

## Einstein addition as 'renormalization'

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The oldest transformation linking relatively moving frames is the Galileo matrix

$$G(v) = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}$$

which leads to infinities. Since composition of two such frames amounts to matrix multiplication,  $G(v)G(w)$ , the total velocity is the simple sum  $G(v,w) = v + w$ . But composition can be done (in principle) ad infinitum, so velocities here add indefinitely beyond bounds. It took some three hundred years to discover a 'renormalization' of this problem by Einstein; the matrix

$$E(v) = \gamma \begin{pmatrix} 1 & -v \\ -v/c^2 & 1 \end{pmatrix} \quad \gamma = \gamma(v) = (1 - v^2/c^2)^{-1/2}$$

does away with infinities because now the product  $E(v)E(w)$  leads to the famous addition-of-velocities rule

$$R(v,w) = (v + w)/(1 + vw/c^2)$$

which never exceeds the (enormous but finite) speed of light  $C$ , since  $V$  and  $W$  don't; a technical leap perhaps unimaginable centuries ago. Or was it?

What assumptions are needed just to derive Einstein addition itself? Of course, the units must be those of speed; most importantly the existence of an invariant upper bound, say  $C$ . These are non-obvious requirements to be sure.

Let  $F(v,w)$  be a candidate rule. We must have

$$F(c,w) = F(v,c) = F(c,c) = c$$

An easy way to bound the growth of a variable  $v$  is

$$v/(1 + v)$$

but to retain the units of speed, use instead

$$v/(1 + v/c)$$

Similarly, bound  $w$  as

$$w/(1 + w/c)$$

and let's try first

$$F(v,w) = v/(1 + v/c) + w/(1 + w/c)$$

Certainly

$$F(c,c) = c$$

though bounds for each variable separately do not hold:

$$F(c,w) \neq c \neq F(v,c)$$

To remedy this, introduce some symmetry by now letting

$$F(v,w) = (v + w)/(1 + v/c)(1 + w/c)$$

Then

$$F(c,w) = F(v,c) = F(c,c) = c/2$$

Not quite; the denominator in this  $F(v,w)$  is too big, as if something is being counted twice. Upon expansion

$$(1 + v/c)(1 + w/c)$$

is obviously

$$(1 + vw/c^2) + (1/c)(v + w)$$

and this makes the solution clear:

$$F(v,w) = (v + w)/\{(1 + v/c)(1 + w/c) - (1/c)(v + w)\} = R(v,w)$$

does it; perhaps not so unimaginable after all. This also shows that

$$1/R(v,w) = (1 + v/c)(1 + w/c)/(v + w) - 1/c$$

one curiosity among other re-arrangements.

Actually, the Galileo matrix  $G(v)$  can lead to the Einstein matrix  $E(v)$ :

$$E(v) = G(v)A(v)B(v)$$

where

$$A(v) = \begin{pmatrix} 1/\gamma & 0 \\ 0 & \gamma \end{pmatrix} \quad \text{and} \quad B(v) = \begin{pmatrix} 1 & 0 \\ -v/c^2 & 1 \end{pmatrix}$$

Clearly a 'bridge-too-far' this time.