value given by

$$P = e^{-rT'}[K \cdot N(-d_2) - F_0 \cdot N(-d_1)]$$

Based on the above closed-form results, one can easily determine the various risk numbers associated with this instrument. For a **call** option, the risk numbers are

$$Delta = e^{-rT} \cdot N(d_1)$$

$$Gamma = \frac{e^{-rT} \cdot n(d_1)}{F_0 \cdot \sigma \cdot \sqrt{T}}$$

$$Vega = e^{-rT} \cdot n(d_1) \cdot F_0 \cdot \sqrt{T}$$

where $n(\cdot)$ denotes the probability density function (pdf) for a standard normal random variable, i.e.,

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Similarly, risk numbers for a **put** option are

$$\begin{aligned} Delta &= -e^{-rT} \cdot N(-d_1) \\ Gamma &= \frac{e^{-rT} \cdot n(d_1)}{F_0 \cdot \sigma \cdot \sqrt{T}} \\ Vega &= e^{-rT} \cdot n(d_1) \cdot F_0 \cdot \sqrt{T} \end{aligned}$$

i.e., Gamma and Vega are identical in both cases. Principal adjustment is necessary if dealing with caps/floors and swaptions.

The current expectation of an underlying rate at maturity must be known. It may be a current forward term rate, a current forward swap rate, a current forward price or a current futures price, depending upon cases.

The volatility of the underlying rate must be provided. Usually, it can be interpolated from a given set of market implied volatilities.