The Trigonometric-Exponential Functions

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Abstract. In this article, we provide definition of the trigonometric-exponential function and it's indentities and properties.

Keywords. composite function; exponential function; trigonometric function.

1. Definition Of The Trigonometric-Exponential Function

To begin, we need to define a trigonometric-exponential function.

Definition 1.1. The trigonometric-exponential functions can defined using trigonometric functions and exponential function. That is

$$\operatorname{sine}\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \sin(\exp x) \tag{1}$$
$$\operatorname{cose}\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \cos(\exp x) \tag{2}$$
$$\operatorname{tane}\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \tan(\exp x) \tag{3}$$
$$\operatorname{cote}\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \cot(\exp x) \tag{4}$$
$$\operatorname{sece}\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \sec(\exp x) \tag{5}$$
$$\operatorname{csce}\colon \mathbb{R} \to \mathbb{R},\ x \mapsto \csc(\exp x). \tag{6}$$

It can be notated as:

$$\operatorname{sine} x = \operatorname{se} x,\ \operatorname{cose} x = \operatorname{cse} x \tag{7}$$
$$\operatorname{tane} x = \operatorname{te} x,\ \operatorname{cote} x = \operatorname{cte} x \tag{8}$$
$$\operatorname{sece} x = \operatorname{scte} x,\ \operatorname{csce} x = \operatorname{csce} x. \tag{9}$$

Range of the trigonometric-exponential functions are followed as:

$$\operatorname{ran}(\operatorname{sine}) = \operatorname{ran}(\operatorname{cose}) = [-1, 1] \tag{10}$$
$$\operatorname{ran}(\operatorname{tane}) = \operatorname{ran}(\operatorname{cote}) = (-\infty, \infty) \tag{11}$$
$$\operatorname{ran}(\operatorname{sece}) = \operatorname{ran}(\operatorname{csce}) = (-\infty, -1] \cup [1, \infty). \tag{12}$$

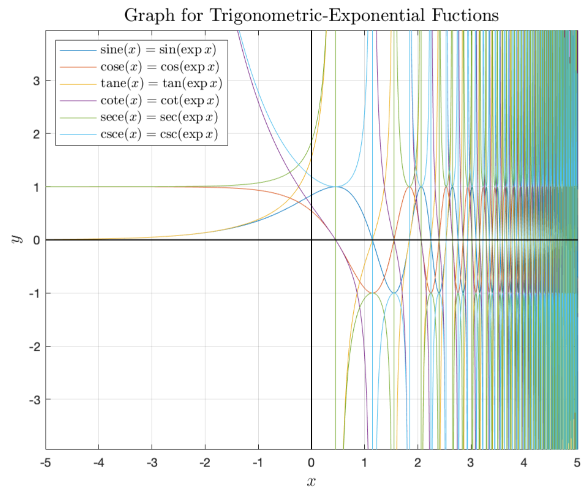

Figure 1: Graph for Trigonometric-Exponential Functions.

2. Basic Identities and Properties

We begin with basic identities of trigonometric-exponential functions.

Lemma 2.1. *For all* $x \in \mathbb{R}$*,*

$$\operatorname{tane} x = \frac{\operatorname{sine} x}{\operatorname{cose} x}, \ \operatorname{cote} x = \frac{\operatorname{cose} x}{\operatorname{sine} x}, \ \operatorname{sece} x = \frac{1}{\operatorname{cose} x}, \ \operatorname{csce} x = \frac{1}{\operatorname{sine} x}. \tag{13}$$

Proof. By the definition of trigonometric functions,

$$\tan x = \frac{\sin x}{\cos x}, \ \cot x = \frac{\cos x}{\sin x}, \ \sec x = \frac{1}{\cos x}, \ \csc x = \frac{1}{\sin x}. \tag{14}$$

Substitute e^x for x,

$$\tan(\mathrm{e}^x) = \frac{\sin(\mathrm{e}^x)}{\cos(\mathrm{e}^x)}, \ \cot \mathrm{e}^x = \frac{\cos(\mathrm{e}^x)}{\sin(\mathrm{e}^x)}, \ \sec(\mathrm{e}^x) = \frac{1}{\cos(\mathrm{e}^x)}, \ \csc x = \frac{1}{\sin(\mathrm{e}^x)}. \tag{15}$$

Therefore,

$$\operatorname{tane} x = \frac{\operatorname{sine} x}{\operatorname{cose} x}, \ \operatorname{cote} x = \frac{\operatorname{cose} x}{\operatorname{sine} x}, \ \operatorname{sece} x = \frac{1}{\operatorname{cose} x}, \ \operatorname{csce} x = \frac{1}{\operatorname{sine} x}. \tag{16}$$

This completes the proof. □

Proposition 2.2. *For all* $x \in \mathbb{R}$*,*

$$\operatorname{sine}^2 x + \operatorname{cose}^2 x = 1 \tag{17}$$

$$\operatorname{tane}^2 x + 1 = \operatorname{sece}^2 x \tag{18}$$

$$\operatorname{cote}^2 x + 1 = \operatorname{csce}^2 x. \tag{19}$$

Proof. For $x \in \mathbb{R}$, we have

$$\operatorname{sine}^2 x + \operatorname{cose}^2 x = \sin^2(\mathrm{e}^x) + \cos^2(\mathrm{e}^x) = 1, \tag{20}$$

$$\operatorname{tane}^2 x + 1 = \tan^2(\mathrm{e}^x) + 1 = \sec^2(\mathrm{e}^x) = \operatorname{sece}^2 x, \tag{21}$$

$$\operatorname{cote}^2 x + 1 = \cot^2(\mathrm{e}^x) + 1 = \csc^2(\mathrm{e}^x) = \operatorname{csce}^2 x. \tag{22}$$

This completes the proof. □

Now, we consider the derivatives and antiderivatives of trigonometric-exponential functions.

Proposition 2.3. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sine} x = \exp x \cdot \operatorname{cose} x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} \operatorname{cose} x = -\exp x \cdot \operatorname{sine} x, \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{tane} x = \exp x \cdot \operatorname{sece}^2 x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} \operatorname{cote} x = -\exp x \cdot \operatorname{csce}^2 x, \tag{24}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sece} x = \exp x \cdot \operatorname{sece} x \cdot \operatorname{tane} x, \qquad \frac{\mathrm{d}}{\mathrm{d}x} \operatorname{csce} x = -\exp x \cdot \operatorname{csce} x \cdot \operatorname{cote} x. \tag{25}$$

Proof. Substitute $t = \mathrm{e}^x$ and using chain rule, we get

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sine} x = \frac{\mathrm{d}}{\mathrm{d}x} \sin(\mathrm{e}^x) = \frac{\mathrm{d}}{\mathrm{d}x} \sin t = \frac{\mathrm{d} \sin t}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = \cos(\mathrm{e}^x) \cdot \mathrm{e}^x \tag{26}$$

$$= \exp x \cdot \operatorname{cose} x, \tag{27}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{cose} x = \frac{\mathrm{d}}{\mathrm{d}x} \cos(\mathrm{e}^x) = \frac{\mathrm{d}}{\mathrm{d}x} \cos t = \frac{\mathrm{d} \cos t}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = -\sin(\mathrm{e}^x) \cdot \mathrm{e}^x \tag{28}$$

$$= -\exp x \cdot \operatorname{sine} x, \tag{29}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{tane} x = \frac{\mathrm{d}}{\mathrm{d}x} \tan(\mathrm{e}^x) = \frac{\mathrm{d}}{\mathrm{d}x} \tan t = \frac{\mathrm{d} \tan t}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = \sec^2(\mathrm{e}^x) \cdot \mathrm{e}^x \tag{30}$$

$$= \exp x \cdot \operatorname{sece}^2 x, \tag{31}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{cote} x = \frac{\mathrm{d}}{\mathrm{d}x} \cot(\mathrm{e}^x) = \frac{\mathrm{d}}{\mathrm{d}x} \cot t = \frac{\mathrm{d} \cot t}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = -\csc^2(\mathrm{e}^x) \cdot \mathrm{e}^x \tag{32}$$

$$= -\exp x \cdot \operatorname{csce}^2 x, \tag{33}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{sece} x = \frac{\mathrm{d}}{\mathrm{d}x} \sec(\mathrm{e}^x) = \frac{\mathrm{d}}{\mathrm{d}x} \sec t = \frac{\mathrm{d} \sec t}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = \sec(\mathrm{e}^x) \tan(\mathrm{e}^x) \cdot \mathrm{e}^x \tag{34}$$

$$= \exp x \cdot \operatorname{sece} x \cdot \operatorname{tane} x, \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \operatorname{csce} x = \frac{\mathrm{d}}{\mathrm{d}x} \csc(\mathrm{e}^x) = \frac{\mathrm{d}}{\mathrm{d}x} \csc t = \frac{\mathrm{d} \csc t}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = -\csc(\mathrm{e}^x) \cot(\mathrm{e}^x) \cdot \mathrm{e}^x \tag{36}$$

$$= -\exp x \cdot \operatorname{csce} x \cdot \operatorname{cote} x. \tag{37}$$

This completes the proof. □

Proposition 2.4. *We have*

$$\int \operatorname{sine} x \,\mathrm{d}x = \operatorname{Si}(\mathrm{e}^x) + C, \quad \int \operatorname{cose} x \,\mathrm{d}x = \operatorname{Ci}(\mathrm{e}^x) + C. \tag{38}$$

Proof. Substitute $t = \mathrm{e}^x,\ \mathrm{d}x = \frac{1}{t}\,\mathrm{d}t$, we have

$$\int \operatorname{sine} x \,\mathrm{d}x = \int \sin(\mathrm{e}^x)\,\mathrm{d}x = \int \frac{\sin t}{t}\,\mathrm{d}t = \operatorname{Si}(\mathrm{e}^x) + C, \tag{39}$$

$$\int \operatorname{cose} x \,\mathrm{d}x = \int \cos(\mathrm{e}^x)\,\mathrm{d}x = \int \frac{\cos t}{t}\,\mathrm{d}t = \operatorname{Ci}(\mathrm{e}^x) + C. \tag{40}$$

This completes the proof. □

3. Inverse Trigonometric-Exponential Functions

Here, we define the inverse functions of trigonometric-exponential functions.

Definition 3.1. The inverse trigonometric-exponential functions defined as inverse functions for trigonometric-exponential functions. Some of them are divided into two functions because the domain of natural logarithm is $(0,\infty)$. It can defined as formula using natural logarithm and inverse trigonometric functions. That is

$$\operatorname{sine}_{-1}^{-1} \colon (0,1] \to \left(-\infty, \ln\frac{\pi}{2}\right],\ x \mapsto \ln(\arcsin(x)) \tag{41}$$

$$\operatorname{sine}_{0}^{-1} \colon [-1,1] \to \left[\ln\frac{\pi}{2}, \ln\frac{3\pi}{2}\right],\ x \mapsto \ln(\pi - \arcsin(x)) \tag{42}$$

$$\operatorname{cose}^{-1} \colon [-1,1] \to (-\infty, \ln\pi],\ x \mapsto \ln(\arccos(x)) \tag{43}$$

$$\operatorname{tane}_{-1}^{-1} \colon (0,\infty) \to \left(-\infty, \ln\frac{\pi}{2}\right),\ x \mapsto \ln(\arctan(x)) \tag{44}$$

$$\operatorname{tane}_{0}^{-1} \colon (-\infty,\infty) \to \left(\ln\frac{\pi}{2}, \ln\frac{3\pi}{2}\right),\ x \mapsto \ln(\pi + \arctan(x)) \tag{45}$$

$$\operatorname{cote}^{-1} \colon (-\infty,\infty) \to (-\infty, \ln\pi),\ x \mapsto \ln(\operatorname{arccot}(x)) \tag{46}$$

$$\operatorname{sece}^{-1} \colon (-\infty,-1] \cup (1,\infty) \to \left(-\infty, \ln\frac{\pi}{2}\right) \cup \left(\ln\frac{\pi}{2}, \ln\pi\right),\ x \mapsto \ln(\operatorname{arcsec}(x)) \tag{47}$$

$$\operatorname{csce}_{-1}^{-1} \colon [1,\infty) \to \left(-\infty, \ln\frac{\pi}{2}\right),\ x \mapsto \ln(\operatorname{arccsc}(x)) \tag{48}$$

$$\operatorname{csce}_{0}^{-1} \colon (-\infty,1] \cup [1,\infty) \to \left[\ln\frac{\pi}{2}, \ln\pi\right) \cup \left(\ln\pi, \ln\frac{3\pi}{2}\right],\ x \mapsto \ln(\pi - \operatorname{arccsc}(x)) \tag{49}$$

In general, inverse functions for sine, tane, csce notated as

$$\operatorname{sine}^{-1} = \operatorname{sine}_0^{-1},\ \operatorname{tane}^{-1} = \operatorname{tane}_0^{-1},\ \operatorname{csce}^{-1} = \operatorname{csce}_0^{-1}. \tag{50}$$

Proposition 3.2. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sine}^{-1} x = -\frac{1}{\sqrt{1-x^2}(\pi - \arcsin(x))}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cose}^{-1} x = -\frac{1}{\sqrt{1-x^2}\arccos(x)}, \tag{51}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} = \operatorname{tane}^{-1} x = \frac{1}{(1+x^2)(\pi + \arctan(x))}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cote}^{-1} x = -\frac{1}{(1+x^2)\operatorname{arccot}(x)}, \tag{52}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sece}^{-1} x = \frac{1}{|x|\sqrt{x^2-1}\operatorname{arcsec}(x)}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{csce}^{-1} x = \frac{1}{|x|\sqrt{x^2-1}(\pi - \operatorname{arccsc}(x))}. \tag{53}$$

Proof. According to the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sine}^{-1}x = \frac{\mathrm{d}}{\mathrm{d}x}\ln(\pi-\arcsin(x)) = \frac{1}{\pi-\arcsin(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\pi-\arcsin(x)) \tag{54}$$

$$= -\frac{1}{\sqrt{1-x^2}(\pi-\arcsin(x))}, \tag{55}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cose}^{-1}x = \frac{\mathrm{d}}{\mathrm{d}x}\ln(\arccos(x)) = \frac{1}{\arccos(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\arccos(x) \tag{56}$$

$$= -\frac{1}{\sqrt{1-x^2}\arccos(x)}, \tag{57}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{tane}^{-1}x = \frac{\mathrm{d}}{\mathrm{d}x}\ln(\pi+\arctan(x)) = \frac{1}{\pi+\arctan(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\pi+\arctan(x)) \tag{58}$$

$$= \frac{1}{(1+x^2)(\pi+\arctan(x))}, \tag{59}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cote}^{-1}x = \frac{\mathrm{d}}{\mathrm{d}x}\ln(\operatorname{arccot}(x)) = \frac{1}{\operatorname{arccot}(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccot}(x) \tag{60}$$

$$= -\frac{1}{(1+x^2)\operatorname{arccot}(x)}, \tag{61}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{sece}^{-1}x = \frac{\mathrm{d}}{\mathrm{d}x}\ln(\operatorname{arcsec}(x)) = \frac{1}{\operatorname{arcsec}(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcsec}(x) \tag{62}$$

$$= \frac{1}{|x|\sqrt{x^2-1}\operatorname{arcsec}(x)}, \tag{63}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{csce}^{-1}x = \frac{\mathrm{d}}{\mathrm{d}x}\ln(\pi-\operatorname{arccsc}(x)) = \frac{1}{\pi-\operatorname{arccsc}(x)}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\pi-\operatorname{arccsc}(x)) \tag{64}$$

$$= \frac{1}{|x|\sqrt{x^2-1}(\pi-\operatorname{arccsc}(x))}. \tag{65}$$

This completes the proof. □

Proposition 3.2. *We have*

$$\int \operatorname{sine}^{-1}x\,\mathrm{d}x = -x\ln(\pi-\arcsin(x)) + \operatorname{Si}(\pi-\arcsin(x)) + C, \tag{66}$$

$$\int \operatorname{cose}^{-1}x\,\mathrm{d}x = x\ln(\arccos(x)) - \operatorname{Ci}(\arccos(x)) + C. \tag{67}$$

Proof. We have

$$\int \operatorname{sine}^{-1}x\,\mathrm{d}x = \int \ln(\pi-\arcsin(x))\,\mathrm{d}x\,. \tag{68}$$

Substitute $t=\pi-\arcsin(x)$, $\mathrm{d}x=-\cos t\,\mathrm{d}t$,

$$\int \ln(\pi-\arcsin(x))\,\mathrm{d}x = -\int \cos t\ln t\,\mathrm{d}t\,. \tag{69}$$

Integration by parts: $u=\ln t,\ \mathrm{d}v=\cos t\,\mathrm{d}t \longrightarrow \mathrm{d}u=1/t,\ v=\sin t$,

$$\int \cos t\ln t\,\mathrm{d}t = \ln t\sin t - \int\frac{\sin t}{t}\,\mathrm{d}t = \ln t\sin t - \operatorname{Si}(t) + C. \tag{70}$$

Undo substitution, since $\sin(\pi - \arcsin(x)) = x$,

$$\int \mathrm{sine}^{-1} x \, \mathrm{d}x = -x \ln(\pi - \arcsin(x)) + \mathrm{Si}(\pi - \arcsin(x)) + C. \tag{71}$$

Also, we have

$$\int \mathrm{cose}^{-1} x \, \mathrm{d}x = \int \ln(\arccos(x)) \, \mathrm{d}x \, . \tag{72}$$

Substitute $t = \arccos(x)$, $\mathrm{d}x = -\sin t$,

$$\int \ln(\arccos(x)) \, \mathrm{d}x = -\int \ln t \sin t \, \mathrm{d}t \, . \tag{73}$$

Integration by parts: $u = \ln t,\ \mathrm{d}v = \sin t \, \mathrm{d}t \longrightarrow \mathrm{d}u = 1/t,\ v = -\cos t$,

$$\int \ln t \sin t \, \mathrm{d}t = -\ln t \cos t + \int \frac{\cos t}{t} \, \mathrm{d}t = -\ln t \cos t + \mathrm{Ci}(t) + C. \tag{74}$$

Undo substitution, since $\cos(\arccos(x))) = x$,

$$\int \mathrm{sine}^{-1} x \, \mathrm{d}x = x \ln(\arccos(x)) - \mathrm{Ci}(\arccos(x)) + C. \tag{75}$$

This completes the proof. □

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Reference

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