

# Note on the Riemann Hypothesis

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## Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . In 2011, Solé and Planat stated that the Riemann Hypothesis is true if and only if the inequality  $\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^\gamma}{\zeta(2)} \times \log \theta(q_n)$  is satisfied for all primes  $q_n > 3$ , where  $\theta(x)$  is the Chebyshev function,  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant and  $\zeta(x)$  is the Riemann zeta function. Using this result, we create a new criterion for the Riemann Hypothesis. We prove the Riemann Hypothesis is true using this new criterion.

*Keywords:* Riemann Hypothesis, Prime numbers, Chebyshev function, Riemann zeta function  
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## 1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . In mathematics, the Chebyshev function  $\theta(x)$  is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers  $p$  that are less than or equal to  $x$ , where  $\log$  is the natural logarithm [1]. We denote the  $n$ th prime number as  $q_n$ . We know the following property for the Chebyshev function and the  $n$ th prime number:

**Proposition 1.1.** For all  $n \geq 2$  [2]:

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \times \left(1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}\right).$$

**Proposition 1.2.** For  $n \geq 8602$  [2]:

$$q_n \leq n \times (\log n + \log \log n - 0.9385).$$

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In mathematics,  $\Psi = n \times \prod_{q|n} \left(1 + \frac{1}{q}\right)$  is called the Dedekind  $\Psi$  function, where  $q | n$  means the prime  $q$  divides  $n$ . Say  $\text{Dedekinds}(q_n)$  holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^\gamma}{\zeta(2)} \times \log \theta(q_n).$$

The constant  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant and  $\zeta(x)$  is the Riemann zeta function. The importance of this inequality is:

**Proposition 1.3.** *Dedekinds( $q_n$ ) holds for all prime numbers  $q_n > 3$  if and only if the Riemann Hypothesis is true [3].*

We define  $H = \gamma - B$  such that  $B \approx 0.2614972128$  is the Meissel-Mertens constant [4]. We know the following formula:

**Proposition 1.4.** *We have that [5]:*

$$\sum_{k=1}^{\infty} \left( \log\left(\frac{q_k}{q_k - 1}\right) - \frac{1}{q_k} \right) = \gamma - B = H.$$

In addition, we know this value of the Riemann zeta function:

**Proposition 1.5.** *It is known that [6]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

Putting all together yields a proof for the Riemann Hypothesis using the Chebyshev function.

## 2. What if the Riemann Hypothesis were false?

**Theorem 2.1.** *If the Riemann Hypothesis is false, then there are infinitely many prime numbers  $q_n$  for which  $\text{Dedekinds}(q_n)$  does not hold.*

*Proof.* The Riemann Hypothesis is false, if there exists some natural number  $x_0 \geq 5$  such that  $g(x_0) > 1$  or equivalent  $\log g(x_0) > 0$  [3]:

$$g(x) = \frac{e^\gamma}{\zeta(2)} \times \log \theta(x) \times \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the bound [3]:

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where  $f$  is introduced in the Nicolas paper [1]:

$$f(x) = e^\gamma \times \log \theta(x) \times \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

When the Riemann Hypothesis is false, then there exists a real number  $b < \frac{1}{2}$  for which there are infinitely many natural numbers  $x$  such that  $\log f(x) = \Omega_+(x^{-b})$  [1]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N} > y_0: \log f(y) \geq k \times y^{-b}.$$

That inequality is equivalent to  $\log f(y) \geq (k \times y^{-b} \times \sqrt{y}) \times \frac{1}{\sqrt{y}}$ , but we note that

$$\lim_{y \rightarrow \infty} (k \times y^{-b} \times \sqrt{y}) = \infty$$

for every possible positive value of  $k$  when  $b < \frac{1}{2}$ . In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N} > y_0: \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann Hypothesis is false, then there are infinitely many natural numbers  $x$  such that  $\log f(x) \geq \frac{1}{\sqrt{x}}$ . Since  $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$ , then it would be infinitely many natural numbers  $x_0$  such that  $\log g(x_0) > 0$  [3]. In addition, if  $\log g(x_0) > 0$  for some natural number  $x_0 \geq 5$ , then  $\log g(x_0) = \log g(q_n)$  where  $q_n$  is the greatest prime number such that  $q_n \leq x_0$ . Actually,

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. □

### 3. A Key Theorem

#### Theorem 3.1.

$$\sum_{k=1}^{\infty} \left( \frac{1}{q_k} - \log\left(1 + \frac{1}{q_k}\right) \right) = \log(\zeta(2)) - H.$$

*Proof.* We obtain that

$$\begin{aligned} \log(\zeta(2)) - H &= \log\left(\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1}\right) - H \\ &= \sum_{k=1}^{\infty} \left( \log\left(\frac{q_k^2}{q_k^2 - 1}\right) \right) - H \\ &= \sum_{k=1}^{\infty} \left( \log\left(\frac{q_k^2}{(q_k - 1) \times (q_k + 1)}\right) \right) - H \\ &= \sum_{k=1}^{\infty} \left( \log\left(\frac{q_k}{q_k - 1}\right) + \log\left(\frac{q_k}{q_k + 1}\right) \right) - H \end{aligned}$$

where

$$\begin{aligned}
&= \sum_{k=1}^{\infty} \left( \log\left(\frac{q_k}{q_k-1}\right) - \log\left(\frac{q_k+1}{q_k}\right) \right) - H \\
&= \sum_{k=1}^{\infty} \left( \log\left(\frac{q_k}{q_k-1}\right) - \log\left(1 + \frac{1}{q_k}\right) \right) - \sum_{k=1}^{\infty} \left( \log\left(\frac{q_k}{q_k-1}\right) - \frac{1}{q_k} \right) \\
&= \sum_{k=1}^{\infty} \left( \log\left(\frac{q_k}{q_k-1}\right) - \log\left(1 + \frac{1}{q_k}\right) - \log\left(\frac{q_k}{q_k-1}\right) + \frac{1}{q_k} \right) \\
&= \sum_{k=1}^{\infty} \left( \frac{1}{q_k} - \log\left(1 + \frac{1}{q_k}\right) \right)
\end{aligned}$$

and the proof is done.  $\square$

#### 4. A New Criterion

**Theorem 4.1.** *Dedekinds( $q_n$ ) holds if and only if the inequality*

$$\sum_{k=1}^{\infty} \left( \frac{1}{q_k} - (\chi_{\{x: x > q_n\}}(q_k)) \times \log\left(1 + \frac{1}{q_k}\right) \right) > B + \log \log \theta(q_n)$$

*is satisfied for the prime number  $q_n$ , where the set  $S = \{x : x > q_n\}$  contains all the real numbers greater than  $q_n$  and  $\chi_S$  is the characteristic function of the set  $S$  (This is the function defined by  $\chi_S(x) = 1$  when  $x \in S$  and  $\chi_S(x) = 0$  otherwise).*

*Proof.* When Dedekinds( $q_n$ ) holds, we apply the logarithm to the both sides of the inequality:

$$\begin{aligned}
&\log(\zeta(2)) + \sum_{q \leq q_n} \log\left(1 + \frac{1}{q}\right) > \gamma + \log \log \theta(q_n) \\
&\log(\zeta(2)) - H + \sum_{q \leq q_n} \log\left(1 + \frac{1}{q}\right) > B + \log \log \theta(q_n) \\
&\sum_{k=1}^{\infty} \left( \frac{1}{q_k} - \log\left(1 + \frac{1}{q_k}\right) \right) + \sum_{q \leq q_n} \log\left(1 + \frac{1}{q}\right) > B + \log \log \theta(q_n).
\end{aligned}$$

Let's distribute the elements of the inequality to obtain that

$$\sum_{k=1}^{\infty} \left( \frac{1}{q_k} - (\chi_{\{x: x > q_n\}}(q_k)) \times \log\left(1 + \frac{1}{q_k}\right) \right) > B + \log \log \theta(q_n)$$

when Dedekinds( $q_n$ ) holds. The same happens in the reverse implication.  $\square$

## 5. The Main Insight

**Theorem 5.1.** *The Riemann Hypothesis is true if the inequality*

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

*is satisfied for all sufficiently large prime numbers  $q_n$ .*

*Proof.* The inequality

$$\sum_{k=1}^{\infty} \left( \frac{1}{q_k} - (\chi_{\{x: x > q_n\}}(q_k)) \times \log\left(1 + \frac{1}{q_k}\right) \right) > B + \log \log \theta(q_n)$$

is satisfied when

$$\sum_{k=1}^{\infty} \left( \frac{1}{q_k} - (\chi_{\{x: x \geq q_n\}}(q_k)) \times \log\left(1 + \frac{1}{q_k}\right) \right) > B + \log \log \theta(q_n)$$

is also satisfied, where the set  $S = \{x : x \geq q_n\}$  contains all the real numbers greater than or equal to  $q_n$ . In the inequality

$$\sum_{k=1}^{\infty} \left( \frac{1}{q_k} - (\chi_{\{x: x \geq q_n\}}(q_k)) \times \log\left(1 + \frac{1}{q_k}\right) \right) > B + \log \log \theta(q_n)$$

only change the values of

$$\log\left(1 + \frac{1}{q_n}\right) + \log \log \theta(q_n)$$

and

$$\log \log \theta(q_{n+1})$$

between the consecutive primes  $q_n$  and  $q_{n+1}$ . It is enough to show that

$$\log\left(1 + \frac{1}{q_n}\right) + \log \log \theta(q_n) \geq \log \log \theta(q_{n+1})$$

for all sufficiently large prime numbers  $q_n$ . Indeed, the inequality

$$\sum_{k=1}^{\infty} \left( \frac{1}{q_k} - (\chi_{\{x: x \geq q_n\}}(q_k)) \times \log\left(1 + \frac{1}{q_k}\right) \right) > B + \log \log \theta(q_n)$$

is the same as

$$\begin{aligned} & \sum_{k=1}^{\infty} \left( \frac{1}{q_k} - (\chi_{\{x: x \geq q_{n+1}\}}(q_k)) \times \log\left(1 + \frac{1}{q_k}\right) \right) \\ & > B + \log \log \theta(q_{n+1}) + \log\left(1 + \frac{1}{q_n}\right) + \log \log \theta(q_n) - \log \log \theta(q_{n+1}) \end{aligned}$$

where  $q_n$  and  $q_{n+1}$  are consecutive primes. If the Riemann Hypothesis is false, then

$$\log\left(1 + \frac{1}{q_n}\right) + \log \log \theta(q_n) \geq \log \log \theta(q_{n+1})$$

must be violated for infinitely many  $n$ 's, since  $\text{Dedekinds}(q_{n+1})$  will not hold for infinitely many  $q_{n+1}$ 's. By contraposition, the Riemann Hypothesis should be true when the previous inequality is satisfied for all sufficiently large prime numbers  $q_n$ . This is

$$\log \left( \left( 1 + \frac{1}{q_n} \right) \times \log \theta(q_n) \right) \geq \log \log \theta(q_{n+1}).$$

That is equivalent to

$$\log \log \theta(q_n)^{1 + \frac{1}{q_n}} \geq \log \log \theta(q_{n+1}).$$

To sum up, the Riemann Hypothesis is true when

$$\theta(q_n)^{1 + \frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers  $q_n$ . □

## 6. The Main Theorem

**Theorem 6.1.** *The Riemann Hypothesis is true.*

*Proof.* The Riemann Hypothesis is true when

$$\theta(q_n)^{1 + \frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers  $q_n$ . That is the same as

$$\theta(q_n)^{1 + \frac{1}{q_n}} \geq \theta(q_n) + \log(q_{n+1})$$

$$\theta(q_n)^{\frac{1}{q_n}} \geq 1 + \frac{\log(q_{n+1})}{\theta(q_n)}$$

after dividing the both sides of the inequality by  $\theta(q_n)$ . We would only need to prove that

$$1 + \frac{\log \theta(q_n)}{q_n} \geq 1 + \frac{1}{n \times \left( 1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n} \right)}$$

because of

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \times \left( 1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n} \right)$$

$$\theta(q_n)^{\frac{1}{q_n}} = e^{\frac{\log \theta(q_n)}{q_n}} \geq 1 + \frac{\log \theta(q_n)}{q_n}.$$

That is equivalent to

$$\left( n \times \left( 1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n} \right) \right) \times \log \theta(q_n) \geq q_n.$$

Therefore,

$$\left( n \times \left( 1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n} \right) \right) \times \log \theta(q_n) \geq n \times (\log n + \log \log n - 0.9385)$$

which is

$$\begin{aligned} \left(1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}\right) \times \log \theta(q_n) + 0.9385 &\geq \log n + \log \log n \\ \theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}} \times e^{0.9385} &\geq n \times \log n \\ e^{0.9385} &\geq \frac{n \times \log n}{\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}}}. \end{aligned}$$

However, we know that

$$\overline{\lim}_{n \rightarrow \infty} \frac{n \times \log n}{\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}}} = \lim_{n \rightarrow \infty} \frac{n \times \log n}{\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}}} = 1$$

since

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}\right) &= 1 \\ \theta(q_n) &\sim q_n, \quad (n \rightarrow \infty) \\ q_n &\sim n \times \log n, \quad (n \rightarrow \infty). \end{aligned}$$

For any positive real number  $\varepsilon$ , there exists a natural number  $m$  such that

$$\frac{n \times \log n}{\theta(q_n)^{1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}}} < 1 + \varepsilon$$

for all  $n > m$ , because of the definition of limit superior. Moreover, we can see that  $e^{0.9385} > 2.5561$ . Consequently, it is enough to take any positive real number  $\varepsilon \leq 1.5561$ . By the definition of the limit superior yields the proof of the Riemann Hypothesis.  $\square$

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