Phase and Group Velocity with a Free Particle Action

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The free particle action (relativistic or nonrelativistic) may be written as A = -Et+px. This same expression is also associated with a photon which has no rest mass mo and is linked to the classical wave result E=pv. dA=0 for a photon suggests one may have both a group velocity dE/dp as well as a phase velocity E=pv which are the same and both portions are 0 in dA, so dA=0 overall. Given that dE/dp and E=pv yield the same velocity, one may argue that either a wave or particle approach to a photon holds. This argument is given in (1). We show using dx/dt =x/t for a free particle that this must follow if dA=0.

For a particle with rest mass mo, a group velocity portion of dA is 0 if dE/dp is taken as dE/dv / dp/dv. The overall dA, however, is not 0 for constant mo. A phase velocity portion is completely at odds with p=Ev. Thus we suggest that the form exp(-iEt+ipx) used in quantum mechanics for a free particle be separated into exp(-Et) and exp(ipx) and not be combined to describe an overall wave.

Thus for the photon and particle with a mass, dA portions corresponding to velocities are 0. For the photon, both dA(group velocity)=0 and dA(phase velocity)=0 and vgroup=vphase, while for a particle with rest mass only dA (group velocity)=0 and dA (overall) is not zero for fixed mass. Thus the photon and particle with a free mass have a group velocity portion of dA being 0 in common. This suggests a similarity of the two at a “particle” level as group velocity is usually associated with particle motion. In (1) the particle properties of the photon are also stressed.

Free Particle Action

The free particle action A= -mo t sqrt(1-vv) (c=1) (relativistic) or A = t .5mo vv may be written with x/t=v and x and t varied independently. This leads to:

dA/dx partial = p and dA/dt partial = -E ((1a)) or A = -Et+px ((1b))

A question arises as to what happens if dA=0. We consider two scenarios. The first follows from wave theory for which E=pv (e.g. this follows from Maxwell’s electromagnetic waves with v=c). The second approach is that of classical mechanics for a particle with a rest mass. Nonrelativistically one has p=mov and relativistically p=Ev which is completely different from the wave result. Interestingly, we see that it is the group velocity portion of dA which is common to both a photon and free particle with mo.

Velocity of a Free Particle

We first consider the velocity of a free particle i.e.

dx/dt = x/t = v = constant ((2))

((2)) may be written as: $dx/x = dt/t$ or integrating: $x=C2\ t$. This is consistent with a free particle, thus we argue one must have $dx/dt = x/t$.

Case of E=pv

Consider a wave with v, the velocity being determined by the medium or theory i.e. for light in a vacuum on has v=c even though p and E may take on different values. In other words p and E do not depend on v. Then if one imposes dA=0:

$dA = -(dE)\ t - E\ dt + p\ dx + x\ (dp) = 0$ ((3))

Using the wave relationship: E=pv with v=x/t yields: $dA = -(dE)\ t + x\ dp$. From E=pv with v being independent of E and p, one has $dE = v\ dp$. This yields:

$dA = -\ vt\ dp + x\ dp$ but $v=x/t$ so $dA=0$

In other words one makes use of both the idea of group velocity $dE/dp = x/t = v$ and phase velocity $dx/dt= v = E/p$, but both are the same. Both equally contribute to dA=0. In other words one does not need to only think of a phase velocity linked to light. For example, it seems to be traditional to consider a classical wave cos(-Et+px) and write:

$-E\ dt + p\ dx =0$ and call this the velocity of the wave (phase velocity), ignoring changes to E and p

In (1) the group velocity of a photon is particularly emphasized.

Case of p=Ev

For a particle with rest mass p=mov (nonrelativistic) or p=Ev (relativistic). There are two parameters which may change i.e. mo and v and these both change p and E. Neither of these changed p or E in the wave case in the above section. Thus:

$dA = -\ (dE/dv\ dv)\ t - (dE/dmo\ dmo)\ t - E\ dt + p\ dx + x\ (dp/dv\ dv) + x\ (dp/dmo\ dmo)$ ((4))

If p and E are proportional to mo then p=Ev suggests: $dp/dmo = v\ dE/dmo$ and

$-\ (dE/dmo\ dmo)\ t + x\ (dp/dmo\ dmo) = E/mo\ dmo\ (-t + xv)$ ((5a))

This may be combined with $pdx -E\ dt$ the phase velocity piece i.e.

$Ev\ dx - Edt = E(vdx-dt) = E\ dt(-1+vv)$ ((5b)) Using $dx/dt = x/t$

((5a)) and ((5b) add to give: $E/mo(-1+vv)\ \{ mo\ dt + t\ dmo \}$

It is possible to make this zero (and hence dA=0) if mo t is a constant. "t" increases positively so mo would need to decrease, but this is a problem because it would reach 0 at some time. Alternatively, one may hold mo constant in which case dA is not 0. Thus for a particle with fixed rest mass dA is not 0 overall as it is in the case of the photon (wave) for which both group velocity and phase velocity represent the same velocity. In fact, for constant mo:

dA = -mo dt sqrt(1-vv) (c=1, v constant)

This, however, follows directly from A = -mo t sqrt(1-vv) the relativistic free particle action. Thus, the phase velocity portion of dA i.e. -Edt + pdx represents the entire dA so the group velocity portion: dA (group) = -t dE/dv dv + x dp/dv dv must be zero. Thus for both the photon and the particle with rest mass, velocity is associated with a dA (portion) =0. For the photon, both the group and phase velocity portions lead separately to dA (portion)=0, while for the particle with rest mass, only the group velocity portion dA (portion)=0. Thus the similarity between a photon and a particle with rest mass is the group velocity portion which is usually associated with a "particle".

We verify that: dA (group velocity portion) = -dE/dv t + x dp/dv = 0
This holds if one uses p=Ev and E=mo/sqrt(1-vv)

How does one deal with the idea that for a wave (mo=0) both phase and group velocity exist and both give the same v and dA is 0. For mo not 0, group velocity describes the motion, dA is not zero and phase velocity E=pv completely contradicts p=Ev. This we argue is the situation of quantum mechanics. For a free particle p=Ev and v=dE/dp, but one writes exp(-iEt+ipx). A phase velocity is often "mentionned", but it is completely at odds with p=Ev and v=dE/dp. A solution we suggest is to keep t and x independent in the wave quantum portion i.e. have exp(-iEt) and exp(ipx) separate with no link between x and t. For example, for a bound state with potential V(x) one writes W(x)exp(-iEn t) with W(x)=Sum over p a(p)exp(ipx). One does not include exp(-i pp/2m t) for each plane wave. For two slit interference or single slit diffraction one does not include time when calculating a spatial distribution which does not change in time.

Case of a Photon

As noted above, for a photon or classical wave, cos(-Et+px) is immediately associated with a phase velocity: -E dt + p dx=0 or E=pv. There is no reason, however, not consider the full dA expression which includes both a phase velocity part and group velocity one. Both yield 0 terms so that dA=0 = 0 (from the phase velocity portion) + 0 (from the group velocity portion). Thus for a photon (which has mo=0), there is no reason to favour phase velocity over group velocity (which is an argument presented in (1)). In other words, it is argued that light may behave as either a wave or a particle, but one sees that the phase velocity E/p (normally associated with a wave) is the same as the group velocity dE/dp which is normally associated with a particle. Thus the wave and particle seem to be on the same footing in terms of translational motion.

This is not the case for mo>0 as argued above. Only the group velocity portion of dA yields 0. The phase velocity portion does not because it is completely at odds with p=Ev. Thus we suggest that one does not need to consider exp(-iEt+ipx) as an x-t wave, but rather keep x and t independent. There is a group velocity v=dE/dp and associated with it is a momentum wave exp(ipx) which governs impulse hits. There is no reason, we argue, to associate time with exp(ipx). Time may be treated separately by exp(-i E t) and indeed it appears, but not by necessarily being associated with exp(ipx). For example, for a bound state one has an overall exp(-iEn t), but does not consider time in the plane waves which constitute W(x). For the Bohm Aharonov effect, one may consider a shift in energy i.e. in the exp(-iE t) piece OR a shift in space which does not change momentum i.e. a phase in exp(ipx).

Conclusion

In conclusion, we note that both a photon and a free particle with rest mass mo are associated with -Et+px. For a particle with rest mass this is the classical action (both in the relativistic and nonrelativistic cases). For a photon, one usually thinks in terms of a phase velocity i.e. -E dt + p dx=0, but for dA =0 = (-t dE + x dp) + (-E dt + p dx), both the group and phase portions of dA are separately 0. Furthermore both represent the same velocity. Thus it seems there is no reason to prefer the phase velocity (associated with a wave) over the group velocity (usually associated with a particle. This point is made in (1).

For a particle with rest mass, the group velocity portion of dA i.e. (-t dE/dv dv + x dp/dv dv) for mo constant is 0, but the phase part is not because p=Ev is completely different from the phase velocity result. Thus for constant mo, dA is not 0 overall, only the group velocity portion.

For a photon and a particle with rest mass, dA (portion)=0 for all velocity portions. The photon and free particle with mo both have a group velocity portion, usually associated with a particle, in common. Thus in terms of velocity motion, the photon is both a particle and a wave, but the free particle is a particle i.e. only has a group velocity. In quantum mechanics, however, there is the wavefunction exp(-iEt+ipx). We suggest separating exp(-iEt) from exp(ipx) as the phase velocity is not linked with translational motion in space it seems. Rather there is a group velocity which describes the free quantum particle's motion and an exp(ipx), independent of time, but linked to momentum and hence impulse reactions, which keeps track of the x space covered (regardless of time). We argue that it is this exp(ipx) portion which is linked to reactions. It is a probability and so one adds OR situations without following the particle in x,t as it interacts. Thus the question: Which slit does a photon pass through? does not arise.

References

1. Buenker, R.J. and Muino, P. Quantum Mechanical Relations for the Energy, Momentum and Velocity of Single Photons in Dispersive Media (2000 or later) https://arxiv.org/pdf/physics/0607094.pdf