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Spline-Fourier Approximations of Discontinuous Waves

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Abstract: In the Fourier series approximation of real functions discontinuities of the functions or their derivatives cause problems like Gibbs phenomenon or slow uniform convergence. In the case of a finite number of isolated discontinuities the problems can be to a large extend rectified by using periodic splines in the series. This modified Fourier series (Spline-Fourier series) is applied to the numerical solution of the wave equation (in periodic form) where discontinuities in the data functions or their derivatives appear quite often. The solution is sought in the form of a Spline-Fourier series about the space variable and close bounds are obtained using a certain iterative procedure of Newton type.

Key Words: Validated numerics, Fourier hyper functoid, Wave equation

1 Introduction

This work is inspired by the ideas for validated approximations of function space problems [3] and essentially it is a further application of the Fourier Hyper Functoid described in [2] where the coefficient vector $c = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)$ of the Fourier series

$$f(x) \sum_{k=-\infty}^{\infty} c_k e^{ik\pi x}$$

of a function $f : [-1, 1] \to R$ is approximated by a vector of the form

$$c(k) = \begin{cases} c_k & : 0 \le k \le N \\ \overline{c_{-k}} & : -N \le k \le -1 \\ (-1)^k \sum_{j=1}^{p} a_j \frac{1}{k^j} & : |k| > N \end{cases} \qquad (1)$$

Let $\{s_j : j = 1, 2, \ldots\}$ be a set of periodic splines defined as polynomials on $(-1, 1)$ such that $s_1(x) = x,\ x \in (-1, 1),\ s'_j = s_{j-1},\ s_j \in C_{j-2}(-\infty, \infty)$, $j = 2, 3, \ldots$. These splines are closely related to the monosplines, which are well-known in the spline theory, as well as the Bernoulli polynomials. In fact $n!s_n$ and the nth monospline differ by a constant and $s_n(x) = \frac{2^n}{n!} B_n(\frac{x+1}{2})$ where B_n is the nth Bernoulli polynomial. From the Fourier series expansions

$$s_j(x) = -\frac{1}{(i\pi)^j} \sum_{\substack{k=-\infty \\ k \ne 0}}^{\infty} \frac{(-1)^k}{k^j} e^{ik\pi x}\ ,\ j \ge 1$$

we can see that the approximation of the Fourier coefficients of the form (1)is equivalent to approximating the function f by linear combinations of

$$\{s_j : j = 1, \ldots, p\} \cup \{e^{ik\pi x} : k = 0, \pm 1, \ldots, \pm N\}$$

For practical reasons that can be seen below we would prefer to use the spline notations rather then the above ansatz (1) for the coefficients.

2 Spline-Fourier Series

Let $\mathcal{M}$ be the set of all periodical function with period 2 which are p-times differentiable at every point of $(-1, 1]$ except for a finite number of points $\alpha_1, \alpha_2, \ldots, \alpha_M$ and let $\frac{d^p f}{dx^p} \in L_2[-1, 1]$. There is a unique representation of every function $f \in \mathcal{M}$ in the form

$$f(x) = a_0 + \sum_{m=1}^{M} \sum_{j=1}^{p} a_{mj} s_j(x + 1 - \alpha_m) + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} b_k e^{ik\pi x} \tag{2}$$

$$where \quad \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} b_k e^{ik\pi x} \in C_{p-1}(-\infty, \infty)$$

The coefficients are obtained as follows

$$\begin{aligned} a_j &= \frac{1}{2} \left(\frac{d^{j-1} f}{dx^{j-1}}(1-0) - \frac{d^{j-1} f}{dx^{j-1}}(1+0) \right) = \frac{1}{2} \int_{-1}^{1} \frac{d^j f(x)}{dx^j} dx \ , \ j = 1, \ldots, p \\ a_0 &= \frac{1}{2} \int_{-1}^{1} f(x) dx \\ b_k &= \frac{1}{2(ik\pi)^p} \int_{-1}^{1} \frac{d^p f(x)}{dx^p} e^{-ik\pi} dx \ , \ k = \pm 1, \pm 2, \ldots \end{aligned} \tag{3}$$

The rounding $S_N : \mathcal{M} \to S_N(\mathcal{M})$ is defined by

$$S_N(f; x) = a_0 + \sum_{m=1}^{M} \sum_{j=1}^{p} a_{mj} s_j(x + 1 - \alpha_m) + \sum_{\substack{k=-N \\ k \neq 0}}^{N} b_k e^{ik\pi x}$$

i.e. by truncating the series at $k = N$. The error is estimated as follows

$$|f(x) - S_N(f; x)| \ \leq \ \varepsilon_N \ =$$

$$\left(\frac{1}{2} \int_{-1}^{1} \left(\frac{d^p f(x)}{dx^p} \right)^2 dx - \sum_{m=1}^{M} a_{mp}^2 - 2 \sum_{k=1}^{N} (k\pi)^{2p} |b_k|^2 \right)^{\frac{1}{2}} \left(\frac{2}{(2p-1)\pi^{2p} N^{2p-1}} \right)^{\frac{1}{2}}$$

Therefore we have the inclusion

$$f(x) \in a_0 + \sum_{m=1}^{M} \sum_{j=1}^{p} a_{mj} s_j(x + 1 - \alpha_m) + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} b_k e^{ik\pi x} + [-\varepsilon_N, \varepsilon_N]$$

3 Numerical Method for the Wave Equation

We consider a wave equation of the form

$$\begin{aligned} & u(x,t)_{tt} - u(x,t)_{xx} = \rho(t)u(x,t) + \phi(x,t)) \\ & u(x,0) = g_1(x)\ , \quad u_t(x,0) = g_2(x) \end{aligned}$$

with periodic boundary conditions at $x = -1$ and $x = 1$, assuming that ϕ, g_1, g_2 or some of their derivatives may be discontinuous but they can be represented as a spline-Fourier series (2).

An interval enclosure of the solution is calculated in the form

$$U(x,t) = A_0(t) + \sum_{m=1}^{M}\sum_{j=1}^{p}\sum_{\varepsilon=-1}^{1} A_{mj\varepsilon}(t)s_j(x+\varepsilon t+1-\alpha_m) + \sum_{\substack{k=-N \\ k\neq 0}}^{N} B_k(t)e^{ik\pi x}$$

using an iterative procedure

$$U^{(k+1)} = (1-\lambda)U^{(k)} + \lambda\left(g + \tfrac{1}{2}\iint_{G(x,t)} \rho U^{(k)}\right)$$

where $G(x,t)$ is a triangle with vertices (x,t), $(x-t,0)$, $(x+t,0)$ and

$$g(x,t) = \frac{1}{2}\left(g_1(x+t) + g_1(x-t) + \int_{x-t}^{x+t} g_2(\theta)d\theta + \iint_{G(x,t)} \phi(y,\theta)dyd\theta\right)$$

Essential part of each iteration is the evaluation of the integral. The following formulas are used:

$$\iint_{G(x,t)} \frac{\theta^q}{q!} s_j(y)dyd\theta = s_{j+q+2}(x+t) + (-1)^q s_{j+q+2}(x-t) - 2\sum_{\substack{l=0 \\ l-even}}^{q} \frac{t^{q-l}}{(q-l)!} s_{j+l+2}(x)$$

$$\iint_{G(x,t)} \frac{\theta^q}{q!} s_j(y+\theta)dyd\theta = \sum_{l=0}^{q+1}\left(-\frac{1}{2}\right)^l \frac{t^{q+1-l}}{(q+1-l)!} s_{j+l+1}(x+t) - \left(-\frac{1}{2}\right)^{q+1} s_{j+q+2}(x-t)$$

$$\iint_{G(x,t)} \frac{\theta^q}{q!} s_j(y-\theta)dyd\theta = -\sum_{l=0}^{q+1}\left(\frac{1}{2}\right)^l \frac{t^{q+1-l}}{(q+1-l)!} s_{j+l+1}(x+t) + \left(\frac{1}{2}\right)^{q+1} s_{j+q+2}(x-t)$$

$$\begin{aligned} \iint_{G(x,t)} \frac{\theta^q}{q!} e^{ik\pi y}dyd\theta &= \left(\frac{1}{ik\pi}\right)^{q+2} e^{ik\pi(x+t)} + \left(\frac{1}{ik\pi}\right)^{q+2} e^{ik\pi(x-t)} - 2\sum_{\substack{l=0 \\ l-even}}^{q} \frac{t^{q-l}}{(q-l)!} e^{ik\pi x} \\ &= 2\sum_{\substack{l=0 \\ l-even}}^{\infty} (ik\pi)^l \frac{t^{q+l+2}}{(q+l+2)!} e^{ik\pi x} \end{aligned}$$

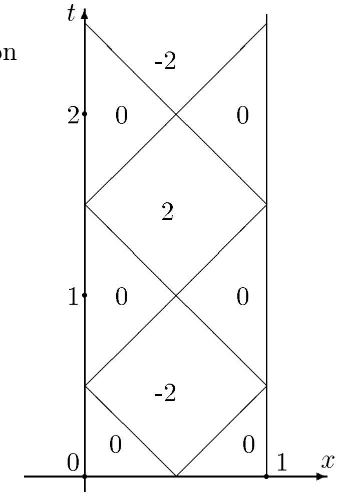
The integration over $G(x,t)$ produces splines with larger indexes. The splines s_j that are sufficiently smooth (e.g. $j > p$) are rounded as follows

$$s_j(x) \in -\frac{1}{(i\pi)^j}\sum_{\substack{k=-N \\ k\neq 0}}^{N} \frac{(-1)^k}{k^j} e^{ik\pi x} + \frac{2}{(j-1)\pi^j N^{j-1}}[-1,1]$$

4 Numerical Examples

Example 1. $u_{tt} - u_{xx} = \frac{2}{(t+3)^2}u + \pi^2(t+3)^2\sin(\pi x) + 2(t+3)\phi(x,t)$

where $\phi(x,t)$ is a piecewise constant function in $[0,1]\times[0,\infty)$ as defined on the sketch



$$u(0,t) = 0$$
$$u(1,t) = 0 \ , \quad t \geq 0$$

$$u(x,0) = 9\sin(\pi x) + 18\psi(x)$$
$$u_t(x,0) = 6\sin(\pi x) + 12\psi(x), \ 0 \leq x \leq 1$$

where

$$\psi(x) = \begin{cases} x & 0 \leq x \leq 0.5 \\ 1-x & 0.5 \leq x \leq 1 \end{cases}$$

Exact Solution

$$u(x,t) = (t+3)^2(\sin(\pi x) + s_2(x+t+0.5) + s_2(x-t+0.5) - s_2(x+t-0.5) - s_2(x-t-0.5))$$

Example 2. $$u_{tt} - u_{xx} = \left(\pi^2 + \frac{2}{(t+3)^2}\right)u$$
$$u(x,0) = 2(s_3(x+1) - s_3(x))$$
$$u_t(x,0) = 0$$

In both examples it is demonstrated that using relatively small values of p and N,e.g.$p=5, N=5$ close bounds for the solutions are obtained. In example 1 since the exact solution is in $S_N(\mathcal{M})$ the computed bounds are particularly close giving up to 10 correct digits of the solution. In example 2 using similar values of p and N 4-5 correct digits of the solution are obtained. All programs are written in PASCAL-XSC.

References

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