Non-dimensionalization scheme, Boussinesq approximation

June 18, 2020

1 The Navier-Stokes equation

The Navier-Stokes equation (NS), *dimensional* form:

$$\rho\partial_t \mathbf{u} + 2\rho\mathbf{\Omega} \times \mathbf{u} = -\boldsymbol{\nabla} P + \rho\mathbf{g} + \rho\nu\nabla^2\mathbf{u}. \tag{1}$$

Assume small density variations ρ' following

$$\rho = \rho_0 + \rho' = \rho_0 - \rho_0\alpha\theta,$$

where α is the thermal expansion coefficient and θ is the temperature variation from the isentropic profile $T(r)$. Assume also

$$\mathbf{g} = -g_0\frac{r}{R}\hat{\mathbf{r}}.$$

Within the Boussinesq approximation, the density variations enter only through the buoyancy force, so

$$\rho\mathbf{g} \longrightarrow \rho'\mathbf{g} = \rho_0\alpha g_0\frac{r}{R}\theta\hat{\mathbf{r}}.$$

Now we make the dimensional units explicit. With the unit of length being R, the unit of time $1/\Omega$, the unit of temperature θ^*, the unit of pressure P^* then the NS equation, after dividing by ρ_0, is

$$\Omega^2 R\,\partial_t\mathbf{u} + \Omega^2 R\,2\hat{\mathbf{z}} \times \mathbf{u} = -\frac{P^*}{\rho_0 R}\boldsymbol{\nabla} P + \alpha g_0\theta^* r\theta\hat{\mathbf{r}} + \frac{\nu\Omega}{R}\nabla^2\mathbf{u},$$

where now the variables $r, t, \mathbf{u}, P, \theta$ have been rendered *dimensionless*. Divide now the equation by $\Omega^2 R$ and get

$$\partial_t\mathbf{u} + 2\hat{\mathbf{z}} \times \mathbf{u} = -\frac{P^*}{\rho_0\Omega^2 R^2}\boldsymbol{\nabla} P + \frac{\alpha g_0\theta^*}{\Omega^2 R}r\theta\hat{\mathbf{r}} + \frac{\nu}{\Omega R^2}\nabla^2\mathbf{u}.$$

We are free to choose the scales P^* and θ^* as we wish. We choose then $P* = \rho_0\Omega^2 R^2$. For the time being we leave θ^* unspecified and simply define the *dimensionless* Brunt-Väisälä frequency N such that

$$N^2 \equiv \frac{\alpha g_0\theta^*}{\Omega^2 R}, \tag{2}$$

so that the NS equation becomes

$$\partial_t \mathbf{u} + 2\hat{\mathbf{z}} \times \mathbf{u} = -\boldsymbol{\nabla} P + N^2\,\theta r\,\hat{\mathbf{r}} + E\nabla^2\mathbf{u}, \tag{3}$$

where E is the Ekman number.

2 The heat equation

The heat equation, *dimensional* form is:

$$\partial_t \theta = -u_r \frac{\mathrm{d}T}{\mathrm{d}r} + \kappa\nabla^2\theta, \tag{4}$$

where κ is the thermal diffusivity. Assume an isentropic temperature distribution (dimensional too) such that

$$\frac{\partial T}{\partial r} = \beta\left(\frac{r}{R}\right)^q, \tag{5}$$

where β is a positive constant and the exponent q is either $q = 1$ or $q = -2$. Following Dormy(2004) and Dintrans(1999) the case $q = 1$ corresponds to a uniform distribution of heat *sinks* across the fluid volume, and the case $q = -2$ corresponds to a differential heating case. In Dintrans(1999) they focus only on the $q = 1$ case.
Analogously as for the NS equation, we make the dimensional units explicit and heat equation becomes

$$\Omega\theta^*\partial_t\theta = -\Omega R\,u_r\beta r^q + \frac{\kappa\theta^*}{R^2}\nabla^2\theta,$$

where it is understood that now the variables r, t, u_r, θ became *dimensionless*. Divide the equation by $\Omega\theta^*$ and get

$$\partial_t\theta = -\frac{R}{\theta^*}\beta\,r^q\,u_r + \frac{\kappa}{\Omega R^2}\nabla^2\theta.$$

Choosing the temperature scale as $\theta^* = R\beta$ and using the Prandtl number $\mathcal{P} \equiv \nu/\kappa$ then the heat equation is

$$\partial_t\theta = -r^q\,u_r + \frac{E}{\mathcal{P}}\nabla^2\theta. \tag{6}$$

We can go back now to Eq. 2 and replace θ^* by $R\beta$ and write the dimensionless Brunt-Väisälä frequency (squared) as

$$N^2 = \frac{\alpha g_0\beta}{\Omega^2}. \tag{7}$$