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Research Article

# HIGHER ORDER QUASI CUBIC B-SPLINE SOLUTION OF ADVECTION EQUATION

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Abstract. In this study, the advection equation (AE) will be solved numerically by using the high order method based on cubic B-spline quasi-interpolation for space discretization and second and fourth order single step method for time discretization. The pure advection test problem is studied and the accuracy of the numerical results are measured by computing the maximum error norm and the rate of convergence for both of the proposed methods.

*Keywords*: Spline; Numerical methods for PDEs; Advection equation.

*AMS Subject Classification*: 65D07; 65N22.

## 1. Introduction

We consider the following one dimensional AE

$$u_t + \alpha u_x = 0, \ a \le x \le b \tag{1.1}$$

with the boundary conditions

$$u(a,t) = u(b,t) = 0, t \in [0,T] \tag{1.2}$$

and initial condition

$$u(x,0) = f(x), \ a \le x \le b \tag{1.3}$$

in a restricted solution domain over a space/time interval $[a,b] \times [0,T]$. In the one dimensional linear AE, $\alpha$ is the steady uniform fluid velocity and $u = u(x,t)$ is a function of two independent variables $t$ and $x$, which generally denote time and space, respectively.
The AE is the part of well known advection diffusion equation (ADE)

$$u_t + \alpha u_x - \mu u_{xx} = 0, \ a \le x \le b \tag{1.4}$$

and they are the basis of many physical and chemical phenomena. Various numerical techniques have been developed and compared for solving the one dimensional AE and ADE with constant coefficient so far including Taylor-Galerkin method [1], least-squares finite element method [2], the Galerkin finite element method [3], Weighted finite difference methods [4], high-order finite

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difference schemes [5] and differential quadrature methods based on B-spline functions of fourth and fifth degrees [6].
The main idea of using this method is to obtain high-order approximate solution for AE. This study presents high order numerical method for the AE based on the cubic B-spline quasi-interpolation for space discretization and second and fourth order single step method for time discretization. The structure of the study is as follows. In the next section, after the time discretization of the AE is performed by using higher accurate finite difference method, a cubic B-spline quasi-interpolation for space discretization is used to obtain a system of algebraic equation. In the numerical experiment section, proposed methods are tested for the test problem and finally, a summary of main findings of the work is presented in the last section.

## 2. Application of the Method

The space interval $[a, b]$ is divided into uniformly sized finite subelements of equal length $h$ at the knots

$$\{x_m = a + mh,\ m = 0, \ldots, N\}$$

where $h = (b - a)/N$ and $x_N = b$. For computational work, the space-time plane is discretized by grids with the time step $k$ and space step $h$. The exact solution of the unknown function at the grid points is denoted by

$$u(x_m, t_n) = u_m^n,\ m = 0, 1, \ldots, N; \qquad n = 0, 1, 2, \ldots$$

where $x_m = a + mh$, $t_n = nk$ and the notation $U_m^n$ is used to represent the numerical value of $u_m^n$.

### 2.1. Time Discretization.

Using the advection equation of the form

$$u_t = -\alpha u_x \tag{2.1}$$

and the following one-step method

$$u^{n+1} = u^n + \theta_1 u_t^{n+1} + \theta_2 u_t^n + \theta_3 u_{tt}^{n+1} + \theta_4 u_{tt}^n, \tag{2.2}$$

we have the time discretization of the Eq. (2.1). Using the (2.1) then we have the following equation:

$$u_{tt} = -\alpha\left(u_t\right)_x = -\alpha\left(-\alpha u_x\right)_x = \alpha^2 u_{xx}. \tag{2.3}$$

and then using the Eqs. (2.1-2.3) in the one step method (2.2), we obtain the time discretization form of the AE equation as

$$u^{n+1} + \alpha\theta_1\left(u_x\right)^{n+1} - \alpha^2\theta_3\left(u_{xx}\right)^{n+1} = u^n - \alpha\theta_2\left(u_x\right)^n + \alpha^2\theta_4\left(u_{xx}\right)^n. \tag{2.4}$$

### 2.2. Cubic B-spline Quasi Interpolants Method.

The cubic B-splines $B_m$, $m = -1, \ldots, N+1$, have the following form [7, 8]:

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} (z_{m-2})^3 & , x_{m-2} \le x < x_{m-1} \\ h^3 + 3h^2 z_{m-1} + 3h(z_{m-1})^2 - 3(z_{m-1})^3 & , x_{m-1} \le x < x_m \\ h^3 - 3h^2 z_{m+1} + 3h(z_{m+1})^2 + 3(z_{m+1})^3 & , x_m \le x < x_{m+1} \\ -\left(z_{m+2}\right)^3 & , x_{m+1} \le x < x_{m+2} \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

where $z_m = x - x_m$. The set of cubic B-splines $B_m(x)$, $m = -1, \ldots, N+1$ forms a basis over the space interval $a \le x \le b$ [9]. Over the space interval $a \le x \le b$, the approximate solution $U(x, t)$ to the exact solution $u(x, t)$ can be written as a combination of the cubic B-splines

$$U(x, t) = \sum_{j=-1}^{N+1} \delta_j B_j \tag{2.6}$$

where $\delta_j$ are time dependent unknown parameters. Then the cubic B-spline quasi interpolants (QIs) can be defined as

$$Q_3 f = \sum_{j=1}^{N+3} \mu_j(f) B_j \tag{2.7}$$

where the coefficients are

$$\begin{aligned} \mu_1(f) &= f_0 \\ \mu_2(f) &= \frac{1}{18}\left(7f_0 + 18f_1 - 9f_2 + 2f_3\right) \\ \mu_j(f) &= \frac{1}{6}\left(-f_{j-3} + 8f_{j-2} - f_{j-1}\right), \ j = 3, \ldots, N+1 \\ \mu_{N+2}(f) &= \frac{1}{18}\left(2f_{N-3} + 18f_1 - 9f_2 + 2f_3\right) \\ \mu_{N+3}(f) &= f_N. \end{aligned} \tag{2.8}$$

The main advantage of QIs is that they have a direct construction without solving any system of linear equations. Using the approximation

$$u \simeq U = \sum_{j=1}^{N+3} \mu_j(u) B_j, \quad u' \simeq U' = \sum_{j=1}^{N+3} \mu_j(u) B'_j, \quad u'' \simeq U'' = \sum_{j=1}^{N+3} \mu_j(u) B''_j, \tag{2.9}$$

we have the approximation for first and second derivatives of unknown function $u$ as follows ([10]):

$$\begin{aligned} U'(x_0) &= \frac{1}{h}\left(-\frac{11}{6}U_0 + 3U_1 - \frac{3}{2}U_2 + \frac{1}{3}U_3\right), \\ U'(x_1) &= \frac{1}{h}\left(-\frac{1}{3}U_0 - \frac{1}{2}U_1 + U_2 - \frac{1}{6}U_3\right), \\ U'(x_j) &= \frac{1}{h}\left(\frac{1}{12}U_{j-2} - \frac{2}{3}U_{j-1} + \frac{2}{3}U_{j+1} - \frac{1}{12}U_{j+2}\right), \\ U'(x_{N-1}) &= \frac{1}{h}\left(\frac{1}{6}U_{N-3} - U_{N-2} + \frac{1}{2}U_{N-1} + \frac{1}{3}U_N\right), \\ U'(x_N) &= \frac{1}{h}\left(-\frac{1}{3}U_{N-3} + \frac{3}{2}U_{N-2} - 3U_{N-1} + \frac{11}{6}U_N\right) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} U''(x_0) &= \frac{1}{h^2}(2U_0 - 5U_1 + 4U_2 - U_3), \\ U''(x_1) &= \frac{1}{h^2}(U_0 - 2U_1 + U_2), \\ U''(x_j) &= \frac{1}{h^2}\left(-\frac{1}{6}U_{j-2} + \frac{5}{3}U_{j-1} - 3U_j + \frac{5}{3}U_{j+1} - \frac{1}{6}U_{j+2}\right), \\ U''(x_{N-1}) &= \frac{1}{h^2}(U_{N-2} - 2U_{N-1} + U_N), \\ U''(x_N) &= \frac{1}{h^2}(-U_{N-3} + 4U_{N-2} - 5U_{N-1} + 2U_N). \end{aligned} \tag{2.11}$$

Using quasi spline approximations for the first and the second derivatives (2.10-2.11) in the equation (2.4) yields

$$\begin{aligned}
&U_0^{n+1}\left[1+\alpha\theta_1\left(\frac{-11}{6h}\right)-\alpha^2\theta_3\left(\frac{2}{h^2}\right)\right]+U_1^{n+1}\left[\alpha\theta_1\left(\frac{3}{h}\right)-\alpha^2\theta_3\left(\frac{-5}{h^2}\right)\right]+\\
&U_2^{n+1}\left[\alpha\theta_1\left(\frac{-3}{2h}\right)-\alpha^2\theta_3\left(\frac{4}{h^2}\right)\right]+U_3^{n+1}\left[\alpha\theta_1\left(\frac{1}{3h}\right)-\alpha^2\theta_3\left(\frac{-1}{h^2}\right)\right]=\\
&U_0^{n}\left[1-\alpha\theta_2\left(-\frac{11}{6h}\right)+\alpha^2\theta_4\left(\frac{2}{h^2}\right)\right]+U_1^{n}\left[-\alpha\theta_2\left(\frac{3}{h}\right)+\alpha^2\theta_4\left(\frac{-5}{h^2}\right)\right]+\\
&U_2^{n}\left[-\alpha\theta_2\left(\frac{-3}{2h}\right)+\alpha^2\theta_4\left(\frac{4}{h^2}\right)\right]+U_3^{n}\left[-\alpha\theta_2\left(\frac{1}{3h}\right)+\alpha^2\theta_4\left(-\frac{1}{h^2}\right)\right],
\end{aligned}\tag{2.12}$$

$$\begin{aligned}
&U_0^{n+1}\left[\alpha\theta_1\left(\frac{-1}{3h}\right)-\alpha^2\theta_3\left(\frac{1}{h^2}\right)\right]+U_1^{n+1}\left[1+\alpha\theta_1\left(\frac{-1}{2h}\right)-\alpha^2\theta_3\left(\frac{-2}{h^2}\right)\right]+\\
&U_2^{n+1}\left[\alpha\theta_1\left(\frac{1}{h}\right)-\alpha^2\theta_3\left(\frac{1}{h^2}\right)\right]+U_3^{n+1}\left[\alpha\theta_1\left(\frac{-1}{6h}\right)\right]=\\
&U_0^{n}\left[-\alpha\theta_2\left(\frac{-1}{3h}\right)+\alpha^2\theta_4\left(\frac{1}{h^2}\right)\right]+U_1^{n}\left[1-\alpha\theta_2\left(\frac{-1}{2h}\right)+\alpha^2\theta_4\left(\frac{-2}{h^2}\right)\right]+\\
&U_2^{n}\left[-\alpha\theta_2\left(\frac{1}{h}\right)+\alpha^2\theta_4\left(\frac{1}{h^2}\right)\right]+U_3^{n}\left[-\alpha\theta_2\left(\frac{-1}{6h}\right)\right],
\end{aligned}\tag{2.13}$$

$$\begin{aligned}
&U_{j-2}^{n+1}\left[\alpha\theta_1\left(\frac{1}{12h}\right)-\alpha^2\theta_3\left(\frac{-1}{6h^2}\right)\right]+U_{j-1}^{n+1}\left[1+\alpha\theta_1\left(\frac{-2}{3h}\right)-\alpha^2\theta_3\left(\frac{5}{3h^2}\right)\right]+\\
&U_{j}^{n+1}\left[1-\alpha^2\theta_3\left(\frac{-3}{h^2}\right)\right]+\\
&U_{j+1}^{n+1}\left[\alpha\theta_1\left(\frac{2}{3h}\right)-\alpha^2\theta_3\left(\frac{5}{3h^2}\right)\right]+U_{j+2}^{n+1}\left[\alpha\theta_1\left(\frac{-1}{12h}\right)-\alpha^2\theta_3\left(\frac{-1}{6h^2}\right)\right]\\
=\;&\\
&U_{j-2}^{n}\left[-\alpha\theta_2\left(\frac{1}{12h}\right)+\alpha^2\theta_4\left(\frac{-1}{6h^2}\right)\right]+U_{j-1}^{n}\left[1-\alpha\theta_2\left(\frac{-2}{3h}\right)+\alpha^2\theta_4\left(\frac{5}{3h^2}\right)\right]+\\
&U_{j}^{n}\left[1+\alpha^2\theta_4\left(\frac{-3}{h^2}\right)\right]\\
&U_{j+1}^{n}\left[-\alpha\theta_2\left(\frac{2}{3h}\right)+\alpha^2\theta_4\left(\frac{5}{3h^2}\right)\right]+U_{j+2}^{n}\left[-\alpha\theta_2\left(\frac{-1}{12h}\right)+\alpha^2\theta_4\left(\frac{-1}{6h^2}\right)\right],
\end{aligned}\tag{2.14}$$

$$\begin{aligned}
&U_{N-3}^{n+1}\left[\alpha\theta_1\left(\frac{6}{h}\right)\right]+U_{N-2}^{n+1}\left[\alpha\theta_1\left(\frac{-1}{h}\right)-\alpha^2\theta_3\left(\frac{1}{h^2}\right)\right]+\\
&U_{N-1}^{n+1}\left[1+\alpha\theta_1\left(\frac{-1}{2h}\right)-\alpha^2\theta_3\left(\frac{-3}{h^2}\right)\right]+U_{N}^{n+1}\left[\alpha\theta_1\left(\frac{1}{3h}\right)-\alpha^2\theta_3\left(\frac{1}{h^2}\right)\right]=\\
&U_{N-3}^{n}\left[-\alpha\theta_2\left(\frac{6}{h}\right)\right]+U_{N-2}^{n}\left[-\alpha\theta_2\left(\frac{-1}{h}\right)+\alpha^2\theta_4\left(\frac{1}{h^2}\right)\right]+\\
&U_{N-1}^{n}\left[1-\alpha\theta_2\left(\frac{-1}{2h}\right)+\alpha^2\theta_4\left(\frac{-3}{h^2}\right)\right]+U_{N}^{n}\left[-\alpha\theta_2\left(\frac{1}{3h}\right)+\alpha^2\theta_4\left(\frac{1}{h^2}\right)\right],
\end{aligned}\tag{2.15}$$

$$\begin{aligned}
&U_{N-3}^{n+1}\left[\alpha\theta_1\left(\frac{-1}{3h}\right)-\alpha^2\theta_3\left(\frac{-1}{h^2}\right)\right]+U_{N-2}^{n+1}\left[\alpha\theta_1\left(\frac{3}{2h}\right)-\alpha^2\theta_3\left(\frac{4}{h^2}\right)\right]+\\
&U_{N-1}^{n+1}\left[\alpha\theta_1\left(\frac{-3}{h}\right)-\alpha^2\theta_3\left(\frac{-5}{h^2}\right)\right]+U_{N}^{n+1}\left[1+\alpha\theta_1\left(\frac{11}{6h}\right)-\alpha^2\theta_3\left(\frac{2}{h^2}\right)\right]=\\
&U_{N-3}^{n}\left[-\alpha\theta_2\left(\frac{-1}{3h}\right)+\alpha^2\theta_4\left(\frac{-1}{h^2}\right)\right]+U_{N-2}^{n}\left[-\alpha\theta_2\left(\frac{3}{2h}\right)+\alpha^2\theta_4\left(\frac{4}{h^2}\right)\right]+\\
&U_{N-1}^{n}\left[-\alpha\theta_2\left(\frac{-3}{h}\right)+\alpha^2\theta_4\left(\frac{-5}{h^2}\right)\right]+U_{N}^{n}\left[1-\alpha\theta_2\left(\frac{11}{6h}\right)+\alpha^2\theta_4\left(\frac{2}{h^2}\right)\right]
\end{aligned}\tag{2.16}$$

for $j = 2, \cdots, N-2$. The above system (2.12-2.16) contains $N+1$ unknowns and $N+1$ equations. Boundary conditions $u(a,t) = u(b,t) = 0$ can be applied by deleting the first and last equations in the system (2.12-2.16). After initial vector

$$\mathbf{U}^0 = (U_0^0, \ldots, U_N^0)$$

is found with the help of the initial condition,

$$\mathbf{U}^{n+1} = (U_0^{n+1}, \ldots, U_N^{n+1}), (n = 0, 1, \ldots)$$

unknown vectors can be found repeatedly by solving the system (2.12-2.16) using previous $\mathbf{U}^n$ unknown vector. In the the system (2.12-2.16), taking $\theta_1 = \theta_2 = k/2$, $\theta_3 = \theta_4 = 0$, yields a second order method in time known as the Crank-Nicolson method and then taking $\theta_1 = \theta_2 = k/2$, $\theta_3 = -k^2/12$, $\theta_4 = k^2/12$, yields fourth order method in time.

## 3. Pure Advection Test Problem

For the pure advection test problem, accuracy of the proposed two algorithms is worked out by measuring error norm $L_\infty$

$$L_\infty = \max_m |u_m - U_m|,$$

and the order of convergence in time is computed by the formula

$$\text{order}=\frac{\log\left|\dfrac{(L_\infty)_{k_i}}{(L_\infty)_{k_{i+1}}}\right|}{\log\left|\dfrac{k_i}{k_{i+1}}\right|},$$

where $(L_\infty)_{k_i}$ is the error norm $L_\infty$ for time step $k_i$.
In the test problem, AE has the exact solution and initial conditions

$$u(x,t) = 10\exp\left(-\frac{(x-\tilde{x}_0-\alpha t)^2}{2\rho^2}\right), \tag{3.1}$$

$$u(x,0) = 10\exp\left(-\frac{(x-\tilde{x}_0)^2}{2\rho^2}\right). \tag{3.2}$$

The numerical simulation will be performed by selecting the flow velocity $\alpha = 0.5m/s$ of the wave, initial peak location $\tilde{x}_0 = 2km$ and $\rho = 264$ by the terminating time $t = 10000s$. In this case, the wave initially located with its peak at $\tilde{x}_0 = 2km$ will move to the right in a long channel without change in shape or size by the time $t = 10000s$ with flow velocity $\alpha = 0.5m/s$. So the initial condition travels from the initial position to a distance of $5km$ and the peak value of the solution remains constant 10 for all time. The proposed algorithms are run until $t = 10000s$ and the figures of the initial solutions and waves at $t = 2000$, 4000, 6000, 8000, 10000 are drawn in Fig. 1 for the M2 with $h = k = 2$. It can be seen from the figure that wave propagates without any change in its shape.
The error norms $L_\infty$ and rate of convergence for the both proposed methods are listed in Table 1. According to the table, when time and space steps are reduced from 200 to 5, the error norms decrease for the both presented methods. It can also be seen that the rate of convergence is almost two for M1 and almost four for the M2. Therefore the proposed methods especially M2

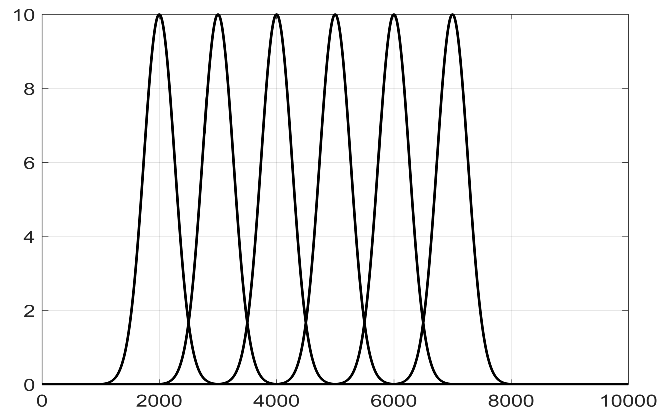


FIGURE 1. Waves at $t = 0, 2000, 4000, 6000, 8000, 10000$.

are quite satisfactory.

| | Method 1 | | Method 2 | |
|---|---|---|---|---|
| $h = k$ | $L_\infty$ | Order | $L_\infty$ | Order |
| 200 | 3.53 | 1.45 | 2.79 | 2.31 |
| 100 | 1.29 | 2.40 | $5.61\times10^{-1}$ | 3.70 |
| 50 | $2.45\times10^{-1}$ | 2.21 | $4.32\times10^{-2}$ | 3.97 |
| 20 | $3.25\times10^{-2}$ | 2.04 | $1.14\times10^{-3}$ | 4.00 |
| 10 | $7.89\times10^{-3}$ | 2.01 | $7.13\times10^{-5}$ | 4.00 |
| 5 | $1.96\times10^{-3}$ | 2.00 | $4.46\times10^{-6}$ | 4.00 |
| 2 | $3.13\times10^{-4}$ | | $1.14\times10^{-7}$ | |

Absolute error (difference between the exact and numerical solutions) distribution at $t = 10000s$ is also depicted in Figs. 2 and 3 for both of the presented methods. Since the maximum error occurs at about peak value of the wave at time $t = 10000s$, we can say that the effect of boundary conditions is negligible for the presented both methods.

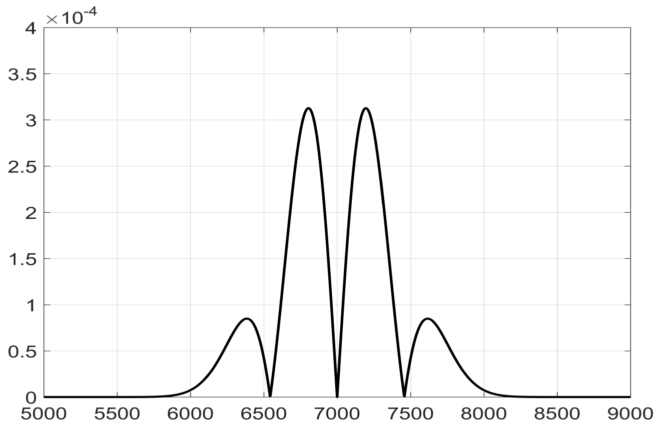


FIGURE 2. Absolute error for M1.

## 4. CONCLUSION

The high-order method based on Taylor series expansion for the time discretization and cubic B-spline quasi-interpolation for the space discretization was proposed to solve numerically the AE. The test problem was simulated well with the proposed algorithms. Consequently, the

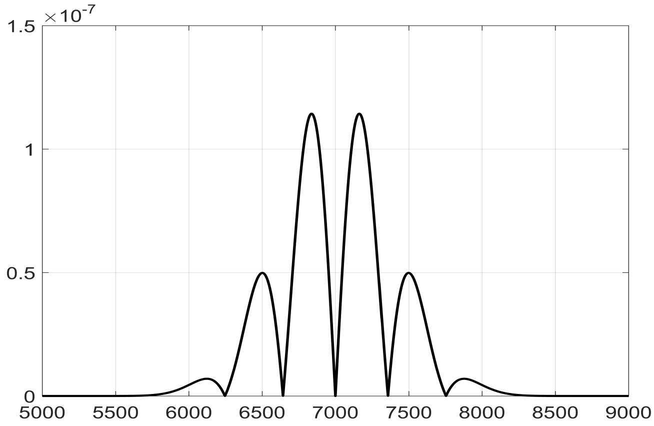


FIGURE 3. Absolute error for M2.

numerical results of this study demonstrate that the proposed fourth order single step method in time are a remarkably successful numerical technique for solving the AE.

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**Authors Contribution Statement.** All authors conceived of the presented idea, developed the study, performed the computations, verified the methods and investigated results and also supervised and edited all the findings of this work. All authors discussed the results and contributed to the final manuscript.

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