Modelling energy changes in the energy harvesting battery of an IoT device

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***Abstract*—The complexity of battery-powered autonomous devices such as Internet of Things (IoT) nodes or Unmanned Aerial Vehicles (UAV) and the necessity to ensure an acceptable quality of service, reliability, and security, have significantly increased their energy demand. In this paper, we discuss using a diffusion approximation process to approximate the dynamic changes in the energy content of a battery. We consider the case when energy harvesting sources are constantly charging the battery. The model assumes a probabilistic consumption and delivery of energy, giving the time-dependent distributions of the energy at the battery, of the time remaining until it becomes empty, the time required to charge the battery to its total capacity, or the time it is operational between two moments of complete depletion. When possible, we compare the diffusion approximation results with corresponding Markovian models.**

***Index Terms*—Energy Harvesting, IoT, Diffusion models, Markovian Models**

I. Introduction

Numerous new IoT systems combine batteries with energy-harvesting and take advantage of ambient energy. They use technologies which derive power from external sources such as solar, thermal, wind, and vibration. This way, the IoT devices may become energy-independent and perform their duties almost perpetually. An effort to ensure higher efficiency of the harvesting and more economical performance of these devices is necessary. It should lead to a balance between consumption and power generation. Several factors are playing a role [1]. The processes of harvesting and consumption are not deterministic and change with time, as they depend on external conditions and the current work of the system. Also, the resulting stochastic process parameters representing stored energy are not constant. The parameters of a battery change as it ages and becomes less efficient. Wireless devices and networks which use energy harvesting are exposed to attacks on different protocol stack layers. They include eavesdropping, energy depletion, flooding, beamforming vector poisoning, side channel, spoofing/replay, and device tampering attacks, increasing energy consumption abruptly. Also, any protection against them is energy consuming, and a trade-off between security and energy efficiency is needed; therefore, hybrid security-energy metrics are introduced. These complex issues need modelling tools to help the performance analysis of IoT devices. This article proposes a mathematical model that might be useful in such studies. We believe that, as a whole, it is more precise than the already existing ones.

A diffusion process models the changes of energy stored in the battery; it is often used to represent more complex stochastic processes. The diffusion process has mean and variance that reflect the means and variances of the energy harvesting and consumption; therefore, it is more accurate than the one-parameter Poisson process. The model results include time-dependent probability distribution of the energy stored in the battery, the distribution of time to the nearest complete depletion, and the distribution of the battery activity time between consecutive depletions. The transient solution to diffusion equations enables us to get results in the form of distributions, therefore giving more precise information than time-dependent mean values furnished by more popular models based on the fluid-flow approach. This approach gives us also the probability that the total depletion will never happen, and the device will work without interruptions due to lack of energy. The model parameters may vary with time, and it is possible to update predictions based on new parameters and use energy distribution at the moment of changes as initial conditions.

Markovian stochastic models have been applied to model the changes in the energy content of a battery, e.g. in [2]–[4] However, the energy Poisson assumption in the arrival of energy packets into the battery and in removing energy packets from the battery may deviate from reality. That is why we apply here a diffusion model where the interarrival times and interdeparture times may follow any distribution, as already proposed in [5], [6]. A useful but seldom used approach in the analysis and optimization of energy, and more broadly for the joint optimization of energy and quality of service in computer systems and networks, named "energy packets" was introduced in [7]. It conveniently represents energy in discrete units, where an energy packet is a minimum energy required to transmit a single data packet or process a single job. This approach was initially applied to the optimization of power flow in multiple node computer networks [8] and joint work and energy in computer systems [9]. The model was applied to the study of sensor nodes [10], and to battery performance, [11].

Indeed, when the energy is quantized, Markovian stochas-

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tic models can be used to model the energy storage and consumption process. In this case, the state probabilities at time t represent the amount of energy stored present in the battery at time t. The authors in [12] developed a mathematical framework for modelling the charging and discharging of the battery of a nanosensor device. The authors used a Markovian process to represent the dynamic changes in the battery's energy content. They then computed the state probabilities of the amount of energy present in the battery (the energy state of the battery). One of the limitations of Markovian models is the assumption that the rate at which energy is drawn from the battery is exponentially distributed, which is not a realistic assumption of the IoT energy consumption patterns.

Since energy is a continuous quantity, the changes in the amount of energy in the battery could be considered analogous to the changes of a fluid in a reservoir and modelled using fluid flow models. The authors in [13] proposed an analytical model of a battery based on the fluid flow queueing model. The authors modelled the battery as a charge or energy reservoir where the charge gets accumulated or depleted over time. By considering that the charge available in the battery at time t is analogous to the fluid available in a reservoir, the authors used fluid flow analytical methods to determine the cumulative distribution function and the mean of the time required for the battery to be discharged entirely. The authors in [14] proposed a fluid queue model for the representation of the dynamic changes in the energy content of a battery and then used it to determine the time required to completely depletes the the energy of the battery. The authors in [15] proposed a Markov fluid queue model for the battery of an energy harvesting IoT device. The authors used their model to compute the probability that the battery's energy level hits zero for the first time within a given finite time horizon. Fluid flow models capture the mean changes in the amount of energy present in the battery but not the variance.

Here, we follow the description of the battery energy content based on the arrivals and departures of unitary energy packets. We also represent the energy by the number of these packets in a queueing system. The diffusion approximation queueing model allows us to assume general distributions of packets' interarrival and consumption times. Its transient analysis gives us the distribution of energy content at any time, also if the parameters of the harvesting and consumption processes are time-varying. In section II we present the queueing model of the battery content, and in section III we derive the distributions of times needed to deplete or charge the battery. We also derive a simple formula for the probability that the depletion will not happen. This section contains the original contributions of the paper. The distributions are compared with exact solutions known in the case of the Markovian model with exponential distributions of interarrival and consumption times of energy packets. The comparison shows high accuracy of the approximation. Section IV concludes the article.

II. Representation of the battery content with the use of a queueing model

Consider a battery equipped with an energy harvesting device. We assume that energy harvesting is represented by the arrival of unitary energy packets and that the distribution of interarrival times has a mean $1/\lambda$ and variance σ_A^2. The energy consumption is represented by the service of energy packets with the mean rate μ and variance σ_B^2. This way, the battery model is equivalent to G/G/1/B station. Following Kendal's notation [16] it denotes the one-server service station with first-in-first-out service, general type of interarrival and service time distributions and queue limited to B customers where customers represent the energy packets. The G/G/1/B model has no effective analytical solution [17]. However, we may use its diffusion approximation, as proposed in [18].

Following this approach, diffusion process $X(t)$, $x \in [0, B]$ denotes the value of energy at the battery (i.e. the number of energy packets in the queue); $x = B$ means that the battery is fully charged; in this case, the coming energy packets are lost. The value $x = 0$ means the battery is empty and may resume its activity after the arrival of the following energy packet.

The parameters α and β of the diffusion equation depend on mean and variance of interarrival and service time distributions, the type of these distributions is not relevant: $\beta = \lambda - \mu$, $\alpha = \sigma_A^2\lambda^3 + \sigma_B^2\mu^3 = C_A^2\lambda + C_B^2\mu$, where C_A^2, C_B^2 are squared coefficients of variation of interarrival and service time distributions, [18].

The diffusion process has two limiting barriers, at $x = 0$ and $x = B$. When it comes to $x = 0$, it stays there for a specific time waiting for the arrival of the next packet, jumps to $x = 1$ when it arrives, and then moves until it approaches any of the barriers again. When it comes to $x = B$, it stays in the barrier waiting for the consumption of an energy packet and then jumps to $x = B - 1$ and resumes the movement. The jumps from $x = 0$ to $x = 1$ are performed with intensity λ (intensity of energy packet arrivals) and from $x = B$ to $x = B - 1$ with intensity μ (intensity of packets departures).

The system of equations defining the density

$$f(x, t; x_0)dx = P[x \leq X(t) < x + dx \mid X(0) = x_0]$$

of the diffusion process is

$$\begin{aligned}
\frac{\partial f(x,t;x_0)}{\partial t} &= \frac{\alpha}{2}\frac{\partial^2 f(x,t;x_0)}{\partial x^2} - \beta\frac{\partial f(x,t;x_0)}{\partial x} + \\
&\quad + \lambda p_0(t)\delta(x-1) + \mu p_B(t)\delta(x-B+1)\ , \\
\frac{dp_0(t)}{dt} &= \lim_{x\to 0}\,[\frac{\alpha}{2}\frac{\partial f(x,t;x_0)}{\partial x} - \beta f(x,t;x_0)] \\
&\quad -\lambda p_0(t)\ , \\
\frac{dp_B(t)}{dt} &= \lim_{x\to B}\,[-\frac{\alpha}{2}\frac{\partial f(x,t;x_0)}{\partial x} + \beta f(x,t;x_0)] \\
&\quad -\mu p_B(t)\ , \qquad (1)
\end{aligned}$$

$\delta(x)$ is the Dirac delta function, x_0 is the initial condition, $p_0(t)$, $p_B(t)$ are probabilities that the process is at time t at the barriers at $x = 0$ or $x = B$. The first equation is the diffusion equation with jumps to $x = 1$ and $x = B - 1$ the others are probability balnce equations for the barriers. The solution give us $f(x,t;x_0)$, $p_0(t)$, and $p_B(t)$. If the battery is fully loaded at the beginning, $x_0 = B$, $p_B(0) = 1$.

The transient solution of (1) may be computationaly obtained with the approach of [19], [20]. In the first step we consider a diffusion process with two absorbing barriers at $x = 0$ and $x = B$, started at $t = 0$ from $x = x_0$. Its probability density function $\phi(x,t;x_0)$ has the following form [21]

$$\begin{aligned}&\phi(x,t;x_0) = \\ &\frac{1}{\sqrt{2\Pi\alpha t}}\sum_{n=-\infty}^{\infty}\{\exp[\frac{\beta x'_n}{\alpha} - \frac{(x-x_0-x'_n-\beta t)^2}{2\alpha t}] \\ &-\exp[\frac{\beta x''_n}{\alpha} - \frac{(x-x_0-x''_n-\beta t)^2}{2\alpha t}]\},\end{aligned} \quad (2)$$

where $x'_n = 2nB$, $x''_n = -2x_0 - x'_n$.

The Laplace transform of $\phi(x,t;x_0)$ is

$$\begin{aligned}&\bar{\phi}(x,s;x_0) = \\ &\frac{\exp[\frac{\beta(x-x_0)}{\alpha}]}{A(s)}\sum_{n=-\infty}^{\infty}\{\exp[-\frac{|x-x_0-x'_n|}{\alpha}A(s)] \\ &-\exp[-\frac{|x-x_0-x''_n|}{\alpha}A(s)]\},\end{aligned} \quad (3)$$

with $A(s) = \sqrt{\beta^2 + 2\alpha s}$.

The probability density function $f(x,t;B)$ of the diffusion process with jumps from the boundaries is composed of a spectrum of functions $\phi(x,t-\tau;1)$, $\phi(x,t-\tau;B-1)$ representing diffusion processes with absorbing barriers at $x = 0$ and $x = B$, started with densities $g_1(\tau)$ and $g_{B-1}(\tau)$ at time $\tau < t$ at points $x = 1$ and $x = B-1$ due to jumps from the barriers:

$$f(x,t;B) = g_1(t) * \phi(x,t;1) + g_{B-1}(t) * \phi(x,t;B-1) \quad (4)$$

where $*$ denotes convolution, and densities $g_1(t)$, $g_{B-1}(t)$, as well as $p_0(t)$ and $p_B(t)$, are obtained from the probability balance equations at the barriers.

The densities $\gamma_0(t)$, $\gamma_B(t)$ of probability that at time t the process enters a barrier at $x = 0$ or $x = B$ are

$$\begin{aligned}\gamma_0(t) &= p_0(0)\delta(t) + g_1(t) * \gamma_{1,0}(t) + g_{B-1}(t) * \gamma_{B-1,0}(t), \\ \gamma_B(t) &= p_B(0)\delta(t) + g_1(t) * \gamma_{1,B}(t)g_{B-1}(t) * \gamma_{B-1,B}(t),\end{aligned} \quad (5)$$

where $\gamma_{1,0}(t)$, $\gamma_{1,B}(t)$, $\gamma_{B-1,0}(t)$, $\gamma_{B-1,B}(t)$ are densities of the first passage times between the points indicated in the index. The densities are obtained in the same way as $\gamma_{x_0,0}(t)$ in the next section, Eq. (12).

The intensities of jumps in Eq. (4) depend on $\gamma_0(t)$, $\gamma_B(t)$ in the following way:

$$g_1(t) = \gamma_0(t) * l_0(t), \qquad g_{B-1}(t) = \gamma_B(t) * l_B(t), \quad (6)$$

where $l_0(t)$, $l_B(t)$ are the densities of sojourn times in $x = 0$ and $x = B$ (they have means $1/\lambda$ and $1/\mu$).

With the use of (5) and (6) we obtain the densities $g_1(t)$, $g_{B-1}(t)$ needed in the solution (4). We use these equations in the Laplace domain, where the convolutions of functions become their products. Then we invert the obtained transform of $\bar{f}(x,s;B)$ numerically.

Probabilities that the proces is at barriers are

$$p_0(t) = \int_0^t [\gamma_0(\tau) - g_1(\tau)]d\tau, \; p_B(t) = \int_0^t [\gamma_B(\tau) - g_{B-1}(\tau)]d\tau. \quad (7)$$

They are convergent to steady state values

$$\begin{aligned}p_0 &= \{1 + \varrho e^{z(B-1)} + \frac{\varrho}{1-\varrho}[1 - e^{z(B-1)}]\}^{-1}, \\ p_B &= \varrho p_0 e^{z(B-1)},\end{aligned} \quad (8)$$

where $\varrho = \lambda/\mu$, $z = 2\beta/\alpha$, and the solution (4) converges to to the known [18] steady-state solution. If the diffusion parameters change, the density $f(x,t;B)$ obtained just before serves as the initial condition for the new one; this way, the model adapts to the parameters changes reflecting time-dependent harvesting and energy consumption.

III. The first passage times

A. First passage times in unlimited diffusion process

The pdf of the unlimited diffusion process (no barriers) is defined by the equation

$$\frac{\partial f(x,t;x_0)}{\partial t} = \frac{\alpha}{2}\frac{\partial^2 f(x,t;x_0)}{\partial x^2} - \beta\frac{\partial f(x,t;x_0)}{\partial x} \quad (9)$$

and its solution

$$f(x,t;x_0) = \frac{1}{\sqrt{2\Pi\alpha t}}\exp(\frac{(x-x_0-\beta t)^2}{2\alpha t}). \quad (10)$$

Suppose the diffusion process value represents the battery's energy content. In that case, the battery's lifetime corresponds to the time the diffusion process needs to pass from the initial point $x_0 = B > 0$ (initial energy of fully charged battery) to $x = 0$. We may determine the distribution of this time by considering a diffusion process with an absorbing barrier at $x = 0$ i.e. the process started at x_0 is ended when it comes to zero. It corresponds to the condition $\lim_{x\to 0} f(x,t;x_0) = 0$.

The problem of diffusion with absorbing barrier was studied e.g. in [21] and the solution is given by Eq. (11). It was obtained with the use of the method of images: the barrier is a mirror, and the solution is a superposition of two unrestricted processes, one of unit strength, starting at the origin, and the other of strength $-\exp(\frac{2\beta x_0}{\alpha})$ starting at $x = 2x_0$. It yields

$$\begin{aligned}f(x,t;x_0) &= \frac{1}{\sqrt{2\Pi\alpha t}}\Big[\exp(-\frac{(x-\beta t)^2}{2\alpha t}) \\ &\quad -\exp(\frac{2\beta x_0}{\alpha} - \frac{(x-2x_0-\beta t)^2}{2\alpha t})\Big]. \end{aligned} \quad (11)$$

The density of the first passage time of a diffusion process that starts from the point $x = x_0$ and ends at $x = 0$ is

$$\begin{aligned}\gamma_{x_0,0}(t) &= \int_{0+}^{\infty} \frac{\partial f(x,t;x_0)}{\partial t} dx \\ &= \int_{0+}^{\infty} [\frac{\alpha}{2}\frac{\partial^2 f(x,t;x_0)}{\partial x^2} - \beta \frac{\partial f(x,t;x_0)}{\partial x}] dx \\ &= \frac{x_0}{\sqrt{2\Pi\alpha t^3}} e^{-\frac{(x_0-\beta t)^2}{2\alpha t}}, \qquad (12)\end{aligned}$$

with the Laplace transform

$$\bar{\gamma}_{x_0,0}(s) = e^{-x_0 \frac{\beta + \sqrt{\beta^2 + 2\alpha s}}{\alpha}}. \qquad (13)$$

Eq. (12) presents the probability density function in case of $\beta < 0$, when probability that the process will reach the barrier $\int_0^\infty \gamma_{x_0,0}(t)dt = 1$. Otherwise, for $\beta > 1$,

$$\int_0^\infty \gamma_{x_0,0}(t)dt = \bar{\gamma}_{x_0,0}(0) = e^{-2\beta x_0/\alpha},$$

that means that the probability that the process ends at the barrier is $e^{-2\beta x_0/\alpha}$ and the conditional pdf of the first passage time is

$$\gamma'_{x_0,0}(t) = \gamma_{x_0,0}(t) e^{2\beta x_0/\alpha} \qquad (14)$$

with its Laplace transform $\bar{\gamma}'_{x_0,0}(s) = \bar{\gamma}_{x_0,0}(s) e^{2\beta x_0/\alpha}$.

The same reasoning on normalisation to $\gamma'_{x_0,0}(t)$ refers to the case $\beta < 0$ with the initial point x_0 left to the absorbing barrier.

We may compute the moments of $\gamma_{x_0,0}$

$$E[\gamma_{x_0,0}] = \frac{x_0}{|\beta|}, \qquad E[\gamma^2_{x_0,0}] = \frac{|\beta| x_0^2 + \alpha x_0}{|\beta|^3}.$$

Note that the probability that the process with positive β started at $x_0 > 0$ never ends at zero is

$$1 - e^{-2\beta x_0/\alpha}, \qquad (15)$$

therefore probability that the fully charged battery ($x_0 = B$) will never become empty if harvesting intensity λ is greater then the cunsumption intensity μ ($\varrho = \lambda/\mu > 1$) is

$$1 - e^{-2\beta B/\alpha}. \qquad (16)$$

Fig. 1 illustrates Eq. (16) showing how the ratio $\varrho = \lambda/\mu$ as well as the coefficient of variation C_A^2 influence this probability.

If the initial condition of the first passage time is not given by a single point x_0, but by a function ψ, then Eq. (12) becomes

$$\gamma_{\psi,0}(t) = \int_0^B \frac{\xi}{\sqrt{2\Pi\alpha t^3}} e^{-\frac{(\xi-\beta t)^2}{2\alpha t}} \psi(\xi) d\xi. \qquad (17)$$

For example, the process starts at time $t = 0$ from $x_0 = B$ with diffusion parameters α and β, and, due to changas of the energy consumption, at time t_1 diffusion parameters take new values α_1 and β_1. The density of the diffusion prosess at the moment t_1 is given by Eq. (11) and presents the initial condition $\psi(x) = f(x, t_1; x_0)$ for the further evolution of the process. With probability $p(t_1) = \int_0^{t_1} \gamma_{x_0,0}(t)dt$ the barrier

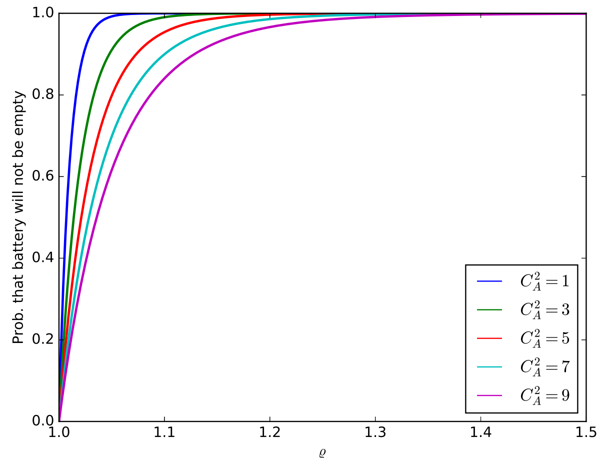

Fig. 1. Probability that the battery will never be completely depleted if the harvesting intensity λ is greater then the cunsumption intensity μ, presented as a function of $\varrho = \lambda/\mu > 1$), see Eq. (16); various C_A^2, $C_B^2 = 1$

was already reached before t_1 and with probability $1 - p(t_1)$ is continued, and the density of the first passage time is

$$\gamma(t) = \begin{cases} \gamma_{x_0,0}(t) & \text{for } t \le t_1 \\ \gamma_{\psi,0}(t - t_1) & \text{for } t > t_1. \end{cases} \qquad (18)$$

Not that

$$\int_0^B \psi(\xi) d\xi = 1 - \int_0^{t_1} f(x, t_1; B) dt.$$

We may extend this approach to any time interval in which the diffusion process has different but constant inside intervals parameters.

B. Probability that the process never reaches a specified point

Let us introduce the function $H(x_0, x_n)$ giving the probability that the diffusion process started at $x = x_0$ and ending at the origin will never reach $x = x_n$. The density of probability that the process is ended at t may be written as

$$\gamma_0(t) = g(t, x_n; x_0) + \int_0^t \gamma_{x_0,x_n}(\tau)\gamma_{x_n,0}(t-\tau)d\tau \qquad (19)$$

where

– $g(t, x_n; x_0)$ density of probability that the process will finish its motion at time t without reaching the point $x_n > x_0$

– $\gamma_{x_0,x_n}(\tau)$ density of probability that the process will reach x_n for the first time at $\tau < t$,

– $\gamma_{x_n,0}(t-\tau)$ density of probability that the process will pass from x_n to $x = 0$ during $t - \tau$.

We look for the distribution function $H(x_0, x_n)$ giving probability that the process will not reach $x_n > x_0$

$$H(x_0, x_n) = \int_0^\infty g(t, x_n; x_0) dt. \qquad (20)$$

Note that for a function $f(x)$, and its Laplace transform $\bar{f}(s)$ the following holds $\bar{f}(0) = \int_0^\infty f(x)dx$, and, if $f(x)$ is a probability density function defined for $x \geq 0$, then $\bar{f}(0) = 1$. Therefore, having in mind Eqs. (13), (19)

$$\begin{aligned} H(x_0, x_n) &= \int_0^\infty g(t, x_n; x_0)dt \\ &= \lim_{s\to 0}[\bar{\gamma}_0(s) - \bar{\gamma}_{x_0,x_n}(s)\bar{\gamma}_{x_n,0}(s)] \\ &= 1 - \lim_{s\to 0}\bar{\gamma}_{x_0,x_n}(s) \\ &= \frac{1-\exp[\frac{2\beta}{\alpha}(x_n - x_0)]}{1-\exp[\frac{2\beta}{\alpha}x_n]}. \end{aligned} \tag{21}$$

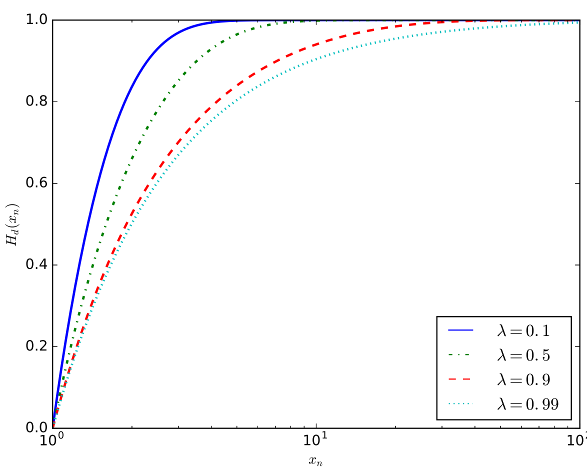

Fig. 2. Distribution function $H(1, x_n)$, representing probability that a diffusion process ($\mu = 1$, $C_A^2 = C_B^2 = 1$) started at $x = 1$ will never reach x_n, see Eq. (21)

Fig. 2 presents $H(1, x_n)$, probability that a diffusion process (various λ, $\mu = 1$, $C_A^2 = C_B^2 = 1$) started at $x = 1$ and ending at $x = 0$ will never attain x_n. Four curves demonstrate the impact of the energy flow intensity on the probability of reaching the point x_n. Of course, the higher intensity λ of arrivals, the higher the probability that the diffusion process will reach this point.

C. First passage times in case of a barrier at $x = B$

The densities in Eqs. (12),(17) refer to the first passage time in case there is no barrier at $x = B$. In the diffusion G/G/1/B model, representing the battery of volume B, the process at x_0, before reaching $x = 0$, may first come to the barrier at $x = B$, stay there, then jump to $x = B - 1$, then go to 0 or again come back to the barrier at B, etc. The number of visits at the barrier at B is not limited. We have to take this into account. The pdf $h_{i,0}(t)$ of the duration of the G/G/1/B busy period which starts at $x = i$ and ends at $x = 0$ having $0, 1, 2, \dots$ visits at $x = B$ and represents the time after which the energy of i packets stored in the battery is completely depleted, is given by (version for $\beta < 0$)

$$\begin{aligned} h_{i,0}(t) &= H(i,B)\gamma_{i,0}(t) + [1 - H(i,B)] \\ & [1 - H(i,B)]\{H(B-1,B)\gamma'_{i,B}(t) * l_B(t) * \gamma_{B-1,0}(t) \\ & +[1 - H(B-1,B)]H(B-1,B) \\ & \quad \gamma'_{i,B}(t) * l_B(t)^{2*}\gamma'_{B-1,B}(t) * \gamma'_{B-1,0}(t) \\ & +[1 - H(B-1,B)]^2 H(B-1,B) \\ & \quad \gamma'_{i,B}(t) * l_B(t)^{3*}\gamma'_{B-1,B}(t)^{2*} * \gamma_{B-1,0}(t) + \dots\} \end{aligned} \tag{22}$$

The right side of the equation summarises all possible trajectories of the process starting at $x = i$: with the probability $H(i, B)$ it is the direct passage from i to 0, with probability $[1 - H(i, B)]H(B - 1, B)$ it is the passage from i to B, stay at B, jump from B to $B-1$ and then passage from $B-1$ to 0; with probability $[1 - H(i, B)][1 - H(B - 1, B)]H(B - 1, B)$ there are two stays at the barrier at B, etc. The symbol ^{n*} denotes n-fold convolution. Naturally, $H_{i,0}(t) = \int_0^t h_{i,0}(\tau)d\tau$ gives us probability that the depletion happens until time t.

The Laplace transform of $h_{i,0}(t)$ is

$$\begin{aligned} \bar{h}_{i,0}(s) &= H(i,B)\bar{\gamma}_{i,0}(s) + [1 - H(i,B)]\bar{\gamma}'_{i,B}(s) \\ & \frac{\bar{\gamma}_{B-1,0}(s)H(B-1,B)\bar{l}_B(s)}{1 - [1 - H(B-1,B)]\bar{\gamma}'_{B-1,B}(s)\bar{l}_B(s)}. \end{aligned} \tag{23}$$

If the impact of the barrier is weak, i.e. if $H(i, B) \approx 1$, then $h_{i,0}(t) \approx \gamma'_{i,0}(t)$.

If $i = B$, Eq. (23) refers to the case of time to deplete the fully charged battery, if $i = 1$, $h_{1,0}(t)$ refers of the battery activity time between successive moments of its complete discharge.

The density $h_{i,0}(t)$ is known for the Markovian case of M/M/1/B station where interarrival and service times are exponentially distributed, see e.g. [22]. However, its form in time-domain is fairly complex, so we cite only the Laplace transform of its density

$$\bar{h}^M_{i,0}(s) = \varrho^{-i}\frac{[\eta(s)]^{B-i}[\eta(s) - 1] + [\xi(s)]^{B-i}[\xi(s) - 1]}{[\eta(s)]^B[\eta(s) - 1] + [\xi(s)]^B[\xi(s) - 1]} \tag{24}$$

where

$$\xi(s) = \frac{s + \lambda + \mu - \sqrt{(s + \lambda + \mu)^2 - 4\lambda\mu}}{2\lambda},$$

and

$$\eta(s) = \frac{s + \lambda + \mu + \sqrt{(s + \lambda + \mu)^2 - 4\lambda\mu}}{2\lambda}.$$

In numerical examples, we use the numerical inversion of $\bar{h}_{i,0}(s)$ and $\bar{h}^M_{i,0}(s)$.

A few examples illustrate the character of these functions. In Fig. 3 we see the densities $h_{100,0}(t)$, Eq. (23), and $h^M_{100,0}(t)$, Eq. (24), given by diffusion (dotted line) and Markov (solid line) models, presenting the density of the discharging time for a battery of the volume $B = 100$ energy units. Two cases $\varrho = 0.6$ and $\varrho = 0.8$ are considered. Evidently, for higher system utilisation (intensity of arrivals), the time to deplete the battery is longer. We observe a perfect match of results given by both models. A slight distortion of the curve in the Markov model is due to the errors of the numerical inversion of the formula (24). The inversion is performed by a simple Stehfest algorithm [23].

Fig. 4 presents $h_{100,0}(t)$ for higher than in Fig. 3 utilisations ϱ, i.e. higher intensities of packets' arrival; longer depletion times are mow much more probable.

Fig. 5 presents the influence of the squared coefficient of variation of interarrival times C_A^2 on the $h_{100,0}(t)$. The $C_A^2 \neq 1$ cases are not available for the Markovian model. We see that the increase of C_A^2 stretches the curve of the pdf, i.e. increases the variance of the depletion time.

Fig. 6 refers to the case when the diffusion process is initially at $i = 10$ and displays the pdfs $h_{10,0}(t)$, $h_{10,0}^M(t)$ of the time remaing to deplete the battery. Several harvesting intensities are considered; as in all examples $\mu = 1$, the utilisation ϱ corresponds to the intensity λ. The time is, of course, shorter than for the fully charged battery. The total volume B of the battery stays unchanged. As we choose the case $C_A^2 = C_B^2 = 1$, the Markov model is available, and the results of both models are very close.

Fig. 7 presents the impact of the starting point $x = i$ on the pdf $h_{i,0}(t)$. The greater i, i.e. the longer distance between the starting point and the barrier at $x = 0$, the longer times to depletion.

Fig. 8 displays the pdfs $h_{1,0}(t)$, $h_{1,0}^M(t)$, that means the densities of the busy period which begins with the arrival of a single energy packet to the G/G/1/100 or M/M/1/100 station and ends when the number of packets drops to 0. It corresponds to the time between two consecutive moments when the battery is depleted. We choose $C_A^2 = C_B^2 = 1$ to be able to compare the diffusion and Markov results: they are practically the same. As the utilisation factor is $\varrho = 0.6$, the probability of long busy periods is weak.

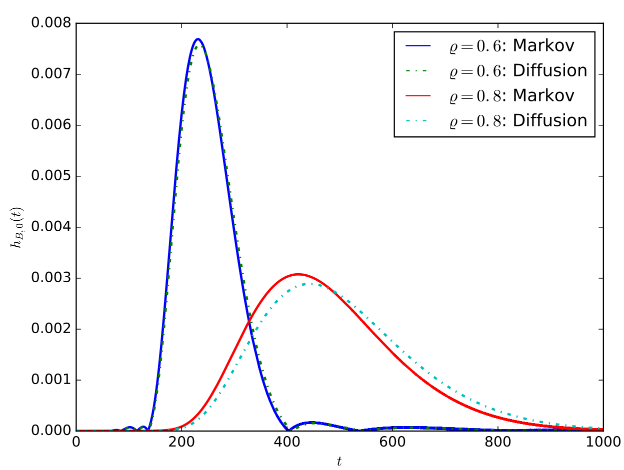

Fig. 3. Comparison of the Markovian and diffusion models of probability densities, $h_{B,0}(t)$, $h_{B,0}^M(t)$, of the time to complete discharge an iniatially fully charged battery, see Eqs. (23), (24). Two cases are considered: $\varrho = 0.6$ and $\varrho = 0.8$; $\mu = 1$, $C_A^2 = C_B^2 = 1$, $B = 100$.

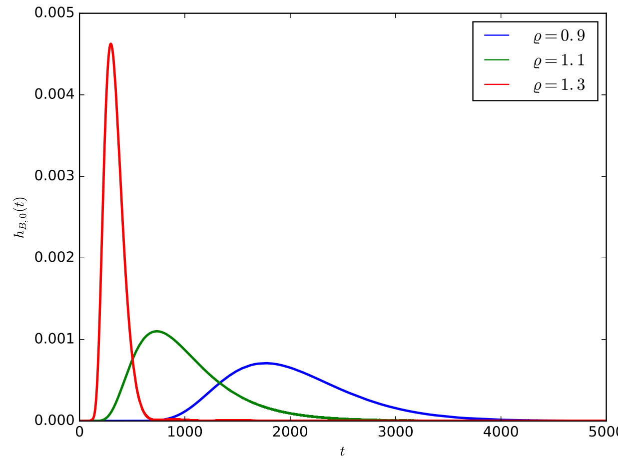

Fig. 4. The influence of the utilisation ϱ on the probability density, $h_{B,0}(t)$, $\mu = 1$, $C_A^2 = C_B^2 = 1$, $B = 100$.

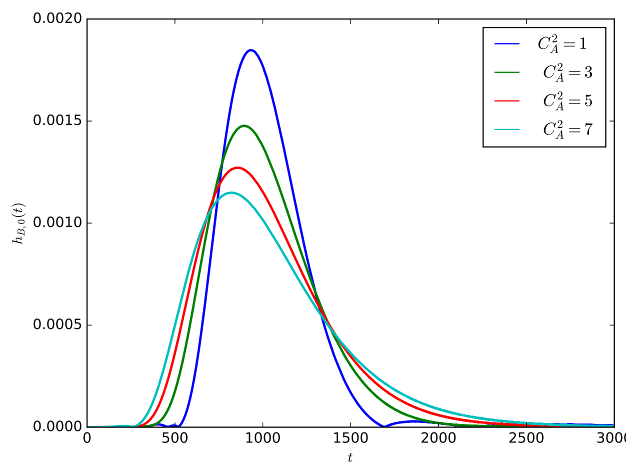

Fig. 5. The influence of C_A^2 on the probability density, $h_{B,0}(t)$, see Eq. (23) $\varrho = 0.8$ $\mu = 1$, $C_B^2 = 1$, $B = 100$.

In the similar way as in case of $h_{i,0}(t)$, we may determine the density $h_{i,B}(t)$ of the first passage time from $x = i$ to $x = B$ with $0, 1, 2, \ldots$ visits at the barrier at $x = 0$. It refers to the time after which the battery having i units of energy may be fully charged again (version for $\beta < 0$)

$$\begin{aligned} h_{i,B}(t) &= H(i,0)\gamma'_{i,B}(t) + [1 - H(i,B)] \\ & [1 - H(i,0)]\{H(1,0)\gamma_{i,0}(t) * l_0(t) * \gamma'_{1,B}(t) \\ & +[1 - H(1,0)]H(1,0) \\ & \quad \gamma_{i,0}(t) * l_0(t)^{2*}\gamma_{1,0}(t) * \gamma'_{1,B}(t) \\ & +[1 - H(B-1,B)]^2 H(B-1,B) \\ & \quad \gamma_{i,0}(t) * l_0(t)^{3*}\gamma_{1,0}(t)^{2*} * \gamma'_{1,B}(t) + \ldots\} \end{aligned} \tag{25}$$

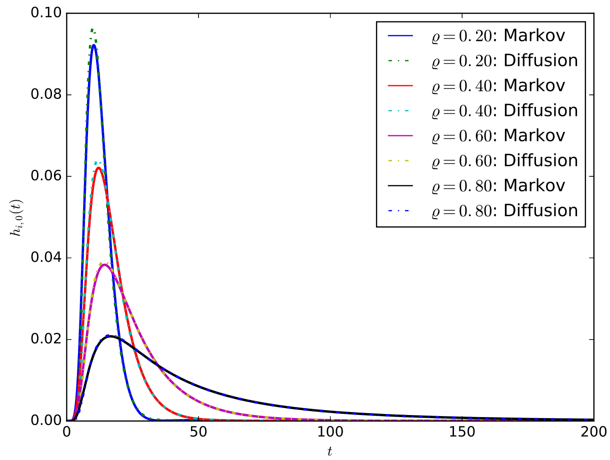

Fig. 6. Comparison of the Markovian and diffusion models of the densities $h_{10,0}(t)$. $h^M_{10,0}(t)$ of the first passage time from $x = 10$ to $x = 0$, see Eqs. (23), (24), for $i = 10$ and various ϱ; $\mu = 1$, $C_A^2 = C_B^2 = 1$, $B = 100$.

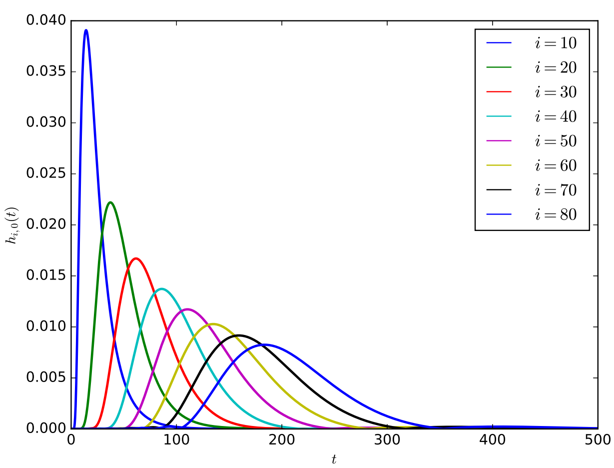

Fig. 7. The impact of the starting point $x = i$ on the first passage time pdf $h_{i,0}(t)$, see Eq. (23), for $\lambda = 0.6$, $\mu = 1$, $C_A^2 = C_B^2 = 1$, $B = 100$.

and

$$\bar{h}_{i,B}(s) = H(i,0)\bar{\gamma}'_{i,B}(s) + [1 - H(i,0)]\bar{\gamma}_{i,0}(s) \frac{\bar{\gamma}'_{1,B}(s)H(1,0)\bar{l}_0(s)}{1 - [1 - H(1,0)]\bar{\gamma}_{1,0}(s)\bar{l}_0(s)} \quad (26)$$

In the case of the Markov M/M/1/B model, the corresponding density is (we cite the result with a minor correction to the original [22] p.223: ϱ is replaced by ϱ^{B-i})

$$\bar{h}^M_{i,B}(s) = \varrho^{-(B-i)} \frac{\{[\eta(s)]^{i+1} - [\xi(s)]^{i+1}\} - \{[\eta(s)]^i - [\xi(s)]^i\}}{\{[\eta(s)]^{B+1} - [\xi(s)]^{B+1}\} - \{[\eta(s)]^B - [\xi(s)]^B\}}$$

Figs. 9, 10 illustrate these results. In the first one we see the pdfs $h_{i,B}(t)$, $h^M_{i,B}(t)$, i.e. the densities of the conditional distribution of time needed for a battery having $i = 10$ energy

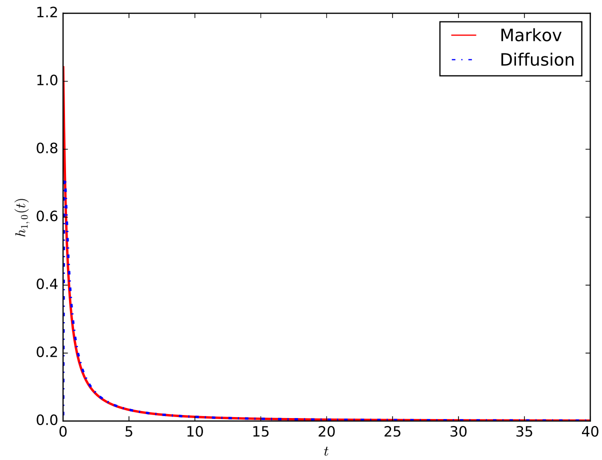

Fig. 8. Comparison of the Markovian and diffusion models of the probability densities $h_{1,0}(t)$, $h^M_{1,0}(t)$, of the time of battery activity between two inactive periods, see Eqs. (23), (24), $\varrho = 0.6$ $\mu = 1$, $C_A^2 = C_B^2 = 1$, $B = 100$.

packets to complete the energy to its full capacity B (100 packets). The curves given by both models are practically the same, even the distortions introduced by numerical inversion coincide. The second figure visualises the impact of the harvesting intensity on $h_{10,100}(t)$: the higher the harvesting intensity, the shorter times to charge the battery.

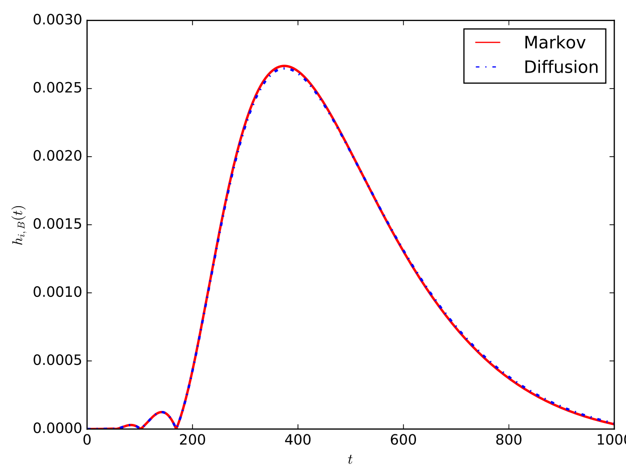

Fig. 9. Comparison of the Markovian and diffusion models of the conditional probability desity, $h_{i,B}(t)$, for $i = 10$, $\varrho = 0.80$, $\mu = 1$, $C_A^2 = C_B^2 = 1$, $B = 100$.

The moments of the distributions $h_{i,0}(x)$, $h_{i,B}(x)$ may be obtained from their Laplace transforms: for any pdf $f_X(x)$ and its Laplace transform $\bar{f}_X(s)$ holds

$$\left.\frac{d^n \bar{f}_X(s)}{ds^n}\right|_{s=0} = -\frac{d^n}{ds^n}\int_0^\infty f_X(x)e^{-sx}dx = = \int_0^\infty f_X(x)(-1)^n x^n e^{-sx}dx = (-1)^n E[X^n]\,. \quad (27)$$

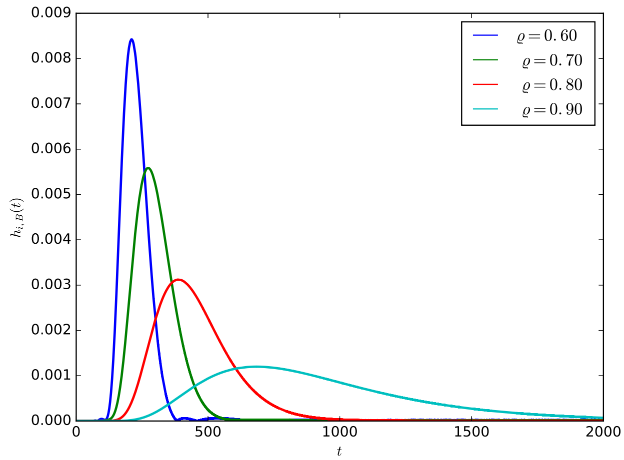

Fig. 10. The influence of ϱ on the conditional probability density, $h_{i,B}(t)$, for $i = 10$, $\mu = 1$, $C_A^2 = C_B^2 = 1$, $B = 100$.

If the initial condition is not given by a single point x_0 but the pdf $\psi(\xi)$, then the densitiets $h_{\psi,0}(t)$, $h_{\psi,B}(t)$ of the first passage time to the barriers at 0 and B are

$$h_{\psi,0}(t) = \int_0^B \psi(x) h_{x,0}(t) dx, \tag{28}$$

$$h_{\psi,B}(t) = \int_0^B \psi(x) h_{x,B}(t) dx.$$

The function ψ may also depend on time, e.g. if at time τ the distribution of energy $f(x,\tau;B)$ is given by the solution of Eq. (4), giving current and time dependent distribution of the diffusion process, then the prognostic for the further life-time distribution is given by Eq. (28)

$$h_{\psi,0}(t,\tau) = \int_0^B f(x,\tau;B) h_{x,0}(t) dx, \tag{29}$$

and

$$h_{\psi,B}(t,\tau) = \int_0^B f(x,\tau;B) h_{x,B}(t) dx.$$

IV. Conclusion

The article presents a study of times needed to empty and recharge a battery feeding an IoT or other autonomous device while using energy-harvesting. It may help estimate its uninterrupted work, also giving the probability that the complete depletion of the battery never happens. The model uses the concept of energy packets coming, queued, and served at a service station. It is based on the known G/G/1/B diffusion approximation model, where we develop the first passage time formulas. If the interarrival and service times of energy packets are exponentially distributed, the diffusion approximation results match the results of existing Markovian models very well. The advantage of the diffusion approach is that the model allows any distribution, both in the input stream of energy packets and their service. It also includes transient cases when the energy delivery and consumption vary in time due to changes in work conditions, battery characteristics or energy depletion attacks.

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