Using a Sharp bound for the Chebyshev function

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Abstract

Under the assumption that the Riemann hypothesis is true, von Koch deduced the asymptotic formula $\theta(x) = x + O(\sqrt{x} \times \log^2 x)$. A precise version of this was given by Schoenfeld. He found under the assumption that the Riemann hypothesis is true that $|\theta(x) - x| < \frac{1}{8\times\pi} \times \sqrt{x} \times \log^2 x$ for every $x \geq 599$. Using this result, we prove that if the Riemann hypothesis is true, then $\prod_{q \leq x} \frac{q}{q-1} > (e^{\gamma} \times \log x) \times \left(1 + \frac{\log(1 - \frac{1}{8\times\pi\times\sqrt{x}} \times \log^2 x)}{\log x}\right)$ for every $x \geq 599$.

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1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x, where log is the natural logarithm. Say $\mathsf{Nicolas}(p_n)$ holds provided

$$\prod_{q \leq p_n} \frac{q}{q-1} > e^{\gamma} \times \log \theta(p_n).$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and p_n is the nth prime number. The importance of this property is:

Theorem 1.1. *[1].* $\mathsf{Nicolas}(p_n)$ *holds for all prime numbers* $p_n > 2$ *if and only if the Riemann hypothesis is true.*

We know the following properties for the Chebyshev function:

Theorem 1.2. *[2]. If the Riemann hypothesis is true, then*

$$\theta(x) = x + O(\sqrt{x} \times \log^2 x).$$

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Theorem 1.3. *[3]. If the Riemann hypothesis is true, then*

$$|\theta(x) - x| < \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x$$

for every $x \geq 599$.

Let's define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant [4]. We have the following formula:

Theorem 1.4. *[5].*

$$\sum_{q} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right) = \gamma - B = H.$$

For $x \geq 2$, the function $u(x)$ is defined as follows

$$u(x) = \sum_{q > x} \left(\log(\frac{q}{q-1}) - \frac{1}{q} \right).$$

Let's define:

$$\delta(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log x - B \right).$$

Definition 1.5. *We define another function:*

$$\varpi(x) = \left(\sum_{q \leq x} \frac{1}{q} - \log \log \theta(x) - B \right).$$

Putting all together yields the proof that the inequality $\varpi(x) > u(x)$ is satisfied for a number $x \geq 3$ if and only if $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $2 < p \leq x$. In this way, we use this well-known criterion and deduce some of its consequences.

2. Results

Theorem 2.1. *The Riemann hypothesis is true if and only if the inequality* $\varpi(x) > u(x)$ *is satisfied for all numbers* $x \geq 3$.

Proof. In the paper [1] is defined the function:

$$f(x) = e^{\gamma} \times (\log \theta(x)) \times \prod_{q \leq x} \frac{q-1}{q}.$$

We know that $f(x)$ is lesser than 1 when $\mathsf{Nicolas}(p)$ holds, where p is the greatest prime number such that $2 < p \leq x$. In the same paper, we found that

$$\log f(x) = U(x) + u(x)$$

where $U(x) = -\varpi(x)$ [1]. When $f(x)$ is lesser than 1, then $\log f(x) < 0$. Consequently, we obtain that

$$-\varpi(x) + u(x) < 0$$

which is the same as $\varpi(x) > u(x)$. Therefore, this is a consequence of the theorem 1.1. □

Theorem 2.2. *If the Riemann hypothesis is true, then*

$$\prod_{q \leq x} \frac{q}{q-1} > (e^{\gamma} \times \log x) \times \left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x}\right)$$

for every $x \geq 599$.

Proof. We use the Theorem 1.3 to show that

$$|\theta(x) - x| < \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x$$

for every $x \geq 599$. That is

$$-\frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x < \theta(x) - x$$

which is

$$\log\left(x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x\right) < \log \theta(x).$$

Hence,

$$\log\log\left(x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x\right) < \log\log \theta(x).$$

We know that

$$\begin{aligned}
\log\log\left(x - \frac{1}{8 \times \pi} \times \sqrt{x} \times \log^2 x\right) &= \log\log\left(x \times (1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)\right) \\
&= \log\left(\log x + \log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)\right) \\
&= \log\left(\log x \times (1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x})\right) \\
&= \log\log x + \log\left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x}\right).
\end{aligned}$$

In this way,

$$\log\log x + \log\left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x}\right) < \log\log \theta(x).$$

That is equivalent to

$$-\log\log \theta(x) + \log\left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x}\right) < -\log\log x.$$

That is the same as

$$\varpi(x) + \log\left(1 + \frac{\log(1 - \frac{1}{8 \times \pi \times \sqrt{x}} \times \log^2 x)}{\log x}\right) < \delta(x)$$

after adding

$$\left(\sum_{q\leq x}\frac{1}{q}-B\right)$$

to the both sides. We can note that

$$u(x)+\log\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^{2}x)}{\log x}\right)<\delta(x)$$

under the assumption that the Riemann hypothesis is true, since we know from the theorem 2.1 that $\varpi(x)>u(x)$ for every $x\geq 599$. Therefore,

$$-u(x)>-\delta(x)+\log\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^{2}x)}{\log x}\right)$$

and

$$H-u(x)>H-\delta(x)+\log\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^{2}x)}{\log x}\right)$$

after adding the constant H to the both sides. So,

$$H-u(x)>H+B+\log\log x-\sum_{q\leq x}\frac{1}{q}+\log\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^{2}x)}{\log x}\right).$$

We use the theorem 1.4 to show that

$$\sum_{q\leq x}\left(\log(\frac{q}{q-1})-\frac{1}{q}\right)=H-u(x)$$

and $\gamma=H+B$. Therefore,

$$\sum_{q\leq x}\left(\log(\frac{q}{q-1})-\frac{1}{q}\right)>\gamma+\log\log x-\sum_{q\leq x}\frac{1}{q}+\log\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^{2}x)}{\log x}\right).$$

Let's remove the value of

$$-\sum_{q\leq x}\frac{1}{q}$$

from the both sides to obtain that

$$\sum_{q\leq x}\left(\log(\frac{q}{q-1})\right)>\gamma+\log\log x+\log\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^{2}x)}{\log x}\right).$$

We can apply the exponentiation to show that

$$\prod_{q\leq x}\frac{q}{q-1}>(e^{\gamma}\times\log x)\times\left(1+\frac{\log(1-\frac{1}{8\times\pi\times\sqrt{x}}\times\log^{2}x)}{\log x}\right)$$

and thus, the proof is done. □

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