Sharp bound for the Chebyshev function

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Abstract

Under the assumption that the Riemann Hypothesis is true, von Koch deduced the asymptotic formula $\theta(x) = x + O(\sqrt{x} \times \log^2 x)$. A precise version of this was given by Schoenfeld. He found under the assumption that the Riemann Hypothesis is true that $|\theta(x) - x| < \frac{1}{8\times\pi} \times \sqrt{x} \times \log^2 x$ for every $x \geq 599$. We prove that if the Riemann Hypothesis is true, then

$$\log\theta(x) > \log x - \log\left(\frac{8\times\pi\times\sqrt{x}}{8\times\pi\times\sqrt{x} - \log^2 x}\right) - \left(\frac{4+\gamma-\log 4\times\pi}{\sqrt{x - \frac{1}{8\times\pi}\times\sqrt{x}\times\log^2 x}}\right)$$

for every $x \geq p_{120569}$, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and p_{120569} is 120569th prime number.

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1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p\leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x, where log is the natural logarithm. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q\leq p_n} \frac{q}{q-1} > e^{\gamma} \times \log\theta(p_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and p_n is the nth prime number. The importance of this property is:

Theorem 1.1. *[1].* $\mathsf{Nicolas}(p_n)$ *holds for all prime numbers* $p_n > 2$ *if and only if the Riemann Hypothesis is true.*

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We know the following properties for the Chebyshev function:

Theorem 1.2. *[2]. If the Riemann Hypothesis is true, then*

$$\theta(x) = x + O(\sqrt{x} \times \log^2 x).$$

Theorem 1.3. *[3]. If the Riemann Hypothesis is true, then*

$$|\theta(x) - x| < \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x$$

for every $x \geq 599$.

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [4]. We have the following formula:

Theorem 1.4. *[5].*

$$\sum_{q} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows

$$u(x) = \sum_{q>x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right).$$

Let's define:

$$\delta(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).$$

Definition 1.5. *We define another function:*

$$\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).$$

Theorem 1.6. *[6]. The Riemann Hypothesis is also true if and only if the inequality*

$$\frac{n}{\varphi(n)} < e^{\gamma} \log \log n + \left(\frac{e^{\gamma} \times (4 + \gamma - \log 4 \times \pi)}{\sqrt{\log n}} \right)$$

is true for all $\log n \geq \theta(p_{120569})$ *where* $\varphi(n)$ *is Euler's totient function and* p_{120569} *is 120569th prime number.*

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $2 < p \leq x$. In this way, we use this well-known criterion and deduce some of its consequences.

2. Results

Theorem 2.1. *The Riemann Hypothesis is true if and only if the inequality $\varpi(x) > u(x)$ is satisfied for all numbers $x \geq 3$.*

Proof. In the Nicolas paper is defined the function [1]:

$$f(x) = e^{\gamma} \times (\log \theta(x)) \times \prod_{q \leq x} \frac{q-1}{q}.$$

We know that $f(x)$ is lesser than 1 when $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $2 < p \leq x$. In the same paper, we found that

$$\log f(x) = U(x) + u(x)$$

where $U(x) = -\varpi(x)$ [1]. When $f(x)$ is lesser than 1, then $\log f(x) < 0$. Consequently, we obtain that

$$-\varpi(x) + u(x) < 0$$

which is the same as $\varpi(x) > u(x)$. Therefore, this is a consequence of the theorem 1.1. □

Theorem 2.2. *If the Riemann Hypothesis is true, then*

$$\prod_{q \leq x} \frac{q}{q-1} > (e^{\gamma} \times \log x) \times \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x}\right)$$

for every $x \geq 599$.

Proof. We use the Theorem 1.3 to show that

$$|\theta(x) - x| < \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x$$

for every $x \geq 599$. That is

$$-\frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x < \theta(x) - x$$

which is

$$\log\left(x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x\right) < \log \theta(x).$$

Hence,

$$\log\log\left(x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x\right) < \log\log \theta(x).$$

We know that

$$\begin{aligned}
\log\log\left(x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x\right) &= \log\log\left(x \times (1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)\right) \\
&= \log\left(\log x + \log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)\right) \\
&= \log\left(\log x \times (1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x})\right) \\
&= \log\log x + \log\left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x}\right).
\end{aligned}$$

In this way,

$$\log\log x + \log\left(1 + \frac{\log(1 - \frac{1}{8\times\pi\times\sqrt{x}} \times \log^2 x)}{\log x}\right) < \log\log\theta(x).$$

That is equivalent to

$$-\log\log\theta(x) + \log\left(1 + \frac{\log(1 - \frac{1}{8\times\pi\times\sqrt{x}} \times \log^2 x)}{\log x}\right) < -\log\log x.$$

That is the same as

$$\varpi(x) + \log\left(1 + \frac{\log(1 - \frac{1}{8\times\pi\times\sqrt{x}} \times \log^2 x)}{\log x}\right) < \delta(x)$$

after adding

$$\left(\sum_{q\leq x}\frac{1}{q} - B\right)$$

to the both sides. We can note that

$$u(x) + \log\left(1 + \frac{\log(1 - \frac{1}{8\times\pi\times\sqrt{x}} \times \log^2 x)}{\log x}\right) < \delta(x)$$

since we know from the theorem 2.1 that $\varpi(x) > u(x)$ for every $x \geq 599$ under the assumption that the Riemann Hypothesis is true. Therefore,

$$-u(x) > -\delta(x) + \log\left(1 + \frac{\log(1 - \frac{1}{8\times\pi\times\sqrt{x}} \times \log^2 x)}{\log x}\right)$$

and

$$H - u(x) > H - \delta(x) + \log\left(1 + \frac{\log(1 - \frac{1}{8\times\pi\times\sqrt{x}} \times \log^2 x)}{\log x}\right)$$

after adding the constant H to the both sides. So,

$$H - u(x) > H + B + \log\log x - \sum_{q\leq x}\frac{1}{q} + \log\left(1 + \frac{\log(1 - \frac{1}{8\times\pi\times\sqrt{x}} \times \log^2 x)}{\log x}\right).$$

We use the theorem 1.4 to show that

$$\sum_{q\leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) = H - u(x)$$

and $\gamma = H + B$. Therefore,

$$\sum_{q\leq x}\left(\log(\frac{q}{q-1}) - \frac{1}{q}\right) > \gamma + \log\log x - \sum_{q\leq x}\frac{1}{q} + \log\left(1 + \frac{\log(1 - \frac{1}{8\times\pi\times\sqrt{x}} \times \log^2 x)}{\log x}\right).$$

Let's remove the value of

$$-\sum_{q\leq x}\frac{1}{q}$$

from the both sides to obtain that

$$\sum_{q\leq x}\left(\log(\frac{q}{q-1})\right) > \gamma + \log\log x + \log\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^2 x)}{\log x}\right).$$

We can apply the exponentiation to show that

$$\prod_{q\leq x}\frac{q}{q-1} > (e^{\gamma}\times\log x)\times\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^2 x)}{\log x}\right)$$

and thus, the proof is done. □

Theorem 2.3. *If the Riemann Hypothesis is true, then*

$$\log\theta(x) > \log x - \log\left(\frac{8\times\pi\times\sqrt{x}}{8\times\pi\times\sqrt{x}-\log^2 x}\right) - \left(\frac{4+\gamma-\log 4\times\pi}{\sqrt{x-\frac{1}{8\times\pi}\times\sqrt{x}\times\log^2 x}}\right)$$

for every $x \geq p_{120569}$*, where* p_{120569} *is 120569th prime number.*

Proof. According to the Theorem 1.6, we obtain that

$$\prod_{q\leq x}\frac{q}{q-1} < e^{\gamma}\log\theta(x) + \left(\frac{e^{\gamma}\times(4+\gamma-\log 4\times\pi)}{\sqrt{\theta(x)}}\right)$$

is true for all $x \geq p_{120569}$ under the assumption that the Riemann Hypothesis is true. We use the Theorem 2.2 to show that

$$(e^{\gamma}\times\log x)\times\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^2 x)}{\log x}\right) < e^{\gamma}\log\theta(x) + \left(\frac{e^{\gamma}\times(4+\gamma-\log 4\times\pi)}{\sqrt{\theta(x)}}\right)$$

and

$$(\log x)\times\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^2 x)}{\log x}\right) < \log\theta(x) + \left(\frac{4+\gamma-\log 4\times\pi}{\sqrt{\theta(x)}}\right)$$

which is

$$\log x + \log\left(1-\frac{\log^2 x}{8\times\pi\times\sqrt{x}}\right) < \log\theta(x) + \left(\frac{4+\gamma-\log 4\times\pi}{\sqrt{\theta(x)}}\right).$$

Finally, this implies that

$$\log\theta(x) > \log x - \log\left(\frac{8\times\pi\times\sqrt{x}}{8\times\pi\times\sqrt{x}-\log^2 x}\right) - \left(\frac{4+\gamma-\log 4\times\pi}{\sqrt{x-\frac{1}{8\times\pi}\times\sqrt{x}\times\log^2 x}}\right)$$

because of $\sqrt{x-\frac{1}{8\times\pi}\times\sqrt{x}\times\log^2 x} < \sqrt{\theta(x)}$ due to the Theorem 1.3. □

References

[1] J.-L. Nicolas, Petites valeurs de la fonction d'Euler, Journal of number theory 17 (3) (1983) 375–388. doi:10.1016/0022-314X(83)90055-0.

[2] H. Von Koch, Sur la distribution des nombres premiers, Acta Mathematica 24 (1) (1901) 159. doi:10.1007/BF02403071.

[3] L. Schoenfeld, Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II, Mathematics of computation 30 (134) (1976) 337–360.

[4] F. Mertens, Ein Beitrag zur analytischen Zahlentheorie., J. reine angew. Math. 1874 (78) (1874) 46–62. doi:10.1515/crll.1874.78.46.

[5] Y. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin's criterion for the Riemann hypothesis, Journal de Théorie des Nombres de Bordeaux 19 (2) (2007) 357–372. doi:10.5802/jtnb.591.

[6] K. Broughan, Equivalents of the Riemann Hypothesis, Vol. 165, Cambridge University Press, 2017.