HBRP PUBLICATION

Journal of Applied Mathematics and Statistical Analysis
Volume 3 Issue 2

A New Technique for Finding Real Roots of Non-Linear Equations Using Arithmetic Mean Formula

Md. Zahangir Alam*
Department of Mathematics, Rajshahi University of Engineering and Technology, (RUET), Rajshahi-6204, Bangladesh

***Corresponding Author**
E-mail Id:-*zahangiramth@gmail.com*

ABSTRACT
The paper describes a new technique for finding real roots of both algebraic and transcendental non-linear equations using arithmetic mean formula. The new technique produces an iterative formula combining arithmetic mean formula and Newton-Rapson method. We have presented some numerical examples to compare the proficiency of proposed method together with results of known methods. The proposed method gives faster convergence and more accurate results than existing methods.

Keywords: *-Arithmetic mean, Taylor's expansion, Iteration*

INTRODUCTION

In science and engineering a very common mater is the occurrence of non-linear equations. Most of the non-linear equations have no exact root. It is very difficult to find analytical solutions of such non-linear equations. Various author used numerical methods to find approximate solution based on iterative techniques.

The most common root finding methods are Iteration method, Bisection method, Newton-Raphson method [11], Regula-false position method, Secant method, Mullers method etc. Newton-Raphson method for computing a root of a nonlinear equation $f(x) = 0$ has been favored for its simplicity and fast rate of convergence. Abbasbandy S., Liao S. [9] introduces a modification of Regula-false position method based on homotopy analysis method. Tanakan, S. [12] imposed a new algorithm of Modified Bisection Method for Nonlinear Equation. Srinivasarao Thota used several techniques [1, 6-8, 10] for finding approximate root. Recently Srinivasarao Thota [1] has presented a new algorithm using exponential series.

In this paper we have presented a new technique combining arithmetic mean formula and Newton-Raphson method for computing real roots of non-linear equation. The results obtain by this technique provides fast and better approximation than the existing methods.

THE METHOD

Let $f(x) = 0$ be a non-linear equation and we define a new function $\varphi(x)$ using Newton-Raphson method [11] of the form

$$\varphi(x) = x - \frac{f(x)}{\dot{f}(x)} \tag{1.1}$$

We define another function $\psi(x)$ of the form

$$\psi(x) = \varphi(\varphi(x)) \tag{1.2}$$

Expanding the functions $\varphi(x)$ and $\psi(x)$ according to Taylor's expansion we obtain

$$\varphi(x_n + h) = \varphi(x_{n)} + h\dot{\varphi}(x_n) + \frac{h^2}{2}\ddot{\varphi}(x_n) + \cdots = 0 \tag{1.3}$$

$$\psi(x_n - h) = \psi(x_{n)} - h\dot{\psi}(x_n) + \frac{h^2}{2}\ddot{\psi}(x_n) + \cdots = 0 \tag{1.4}$$

Then the new iterative formula for initial approximation x_0 is proposed as follows

$$x_{n+1} = \frac{\psi(x_n)w_1 + \varphi(x_n)w_2}{w_1 + w_2}, \tag{1.5}$$

where, the weights $w_1 = \dot{\varphi}(x_n)$ and $w_2 = -\dot{\psi}(x_n)$ are coefficient of h of equations (1.3) and (1.4) respectively.

Then after substituting the values of w_1 and w_2 in equation (1.5) we obtain

$$x_{n+1} = \frac{\psi(x_n)\dot{\varphi}(x_n) - \varphi(x_n)\dot{\psi}(x_n)}{\dot{\varphi}(x_n) - \dot{\psi}(x_n)} \tag{1.6}$$

The function $\varphi(x_n)$ from Newton-Raphson method gives 2nd order of convergence on the other hand the iteration formula (1.6) is obtained by eliminating h from equation (1.3) and (1.4) gives 2nd order of convergence with respect to $\varphi(x_n)$. As a result the new iteration formula shows faster convergence.

NUMERICAL EXAMPLES

Example 2.1. Consider an equation

$$2x^3 + 11x^2 + 12x - 9 = 0 \tag{2.1}$$

Consider the function $\varphi(x)$ according to equation (1.1) of the form

$$\varphi(x) = \frac{3 - x + 4x^2}{4 + 6x} \tag{2.2}$$

$$\begin{aligned}\psi(x) &= \varphi(\varphi(x)) \\ &= \frac{3 - \varphi(x) + 4\varphi(x)^2}{4 + 6\varphi(x)} \\ &= \frac{36 + 53x + 99x^2 - 28x^3 + 32x^4}{68 + 138x + 102x^2 + 72x^3}\end{aligned} \tag{2.3}$$

Substituting the values of $\varphi(x)$, $\psi(x)$, $\dot{\varphi}(x)$ and $\dot{\psi}(x)$ in equation (1.6) and after simplification we get the iteration formula

$$x_{n+1} = \frac{213 + 606x_n + 121{x_n}^2 + 520{x_n}^3 - 176{x_n}^4}{454 + 988x_n + 914{x_n}^2 + 144{x_n}^3 + 96{x_n}^4} \tag{2.4}$$

The equation (2.1) has two real roots -3 and 0.5. Consider initial approximation $x_0 = -2$ then the iteration formula (2.4) generates the successive approximate roots are $x_1 = -2.97498$, $x_2 = -2.99998$, $x_3 = -2.99999999998$.

Again if we chose initial approximation $x_0 = 1$ then the successive approximate roots are $x_1 = 0.494607$, $x_2 = 0.4999999998$.

We see that for initial approximation $x_0 = -2$ only the 3rd iteration gives approximate root correct to 10 decimal places and for initial approximation $x_0 = 1$ only the 2nd iteration gives approximate root correct to 9 decimal places. It is clearly observed that the present method gives faster approximation.

The following Table 1 shows the comparison of approximate roots between existing methods with initial approximations 0 and 1 and present method with initial approximation $x_0 = 1$.

Table 1:*-Comparing approximate roots between existing methods and present method*

Itera. No.	Secant method	Itera. No.	Srinivasarao[1] method	Itera. No.	Present method
1	0.36	1	0.521109333	1	0.494607087
2	0.46558358	2	0.505161774	2	0.499999999
3	0.50297500	3	0.500087229		
4	0.49994069	4	0.500000026		
5	0.49999989	5	0.500000000		
⋮	⋮				
6	0.50000000				

It is clearly observed that the present method solution gives approximate root correct to 9 decimal places by only 2 iterations while existing methods given by 5 or more iterations. It shows that the present method results show a better convergence than existing methods.

Again consider initial approximations-5 and -1 for existing methods and $x_0 = -5$ for present method. Then the following Table 2 shows the comparison of number of iterations between existing methods and present method.

Table 2:*-Comparing number of iteration between existing methods and present method*

Methods	Number of iterations	Exact root
Secant method	8	
Srinivasarao[1] method	8	-3.00000000
Present method	3	

Here also we observed that for finding the root -3.00000000 the present method shows less number of iterations than existing methods. Obviously the present method shows a better convergence than existing methods.

Example 2.2:- Consider the following equations

$(i)\ x^6 - x - 1 = 0$
$(ii)\ e^x - x - 2 = 0$
$(iii) xe^{-x} - 0.1 = 0$

The respective functions $\varphi(x)$ for example 2.2 are

$(i)\varphi(x) = \frac{1+5x^6}{-1+6x^5}$
$(ii)\ \varphi(x) = \frac{2+e^x(-1+x)}{-1+e^x}$ and
$(iii)\ \varphi(x) = \frac{e^x(-1+x)}{-10+e^x}$

The approximate roots of example 2.2 are calculated by similar procedure as example 2.1. The following Table 3 shows comparison between existing methods with initial approximations $(i.1\ and\ 1.5, ii.1\ and\ 2, iii.-0.9\ and\ 0.9)$ and present method with initial approximation $x_0 = 1$.

Table 3:-*Comparing Number of iteration between existing methods and present method*

Equations	Exact root	Bisection method	Secant method	Srinivasarao[1] method	Present method
(i)	1.134724138	16	7	6	3
(ii)	1.146193221	18	7	6	2
(iii)	0.11183256	36	15	11	2

The Table 3 shows a comparison with number of iteration between existing methods and present method for both algebraic and transcendental equations. The present method gives the desired approximation by only 2 or 3 iterations for all equations in example 2.2.

Example 2.3. Consider the following equation

$$x^2 - x - 1 = 0 \tag{2.5}$$

The exact root of equation (2.5) is $\frac{1+\sqrt{5}}{2}$ called the golden ratio the ratio of two consecutive

Fibonacci sequence f_{i+1} and f_i at $i \to \infty$. The Fibonacci sequence are given bellow
$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169, 63245986, 102334155,$
$\cdots$ We consider the function $f(x)$ from equation (2.5) of the form

$$f(x) = x - 1 - \frac{1}{x} \tag{2.6}$$

Again consider the function $\varphi(x)$ from Newton-Raphson method of the form

$$\varphi(x) = x - \frac{f(x)}{\dot{f}(x)} \tag{2.7}$$

and $\psi(x)$ of the form

$$\psi(x) = 1 + \frac{1}{\varphi(x)}, \tag{2.8}$$

then the iteration formula (1.6) yields

$$x_{n+1} = \frac{x_n(2+x_n)(2+2x_n+3{x_n}^2)}{1+6{x_n}^2+4{x_n}^3+2{x_n}^4} \tag{2.9}$$

Consider initial approximation $x_0 = 1$ then the successive iterations are $x_1 = \frac{21}{13}$, $x_2 = \frac{2178309}{1346269}$ and $x_3 = \frac{251728825683549488150424261}{155576970220531065681649693}$. Here we see that the first approximation is the ratio of 8^{th} and 7^{th} Fibonacci number, second approximation is the ratio 32^{th} and 31^{th} Fibonacci number and third approximation is the ratio of 128^{th} and 127^{th} Fibonacci number. It shows that the present technique gives more and more fast convergence.

CONCLUSION

In this paper we have presented a new technique for computing real roots of non-linear equations. The new method will apply for both algebraic and transcendental equations. The new technique shows less number of iterations and gives fast and better results than existing methods. The present method will help to find a real root of different non-linear equations in science and engineering.

REFERENCES

1. Srinivasarao, T. (2019). A new root-finding algorithm using exponential series. *Ural Mathematical Journal*, *5*(1 (8)), 83-90.
2. Chen, J. (2007). New modified regula falsi method for nonlinear equations. *Applied mathematics and computation*, *184*(2), 965-971.
3. Noor, M. A., Noor, K. I., Khan, W. A., & Ahmad, F. (2006). On iterative methods for nonlinear equations. *Applied mathematics and computation*, *183*(1), 128-133.
4. Noor, M. A., & Ahmad, F. (2006). Numerical comparison of iterative methods for solving nonlinear equations. *Applied mathematics and computation*, *180*(1), 167-172.
5. Ehiwario, J. C., & Aghamie, S. O. (2014). Comparative study of bisection, Newton-Raphson and secant methods of root-finding problems. *IOSR Journal of Engineering*, *4*(4), 01-07.
6. Hussain, S., Srivastav, V. K., & Thota, S. (2015). Assessment of interpolation methods for solving the real-life problem. *Int. J. Math. Sci. Appl*, *5*(1), 91-95.
7. Thota, S., & Srivastav, V. K. (2018). Quadratically convergent algorithm for computing real root of non-linear transcendental equations. *BMC research notes*, *11*(1), 1-6.
8. Thota, S., & Srivastav, V. K. (2014). Interpolation based hybrid algorithm for computing real root of non-linear transcendental functions. *Int. J. Math. Comput. Research*, *2*(11), 729-735.
9. Abbasbandy, S., & Liao, S. J. (2008). A new modification of false position method based on homotopy analysis method. *Applied Mathematics and Mechanics*, *29*(2), 223-228.
10. Srivastav, V. K., Thota, S., & Kumar, M. (2019). A new trigonometrical algorithm for computing real root of non-linear transcendental equations. *International Journal of Applied and Computational Mathematics*, *5*(2), 1-8.
11. Ypma, T. J. (1995). Historical development of the Newton–Raphson method. *SIAM review*, *37*(4), 531-551.
12. Tanakan, S. (2013). A new algorithm of modified bisection method for nonlinear equation. *Applied Mathematical Sciences*, *7*(123), 6107-6114.