

Sharp bound for the Chebyshev function

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Abstract

Under the assumption that the Riemann Hypothesis is true, von Koch deduced the asymptotic formula $\theta(x) = x + O(\sqrt{x} \times \log^2 x)$. A precise version of this was given by Schoenfeld. He found under the assumption that the Riemann Hypothesis is true that $|\theta(x) - x| < \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x$ for every $x \geq 599$. We prove that if the Riemann Hypothesis is true, then

$$\log \theta(x) > \log x - \log \left(\frac{8 \times \pi \times \sqrt{x}}{8 \times \pi \times \sqrt{x} - \log^2 x} \right) - \left(\frac{4 + \gamma - \log 4 \times \pi}{\sqrt{x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x}} \right)$$

for every $x \geq p_{120569}$, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and p_{120569} is the 120569th prime number.

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1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x , where \log is the natural logarithm. Say Nicolas(p_n) holds provided

$$\prod_{q \leq p_n} \frac{q}{q-1} > e^\gamma \times \log \theta(p_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and p_n is the n th prime number. The importance of this property is:

Theorem 1.1. [1]. Nicolas(p_n) holds for all prime numbers $p_n > 2$ if and only if the Riemann Hypothesis is true.

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We know the following properties for the Chebyshev function:

Theorem 1.2. [2]. *If the Riemann Hypothesis is true, then*

$$\theta(x) = x + O(\sqrt{x} \times \log^2 x).$$

Theorem 1.3. [3]. *If the Riemann Hypothesis is true, then*

$$|\theta(x) - x| < \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x$$

for every $x \geq 599$.

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [4]. We have the following formula:

Theorem 1.4. [5].

$$\sum_q \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows

$$u(x) = \sum_{q>x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right).$$

Let's define:

$$\delta(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).$$

Definition 1.5. *We define another function:*

$$\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).$$

Theorem 1.6. [6]. *The Riemann Hypothesis is also true if and only if the inequality*

$$\frac{n}{\varphi(n)} < e^\gamma \log \log n + \left(\frac{e^\gamma \times (4 + \gamma - \log 4 \times \pi)}{\sqrt{\log n}} \right)$$

is true for all $\log n \geq \theta(p_{120569})$, where $\varphi(n)$ is the Euler's totient function and p_{120569} is the 120569th prime number.

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\text{Nicolas}(p)$ holds, where p is the greatest prime number such that $2 < p \leq x$. In this way, we use this well-known criterion and deduce some of its consequences.

2. Results

Theorem 2.1. *The Riemann Hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all numbers $x \geq 3$.*

Proof. In the Nicolas paper is defined the function [1]:

$$f(x) = e^\gamma \times (\log \theta(x)) \times \prod_{q \leq x} \frac{q-1}{q}.$$

We know that $f(x)$ is lesser than 1 when Nicolas(p) holds, where p is the greatest prime number such that $2 < p \leq x$. In the same paper, we found that

$$\log f(x) = U(x) + u(x)$$

where $U(x) = -\varpi(x)$ [1]. When $f(x)$ is lesser than 1, then $\log f(x) < 0$. Consequently, we obtain that

$$-\varpi(x) + u(x) < 0$$

which is the same as $\varpi(x) > u(x)$. Therefore, this is a consequence of the theorem 1.1. \square

Theorem 2.2. *If the Riemann Hypothesis is true, then*

$$\prod_{q \leq x} \frac{q}{q-1} > (e^\gamma \times \log x) \times \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right)$$

for every $x \geq 599$.

Proof. We use the Theorem 1.3 to show that

$$|\theta(x) - x| < \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x$$

for every $x \geq 599$. That is

$$-\frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x < \theta(x) - x$$

which is

$$\log \left(x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x \right) < \log \theta(x).$$

Hence,

$$\log \log \left(x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x \right) < \log \log \theta(x).$$

We know that

$$\begin{aligned} \log \log \left(x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x \right) &= \log \log \left(x \times \left(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x \right) \right) \\ &= \log \left(\log x + \log \left(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x \right) \right) \\ &= \log \left(\log x \times \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right) \right) \\ &= \log \log x + \log \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right). \end{aligned}$$

In this way,

$$\log \log x + \log \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right) < \log \log \theta(x).$$

That is equivalent to

$$-\log \log \theta(x) + \log \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right) < -\log \log x.$$

That is the same as

$$\varpi(x) + \log \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right) < \delta(x)$$

after adding

$$\left(\sum_{q \leq x} \frac{1}{q} - B \right)$$

to the both sides. We can note that

$$u(x) + \log \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right) < \delta(x)$$

since we know from the theorem 2.1 that $\varpi(x) > u(x)$ for every $x \geq 599$ under the assumption that the Riemann Hypothesis is true. Therefore,

$$-u(x) > -\delta(x) + \log \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right)$$

and

$$H - u(x) > H - \delta(x) + \log \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right)$$

after adding the constant H to the both sides. So,

$$H - u(x) > H + B + \log \log x - \sum_{q \leq x} \frac{1}{q} + \log \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right).$$

We use the theorem 1.4 to show that

$$\sum_{q \leq x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) = H - u(x)$$

and $\gamma = H + B$. Therefore,

$$\sum_{q \leq x} \left(\log\left(\frac{q}{q-1}\right) - \frac{1}{q} \right) > \gamma + \log \log x - \sum_{q \leq x} \frac{1}{q} + \log \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x} \right).$$

Let's remove the value of

$$- \sum_{q \leq x} \frac{1}{q}$$

from the both sides to obtain that

$$\sum_{q \leq x} \left(\log \left(\frac{q}{q-1} \right) \right) > \gamma + \log \log x + \log \left(1 + \frac{\log \left(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x \right)}{\log x} \right).$$

We can apply the exponentiation to show that

$$\prod_{q \leq x} \frac{q}{q-1} > (e^\gamma \times \log x) \times \left(1 + \frac{\log \left(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x \right)}{\log x} \right)$$

and thus, the proof is done. \square

Theorem 2.3. *If the Riemann Hypothesis is true, then*

$$\log \theta(x) > \log x - \log \left(\frac{8 \times \pi \times \sqrt{x}}{8 \times \pi \times \sqrt{x} - \log^2 x} \right) - \left(\frac{4 + \gamma - \log 4 \times \pi}{\sqrt{x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x}} \right)$$

for every $x \geq p_{120569}$, where p_{120569} is the 120569th prime number.

Proof. According to the Theorem 1.6, we obtain that

$$\prod_{q \leq x} \frac{q}{q-1} < e^\gamma \log \theta(x) + \left(\frac{e^\gamma \times (4 + \gamma - \log 4 \times \pi)}{\sqrt{\theta(x)}} \right)$$

is true for all $x \geq p_{120569}$ under the assumption that the Riemann Hypothesis is true. We use the Theorem 2.2 to show that

$$(e^\gamma \times \log x) \times \left(1 + \frac{\log \left(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x \right)}{\log x} \right) < e^\gamma \log \theta(x) + \left(\frac{e^\gamma \times (4 + \gamma - \log 4 \times \pi)}{\sqrt{\theta(x)}} \right)$$

and

$$(\log x) \times \left(1 + \frac{\log \left(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x \right)}{\log x} \right) < \log \theta(x) + \left(\frac{4 + \gamma - \log 4 \times \pi}{\sqrt{\theta(x)}} \right)$$

which is

$$\log x + \log \left(1 - \frac{\log^2 x}{8 \times \pi \times \sqrt{x}} \right) < \log \theta(x) + \left(\frac{4 + \gamma - \log 4 \times \pi}{\sqrt{\theta(x)}} \right).$$

Finally, this implies that

$$\log \theta(x) > \log x - \log \left(\frac{8 \times \pi \times \sqrt{x}}{8 \times \pi \times \sqrt{x} - \log^2 x} \right) - \left(\frac{4 + \gamma - \log 4 \times \pi}{\sqrt{x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x}} \right)$$

because of $\sqrt{x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x} < \sqrt{\theta(x)}$ due to the Theorem 1.3. \square

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