# Phenomenological models of Cosmic Ray transport in Galaxies


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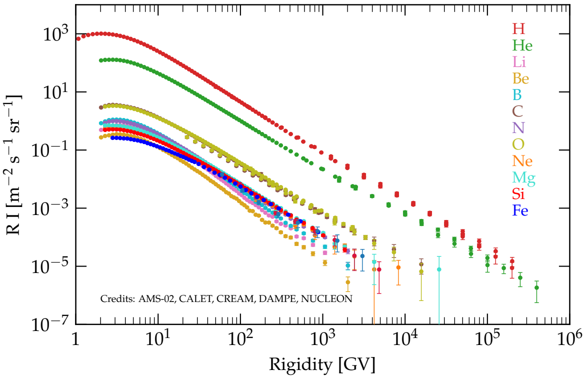
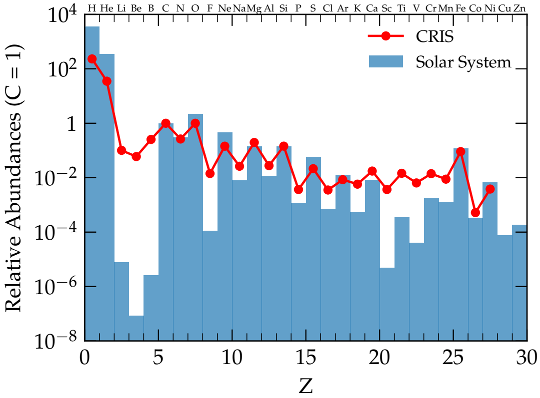
# Galactic Cosmic Rays: unprecedented measurements



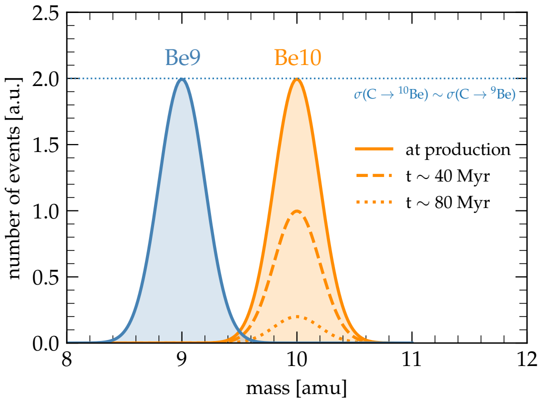

- Amazing new data: The spectrum of each isotope includes contributions from many different parents (both in terms of fragmentation and decays) giving to each observed isotope a potentially very complex history
- In these lectures we will mainly focus on our Galaxy as we take advantage of far more information than any other environment, bearing in mind that the very same physical picture can be straightforwardly applied to the vast majority of astrophysical territories

# Basic indicators of diffusive transport: stable elements




- ▷ Thermal particles in the average interstellar medium are somehow accelerated to relativistic energies becoming CRs $\longrightarrow$ primary CRs
- ▷ It must exist also a second population which is produced during propagation by primary fragmentation $\longrightarrow$ secondary CRs

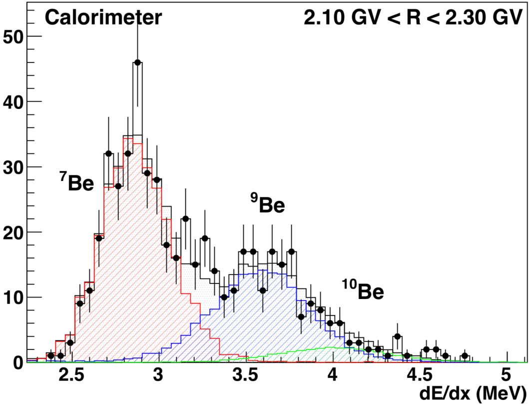
# Basic indicators of diffusive transport: unstable elements




- ▷ ${}^{10}$Be is a $\beta^-$ unstable isotope decaying in ${}^{10}$B with an half-life of $\sim 1.5$ Myr
- ▷ Similar production rates than other (stable) isotopes $\sigma_{\rm Be9} \sim \sigma_{\rm Be10}$
- ▷ Traditionally ${}^{9}$Be/${}^{10}$Be has been used as CR clock pointing to a residence time of $\mathcal{O}(100)$ Myr

# Basic indicators of diffusive transport: unstable elements

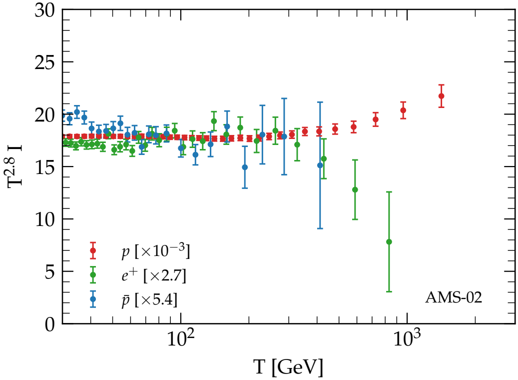
PAMELA coll., ApJ 862 (2018)




- ▷ Traditionally the ratio $^9$Be/$^{10}$Be has been used as CR clock
  $\longrightarrow$ however no measurements of this ratio at $E \gtrsim 1$ GeV/n

# Secondary/primary: the antimatter case

AMS-02 Coll., PRL 117(2016), PRL 122 (2019)




- ▷ CR antimatter is a promising discovery tool for new physics or exotic phenomena.
- ▷ Understanding the irreducible background arising from secondary production by primary CR collisions with interstellar matter is one of the most interesting challenge.

# Prelude

# Basic definitions: The grammage pillar

- The grammage $X$ is the amount of material that the particle go trough along propagation (a sort of column density):

$$X = \int dl \rho(l)$$

- I assume a simple system with one primary species $n_p$ and one secondary $n_s$ only.

- The evolution of primary and secondary along the grammage trajectory is given respectively by:

$$\frac{dn_p}{dX} = -\frac{n_p}{\lambda_p}$$

$$\frac{dn_s}{dX} = -\frac{n_s}{\lambda_s} + P_{p \to s} \frac{n_p}{\lambda_p}$$

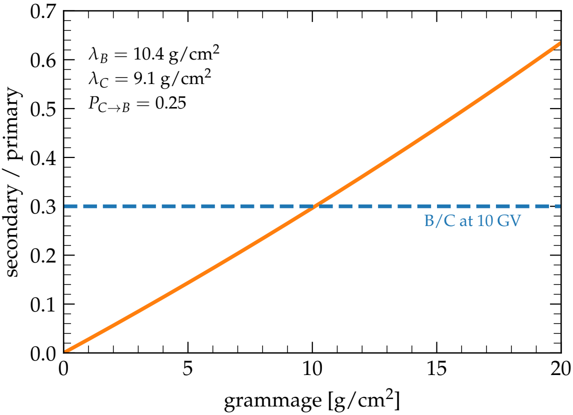
  with initial conditions $n_p(X = 0) = n_0$ and $n_s(X = 0) = 0$, where $\lambda_i$ are some kind of interaction lenght (probability) and $P$ is the fraction resulting in that specific channel.

- Solving this, I can get $n_s/n_p$ in terms of $X$, $\lambda_s$ and $\lambda_p$ only:

$$\frac{n_s}{n_p} = P_{p \to s} \frac{\lambda_s}{\lambda_s - \lambda_p} \left[ \exp\left( -\frac{X}{\lambda_s} + \frac{X}{\lambda_p} \right) - 1 \right]$$

$\longrightarrow$ I quantify the transport process, whatever it is, in something that can be either directly measured in CRs $n_s/n_p$ or provided by a nuclear physics experiment ($\lambda$'s, $P$'s).

## Basic definitions: The grammage pillar




$$\text{B/C} \sim 0.3 \longrightarrow \chi = 10\,\text{g/cm}^2$$

# Basic definitions: The grammage pillar

- Let me assume that the grammage is accumulated in the gas disc of our Galaxy
- At each crossing of the disc $n_{\rm gas} \sim 1$ cm$^{-3}$, $h \sim 200$ pc:

$$\chi_d \sim m_p n_{\rm gas} h_d \sim 10^{-3}\,{\rm g/cm^2} \ll \chi_{\rm B/C}$$

- The grammage accumulated in one crossing is clearly inconsistent with the grammage we estimate from CR measurements $\rightarrow$ the particles have to cross the disk many times
- The time spent in the gas region before escaping the Galaxy must be not less than:

$$t_{\rm esc,min} \sim \frac{\chi_{\rm B/C}}{\chi_d}\frac{h}{v} \sim 7 \times 10^6\,{\rm years} \gg \frac{\rm kpc}{c}$$

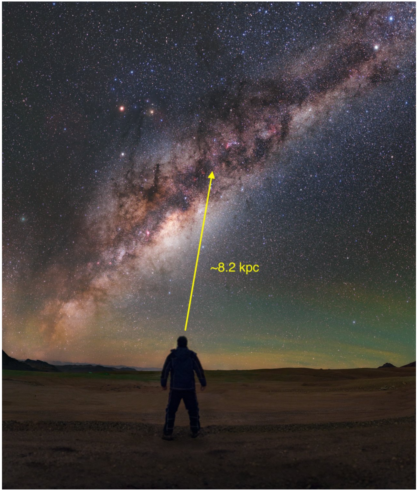
which exceeds by order of magnitudes any possible ballistic timescale in the MW $\sim \mathcal{O}(\frac{\rm kpc}{c})$

- We deduce that CRs follow something more similar to a Brownian motion in the Galaxy

**Open question**

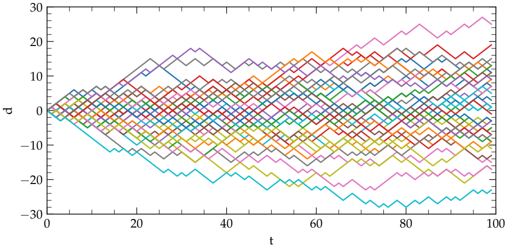
What is the origin of confinement of these particles in the Galaxy?

# Basic definitions: The grammage pillar




Wanderer above the Sea of Stars

# Basic definitions: random walk and diffusion coefficient




- After $N$ steps $\vec{\lambda}_i$ of the same size $||\lambda_i|| = \lambda$ and random direction a particle has reached a distance:

$$\vec{d} = \sum_{i=1}^{N} \vec{\lambda}_i$$

- The scalar product of $d$ with itself is

$$\vec{d} \cdot \vec{d} = \sum_{i=1}^{N} \sum_{j=1}^{N} \vec{\lambda}_i \cdot \vec{\lambda}_j \longrightarrow d^2 = N\lambda^2 + 2\lambda^2 \sum_{i=1}^{N} \sum_{j<1}^{N} \cos\theta_{ij} \sim N\lambda^2$$

as we assumed that the angles $\theta_{ij}$ are chosen randomly and thus the off-diagonal terms are uncorrelated.

# Basic definitions: random walk and diffusion coefficient

- The continuity equation for the number density $n$ and its current $\vec{j}$ reads

$$\frac{\partial n}{\partial t} + \nabla \vec{j} = q$$

  assuming $q$ to be the sum of all sources or losses.

- Combined together with Fick's law for an isotropic flux $\vec{j} = -D\nabla n$ leads to the diffusion equation:

$$\frac{\partial n}{\partial t} - \nabla (D \nabla n) = q$$

- The propagator (Green function) of the 1D diffusion equation with constant $D$ is

$$G(d) = \frac{1}{(4\pi D t)^{1/2}} \mathrm{e}^{-\frac{d^2}{4Dt}}$$

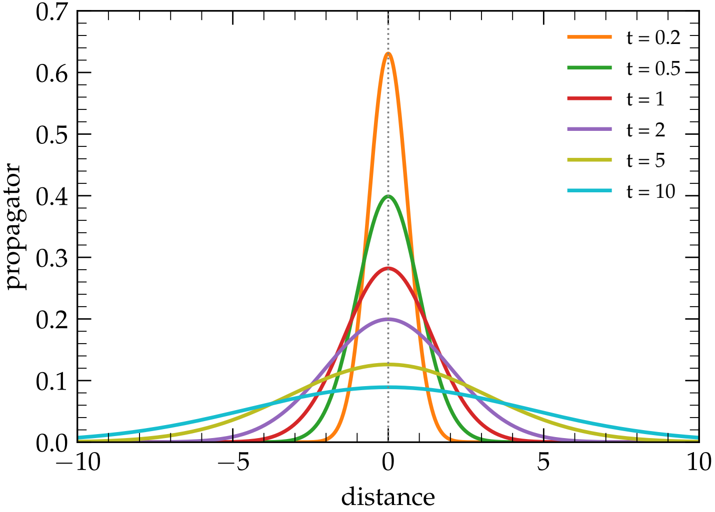
  thus the mean distance traveled outward is $\propto \sqrt{Dt}$

- Connecting the two pictures we obtain that $D$ is the product of particle velocity $v$ and mean free path $\lambda$:

$$\boxed{D \sim \frac{N\lambda^2}{t} \sim \frac{v\lambda}{3}}$$

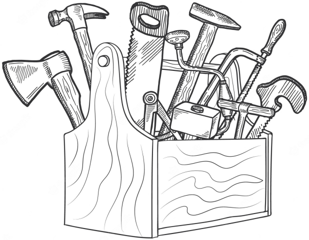
  where the numerical factor is obtained in 3D with a more accurate derivation.

# Basic definitions: random walk and diffusion coefficient



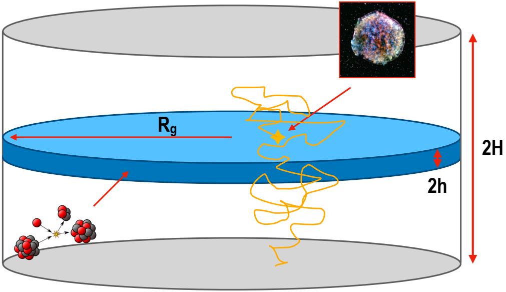
# Lecture I: Nuclei

# Lecture motivations



- The aim of these lectures is to provide you with basic tools to sketch a simple, yet helpful, model of cosmic ray transport easily applied to your best-loved astrophysical environment: milky way, cluster of galaxies, starburst galaxies, ...

# A toy model for protons in our Galaxy: main assumptions




- In the standard model for the origin of Galactic CRs, these particles are accelerated in the disc $h \sim 100$ pc with an injected spectrum $q_p \propto p^{-\gamma}$ where $\gamma \gtrsim 4$
- after injection, CRs propagate diffusively throughout the Galactic halo $H \sim \mathcal{O}(\text{kpc})$ with a diffusion coefficient $D \propto p^{\delta}$ where $\delta \sim 1/3 - 1/2$ and free escape at the boundaries
- $R_{\rm g} \gg H$ is the radius of the Galactic disc $\to$ 1D problem
- Secondary production, e.g. LiBeB, takes place predominantly in the disc $h$ where all the gas is confined.

# A toy model for protons in our Galaxy

- The simplest transport equation for protons, assuming relativistic particles $p \simeq E$:

$$-\frac{\partial}{\partial z}\left[D(E)\frac{\partial n_p}{\partial z}\right] = Q(E,z) = \frac{E_{\rm SN}\mathcal{R}_{\rm SN}}{\pi R_d^2} q_0(E)\delta(z)$$

where $n_p(E)$ is the cosmic ray density, $E_{\rm SN} \simeq 10^{51}$ erg is the SN kinetic energy, and $\mathcal{R}_{\rm SN} \simeq 1/100\ \text{yr}^{-1}$ is the SN galactic rate.

- For $z \neq 0$, and using the boundary condition $n_p(z = \pm H, E) = 0$:

$$D\frac{\partial n_p}{\partial z} = \text{Constant} \longrightarrow n_p(z) = n_0\left(1 - \frac{z}{H}\right)$$

- Since the diffusive flux is constant in $z$, in particular at the disc $z = 0$:

$$D\frac{\partial n_p}{\partial z}\bigg|_{z=0^+} = -D\frac{n_{p,0}}{H}$$

- We now integrate the diffusion equation around $z = 0$

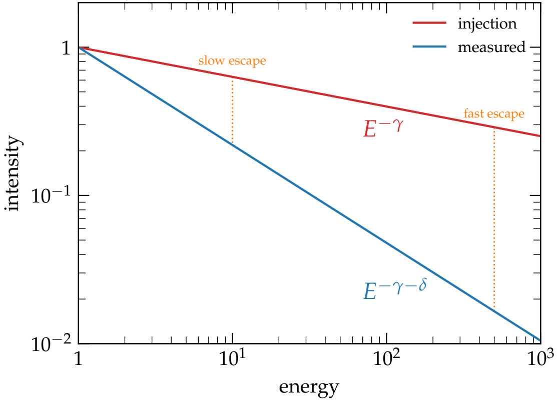
$$\lim_{\epsilon\to 0}\int_{\epsilon^-}^{\epsilon^+} dz \left\{-\frac{\partial}{\partial z}\left[D\frac{\partial n_p}{\partial z}\right] = Q(E,z)\right\} \longrightarrow -2D\frac{\partial n_p}{\partial z}\bigg|_{z=0^+} = \frac{E_{\rm SN}\mathcal{R}_{\rm SN}}{\pi R_d^2} q_0(E)$$

- and using the equation for the flux:

injection rate per unit volume — escape rate

$$n_p(E) = \frac{E_{\rm SN}\mathcal{R}_{\rm SN} q_0(E)}{2\pi R_d^2}\frac{H}{D(E)} = \boxed{\frac{E_{\rm SN}\mathcal{R}_{\rm SN} q_0(E)}{2\pi R_d^2 H}}\boxed{\frac{H^2}{D(E)}} \propto E^{-\gamma-\delta}$$

# A toy model for protons in our Galaxy

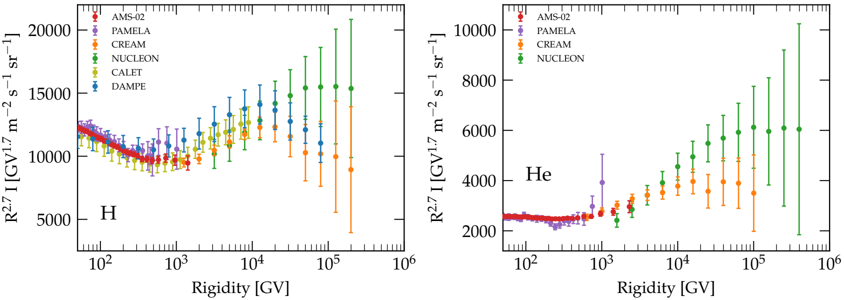



**Open question**

Having evidence of a feature in the proton spectrum, how to distinguish if due to injection or propagation?

# Galactic Cosmic Rays: novel features

PAMELA Coll., Science 2011; AMS-02 Coll., PRL 2015; CREAM Coll., ApJ 2017; NUCLEON Coll., JETP 2018; DAMPE Coll., Science 2019




- ▷ Spectra of protons and helium are not a single power law below the knee $\rightarrow$ some physics kicking in?
- ▷ The hardening at $R = p/Z \sim 300 - 400$ GV is well established since first observation by PAMELA
- ▷ AMS-02 confirmed the same break for almost all nuclei
- ▷ The softening at $R = p/Z \sim 10$ TV is observed by different experiments, first strong evidence in DAMPE

# A toy model for protons in our Galaxy: meaning of free escape boundary?

- The physics of CR transport is as much regulated by diffusion as it is by boundary conditions.
- What does free escape mean?

$$n(z = \pm H, E) = 0$$

- Conservation at the boundaries implies:

$$D \frac{\partial n}{\partial z} = \frac{c}{3} n_{\rm out} \longrightarrow n_{\rm out} = \frac{3D}{cH} n_0 \sim \frac{\lambda(E)}{H} n_0 \ll n_0$$

where $\lambda(E) \simeq 1$ pc $(E/\text{GeV})^{1/2}$ in the GeV-TeV energy range.

- Effectively, it implies a discontinuity where $D \rightarrow \infty$, but why?

**Open question**

despite the great importance of this assumption we do not have any knowledge on what determines the halo size or whether the halo size depends on energy or space.

# The cosmic ray density

- Cosmic rays come from all directions in outer space over large energy intervals.
- The number of particles in volume element $d^3 r$ about $\vec{r}$ and in the momentum interval $d^3 p$ about $\vec{p}$ is given by

$$dn = F(\vec{r}, \vec{p}, t)\, d^3 r\, d^3 p$$

  with $F$ the distribution function.
- Expanding in spherical coordinates $d^3 p$:

$$dn = F(\vec{r}, p, t)\, d^3 r\, p^2\, dp d\Omega$$

- Typically we are not able to measure $F$ but only averages over momentum space, thereby we conveniently introduce the phase-space distribution function as:

$$f(\vec{r}, p, t) = \frac{1}{4\pi} \int_\Omega F(\vec{r}, p, t) d\Omega$$

- Correspondingly, the number of particles $dN$ in $d^3\, r$ and in $(p, p + dp)$ (independent of direction of $\vec{p}$) is:

$$dN = \int_\Omega d\Omega\, F(\vec{r}, p, t)\, d^3 r\, p^2\, dp = 4\pi p^2 f(\vec{r}, p, t)\, d^3 r\, dp$$

# The transport equation

Parker, Planet. Space Sci. (1965); Ginzburg & Syrovatskii (1964); Berezinskii et al. (1980)

The transport of a CR species $\alpha = \mathrm{H}^1, \mathrm{He}^4, \mathrm{C}^{12}, ..., \mathrm{Fe}^{56}$ is well described by an advection-diffusion equation with losses for :

$$\cancelto{0}{\frac{\partial f_\alpha}{\partial t}} - \frac{\partial}{\partial z}\left(D\frac{\partial f_\alpha}{\partial z}\right) + u\frac{\partial f_\alpha}{\partial z} - \frac{du}{dz}\frac{p}{3}\frac{\partial f_\alpha}{\partial p} = q_{\rm SN}\delta(z) - \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2\dot{p}f_\alpha\right] - \frac{f_\alpha}{\tau_\alpha^{\rm in}} + \sum_{\alpha'>\alpha} b_{\alpha'\alpha}\frac{f_{\alpha'}}{\tau_{\alpha'}^{\rm in}}$$

- Stationarity is ensured by proper boundary conditions $f_\alpha(z = \pm H) = 0$
- Spatial diffusion: $\vec{\nabla} \cdot \vec{J}$
- Advection by Galactic winds/outflows: $u = u_w + v_A \sim v_A$
- Source term proportional to Galactic SN rate $\mathcal{R}$: $q_{\rm SN} \propto \frac{E_{\rm SN}\mathcal{R}}{\pi R_g^2} q_\alpha(p)$
- Energy losses: ionization, Coulomb losses, Inverse Compton, Synchrotron, ...
- Production/destruction of nuclei due to inelastic scattering (or decay) $\rightarrow \sigma_\alpha^{\rm in}$ is the inelastic cross-section , $b_{\alpha'\alpha}$ is the fragmentation branching ratio

# Description of transport of nuclei

- For nuclei of mass $A$, it is customary to introduce the **intensity** (this is the quantity given by measurements) as a function of the **kinetic energy per nucleon $T$**:

$$I_\alpha(T)dT = p^2 f_\alpha(p) v(p) dp \longrightarrow I_\alpha(T) = A p^2 f_\alpha(p)$$

**!** spallation preserves the energy per nucleon

- For simplicity we ignore some loss terms (ionization), advection and second oder Fermi acceleration in ISM, all these effects may be relevant at $T \lesssim 10$ GeV/n.

- We explicit $q_{\rm SN} = 2h_d \delta(z) q_\alpha(p)$ and $(\tau_\alpha^{\rm in})^{-1} = 2h_d \delta(z) n_d v \sigma_\alpha$, thereby the transport equation becomes:

diffusion; spallation of nuclei $\alpha$

$$-\frac{\partial}{\partial z}\left[D_\alpha \frac{\partial I_\alpha(T)}{\partial z}\right] + 2h_d n_d v(T) \sigma_\alpha \delta(z) I_\alpha(T) =$$

injection of nuclei $\alpha$; contribution to nuclei $\alpha$ from spallation of $\alpha' > \alpha$

$$= 2Ap^2 h_d q_{0,\alpha}(p) \delta(z) + \sum_{\alpha' > \alpha} 2h_d n_d v(T) \sigma_{\alpha' \to \alpha} \delta(z) I_{\alpha'}(T)$$

# The transport equation for primary Nuclei

- Formally similar to the equation for protons but with spallation taken into account:

$$-\frac{\partial}{\partial z}\left[D_\alpha \frac{\partial I_\alpha(T)}{\partial z}\right] + 2h_d n_d v(T)\sigma_\alpha \delta(z) I_\alpha(T) = 2Ap^2 h_d q_{0,\alpha}(p)\delta(z)$$

- The equation is solved in the same way:
  - first we consider the solution for $z \neq 0$ ($z > 0$ or $z < 0$)
  - then integrate around $z = 0$ between $0^-$ and $0^+$
- It follows:

$$D_\alpha \frac{\partial I_\alpha}{\partial z} = \text{constant} \longrightarrow I_\alpha = I_{0,\alpha}\left(1 - \frac{z}{H}\right)$$

which we use to derive

$$-D_\alpha \frac{\partial I_\alpha}{\partial z}\bigg|_{z=0} = -h_d n_d v(T)\sigma_\alpha I_{0,\alpha} + Ap^2 h_d q_{0,\alpha}(p)$$

# The transport equation for primary Nuclei

- The intensity of a primary nucleus of type $\alpha$ is

$$I_{0,\alpha}(T) = \frac{\frac{Ap^2 h_d q_{0,\alpha}(p)}{H} \frac{H^2}{D_\alpha}}{1 + \frac{\chi_\alpha(T)}{\hat{\chi}_\alpha}} = \frac{\frac{Ap^2 q_{0,\alpha}(p)}{n_d m_p v} \chi_\alpha(T)}{1 + \frac{\chi_\alpha(T)}{\hat{\chi}_\alpha}}$$

- Where the grammage traversed by nuclei of type $\alpha$:

$$\chi_\alpha(T) = n_d \left(\frac{h}{H}\right) m_p v \frac{H^2}{D_\alpha} = \bar{n} m_p v \tau_{\rm esc}(T)$$

- and the critical grammage (energy independent) is:

$$\hat{\chi}_\alpha = \frac{m_p}{\sigma_\alpha}$$

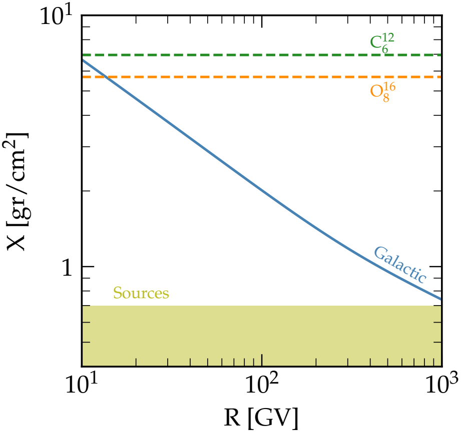
- Relevant limits:

diffusion dominated: for $\chi \ll \hat{\chi}$ the equilibrium spectrum is $I_0 \propto T^{-\gamma-\delta}$

spallation dominated: for $\chi \gg \hat{\chi}$ the equilibrium spectrum is $I_0 \propto T^{-\gamma}$

# The transport equation for primary Nuclei

Evoli et al., PRD 101 (2020)




- Since $\hat{\chi}_{\rm B,C}$ are independent on energy and $\chi(E)$ is a decreasing function of energy we can distinguish the regime where spallation dominates $\lesssim 10$ GeV/n and the regime where diffusion dominates $\gtrsim 10$ GeV/n.

# The transport equation for secondary Nuclei: secondary/primary ratio

- Let's work out a simple case with only Carbon as primary species $\alpha' = \text{C}$, and Boron as secondary $\alpha = \text{B}$:

destruction of B — production of B from C spallation

$$-\frac{\partial}{\partial z}\left[D_\text{B}\frac{\partial I_\text{B}(T)}{\partial z}\right] + 2h_d n_d v \sigma_\text{B} \delta(z) I_\text{B}(T) = 2h_d n_d v \sigma_{\text{C}\to\text{B}} \delta(z) I_\text{C}(T)$$

- following the same approach as before (and assuming $X_\text{B} \simeq X_\text{C} \equiv X$):

$$I_{\text{B},0}(T) = I_{\text{C},0}(T)\frac{X(T)}{\hat{X}_{\text{C}\to\text{B}}}\left(1 + \frac{X(T)}{\hat{X}_\text{B}}\right)^{-1}$$

- which reflects in the following B/C ratio:

$$\frac{\text{B}}{\text{C}} = \frac{\frac{X(T)}{\hat{X}_{\text{C}\to\text{B}}}}{1 + \frac{X(T)}{\hat{X}_\text{B}}}$$

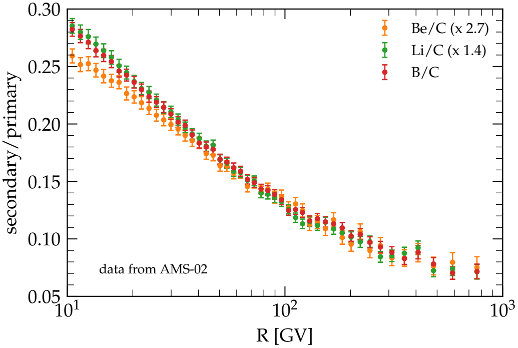
- Relevant limits:

diffusion dominated: for $X \ll \hat{X}$ the ratio is B/C $\propto X(T) \propto 1/D(T)$

spallation dominated: for $X \gg \hat{X}$ the ratio is B/C $\sim$ constant

# The transport equation for secondary Nuclei: the diffusion coefficient slope

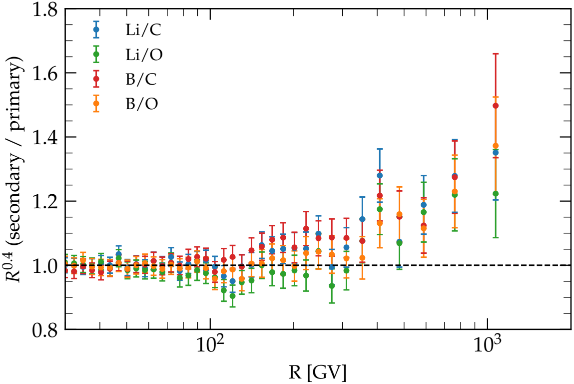
AMS-02 Coll., PRL 120 (2018)




- ▷ Evidence of rigidity dependent grammage $\longrightarrow$ high-energy particles spend less time in our Galaxy than low-energy ones (advection may play a role only at low energies)
- ▷ At $T \gtrsim 50$ GeV/n the B/C ratio scales as the grammage $\longrightarrow$ we can measure the slope of $D(E)$ from the energy dependence of B/C.
- ▷ Notice however that B/C is sensitive only to the H/D ratio

# The transport equation for secondary Nuclei: the origin of the spectral feature

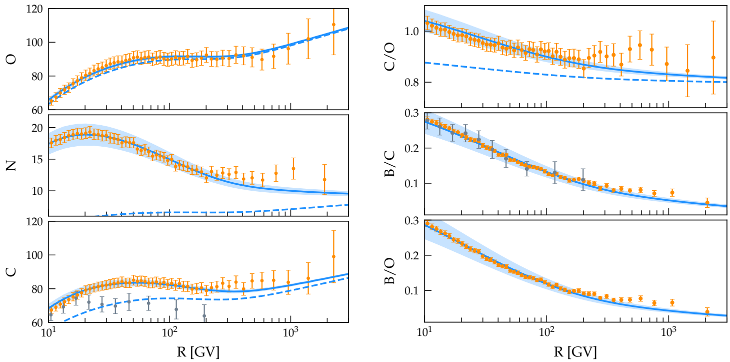
AMS-02 Coll., PRL 120 (2018)




the same feature detected in the primary spectra is observed in the secondary/primary ratio which depends only on the grammage $\longrightarrow$ propagation effect

# Quick look at the data: The CNO element

Evoli et al., PRD 101 (2020), Weinrich et al., A&A 639 (2020), De La Torre Luque et al., JCAP 03 (2021), Schroer et al., PRD 103 (2021)

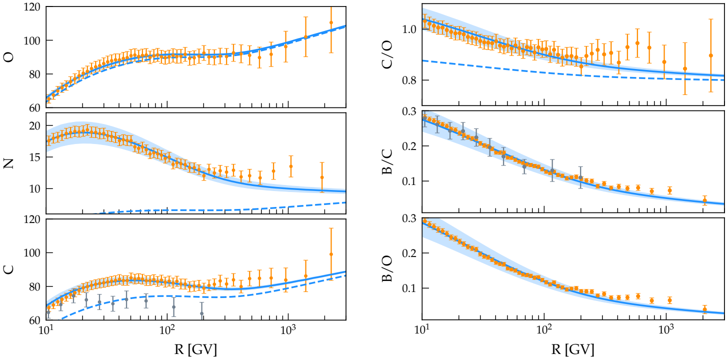



▷ We assume a phenomenological motivated $D(R)$ as a smoothly-broken power-law:

$$D(R) = \boxed{2v_A H} + \frac{\boxed{\beta D_0 (R/\mathrm{GV})^{\delta}}}{\boxed{[1 + (R/R_b)^{\Delta\delta/s}]^s}}$$

# Quick look at the data: The CNO element

Evoli et al., PRD 101 (2020), Weinrich et al., A&A 639 (2020), De La Torre Luque et al., JCAP 03 (2021), Schroer et al., PRD 103 (2021)




- by fitting primary and secondary/primary measurements we found:
  $\delta \sim 0.54$, $D_0/H \sim 0.5 \times 10^{28}$ cm/s$^2$/kpc, $\Delta\delta \sim 0.2$, $v_A \sim 5$ km/s
- All nuclei injected with $\gamma \sim 4.3$
- Shaded areas: uncertainty from cross sections (small for pure primary species as Oxygen).

# Potential sources of galactic CRs

D. Ter Haar, Reviews of Modern Physics, 1950; Ginzburg & Syrovatskii, 1963

- The grammage is also a crucial piece of information to identify galactic CR sources.
- The galactic CR luminosity is:

from B/C

$$L_{\rm CR} \sim \frac{\epsilon_{\rm CR} V_{\rm MW}}{\tau_{\rm esc}} \sim \pi \epsilon_{\rm CR} R_d^2 \boxed{\frac{H}{D}} \sim 10^{41} \text{ erg/s}$$

where

- $\epsilon_{\rm CR} \sim 1$ eV/cm$^3$ is the local CR energy density
- $V_{\rm MW} = \pi R_d^2 2H$ is the Milky Way Volume (for CRs)
- $\tau_{\rm esc} \sim H^2/D$ is the escape time

- This is also the luminosity required (on a timescale of $\sim \tau_{\rm esc}$) to sustain the CR population.
- The SNe energy rate in our Galaxy:

$$L_{\rm SN} = E_{\rm SN} R_{\rm SN} \sim 10^{42} \text{ erg/s} \sim 10 \times L_{\rm CR}$$

- Galactic SNe provide the right energetics if $\sim 10\%$ efficiency in CR acceleration is achieved $\longrightarrow$ a mechanism able to transfer such an energy was discovered in the 70's (DSA).

# Decay of unstable isotopes

B/C only gives the grammage $\propto H/D \rightarrow$ how to break the degeneracy?

- We now look at the ratio of unstable and stable species, as the lifetime introduces a clock breaking the degeneracy
- $^{10}$Be is $\beta^-$ unstable with an half-life $\tau_{1/2} \sim 1.39 \times 10^6$ years $\longrightarrow$ $^{10}$B
- The transport equation for $^{10}$Be is the first case we discuss where the source or loss term is not in the form of a $\delta$-function in $z$:

$$-\frac{\partial}{\partial z}\left[D_{\rm Be}\frac{\partial I_{\rm Be}(T)}{\partial z}\right] + \underbrace{\frac{\mu v \sigma_{\rm Be}}{m}\delta(z) I_{\rm Be}(T)}_{\text{destruction of Be}} + \underbrace{\frac{I_{\rm Be}(T)}{\gamma\tau_d}}_{\text{Be decay}} = \underbrace{\frac{\mu v \sigma_{\rm C\to Be}}{m}\delta(z) I_{\rm C}(T)}_{\text{production of Be from C spallation}} \quad (1)$$

where $\mu = 2 h_d n_d m \sim 10^{-3}$ g/cm$^2$ is the disk surface density.

- $^{10}$Be decays on a time scale $\gamma\tau_d$ that at some high-E becomes longer than $\tau_{\rm esc}$ $\longrightarrow$ stable
- $^{10}$Be decays mainly into $^{10}$B so that it changes the abundance of stable elements.

# Decay of unstable isotopes

- Outside the disk $z \neq 0$ the transport equation becomes

$$-\frac{\partial}{\partial z}\left[D_{\text{Be}}\frac{\partial I_{\text{Be}}(T)}{\partial z}\right] + \frac{I_{\text{Be}}(T)}{\hat{\tau}_d} = 0 \qquad (2)$$

- the solution is in the form

$$I = A\mathrm{e}^{-\alpha z} + B\mathrm{e}^{\alpha z} \qquad (3)$$

which implies $\alpha^{-1} \equiv \sqrt{D\hat{\tau}_d}$

- after imposing the proper boundary conditions we obtain (introducing $y \equiv \mathrm{e}^{\alpha H}$):

$$\frac{I_{\text{Be}}(z)}{I_{\text{Be},0}} = -\frac{y^2}{1-y^2}\mathrm{e}^{-\alpha z} + \frac{1}{1-y^2}\mathrm{e}^{\alpha z} \qquad (4)$$

- the value of the distribution function at $z = 0$ can be obtained by the usual integration above/below disc:

$$-2D_{\text{Be}}\frac{\partial I_{\text{Be}}(T)}{\partial z}\bigg|_{0^+} + \frac{\mu v \sigma_{\text{Be}}}{m} I_{\text{Be},0}(T) = \frac{\mu v \sigma_{\text{C}\to\text{Be}}}{m} I_{\text{C},0}(T) \qquad (5)$$

$$\longrightarrow I_{\text{Be},0}(T)\left[\frac{\sigma_{\text{Be}}}{m} - \frac{2D_{\text{Be}}}{\mu v H}\alpha H\frac{1+y^2}{1-y^2}\right] = \frac{\sigma_{\text{C}\to\text{Be}}}{m} I_{\text{C},0}(T)$$

# Decay of unstable isotopes

- The transport equation in terms of $\mathcal{X}$'s becomes:

$$\frac{I_{\text{Be},0}}{I_{\text{C},0}}(T) = \frac{1}{\hat{\mathcal{X}}_{\text{C}\to\text{Be}}}\left[\frac{1}{\hat{\mathcal{X}}_{\text{Be}}} + \frac{1}{\mathcal{X}'_{\text{Be}}(T)}\right]^{-1}$$

- At high energy: $\frac{H^2}{D_{\text{Be}}} \ll \hat{\tau}_d \longrightarrow \alpha H \to 0$

$$\mathcal{X}'_{\text{Be}}(T) \longrightarrow \mathcal{X}_{\text{Be}}(T)$$

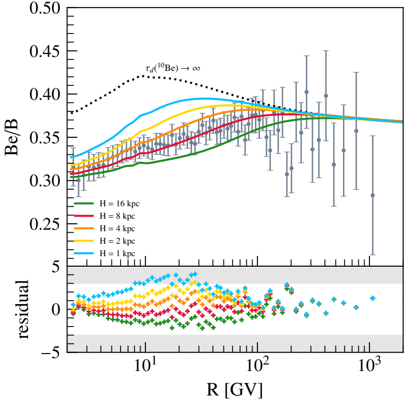
- At low energy: $\frac{H^2}{D_{\text{Be}}} \gg \hat{\tau}_d \longrightarrow \alpha H \to \infty$

$$\mathcal{X}'_{\text{Be}}(T) \longrightarrow \frac{\mu v}{2}\sqrt{\frac{\hat{\tau}_d}{D_{\text{Be}}}} = \frac{\mu v}{2H}\sqrt{\hat{\tau}_d \tau_{\text{esc}}}$$

- It is crucial to consider the additional contribution to B production by Be decay (! homework!).

# Quick look at the data: The Beryllium-over-Boron ratio and the Halo size

Evoli et al., PRD 101 (2020), Weinrich et al., A&A 639 (2020), Korsmeier & Cuoco, PRD 105 (2022), Maurin et al., arXiv:2203.07265




- ▷ Preference for large halos $H \gtrsim 5$ kpc
- ▷ Notice that $H$ and $\tau_{\rm esc}$ are mutual corresponding

$$\tau_{\rm esc}(10\,\mathrm{GV}) \sim \frac{H^2}{2D} \sim 20\,\mathrm{Myr}\left(\frac{H}{\mathrm{kpc}}\right)\left(\frac{0.25\times 10^{28}\,\mathrm{cm}^2/\mathrm{s}/\mathrm{kpc}}{D_0/H}\right)$$

# Cosmic ray transport for the poor physicists

- Generic rule of thumb:

$$\text{Intensity} \sim \text{Injection Rate} \times \frac{\text{Relevant lifetime}}{\text{Relevant volume}}$$

- Primary species equilibrium spectrum:

$$I_p(T) \propto Q(T) \frac{\tau_{\rm esc}(T)}{H}$$

- Secondary stable species equilibrium spectrum:

$$I_s(T) \propto I_p(T) \sigma v n_{\rm d} h_d \frac{\tau_{\rm esc}(T)}{H}$$

- Secondary unstable(*) species equilibrium spectrum:

$$I_s^*(T) \propto I_p(T) \sigma v n_{\rm d} h_d \frac{\tau_{\rm d}(T)}{\sqrt{\tau_{\rm d}(T) D(T)}}$$

- Stable secondary over primary ratio:

$$\frac{I_s(T)}{I_p(T)} \propto X(T) \propto \frac{H}{D(T)}$$

- Unstable secondary over stable secondary ratio:

$$\frac{I_s^*(T)}{I_s(T)} \propto \frac{\sqrt{D(T)}}{H^2}$$ ← break the degeneracy!

# What B/C does imply on scattering micro-physics?

- By reproducing local measurements we obtained:

from B/C — from Be/B

$$\boxed{D(\text{GV})/H \simeq 0.35 \times 10^{28}\ \text{cm}^2/\text{s}} + \boxed{H \simeq 5\ \text{kpc}} \rightarrow \boxed{D(\text{GV}) \simeq 1.8 \times 10^{28}\ \text{cm}^2/\text{s}}$$

- In terms of a diffusion coefficient:

$$D(E) = \frac{1}{3} r_{\text{L}}(E) v \frac{1}{\mathcal{F}(k_{\text{res}})} = \frac{1}{3} v \lambda_{\text{diff}}(E) \quad \text{where} \quad k_{\text{res}} = \frac{1}{r_{\text{L}}(E)}$$

- implying that at $\sim$ GV:

$$\lambda_{\text{diff}} \simeq \text{pc}$$

remember from Blasi's lectures: this is (on average) how much a GV particle has to travel before to deflect by 90°

- the turbulence level required to do so

$$\boxed{r_{\text{L}}(\text{GV}) \simeq 10^{12}\ \text{cm} \rightarrow \mathcal{F}(k) \simeq \frac{r_{\text{L}} c}{3 D_0} \simeq 6 \times 10^{-7} = \left(\frac{\delta B}{B_0}\right)^2_{k_{\text{res}}}}$$

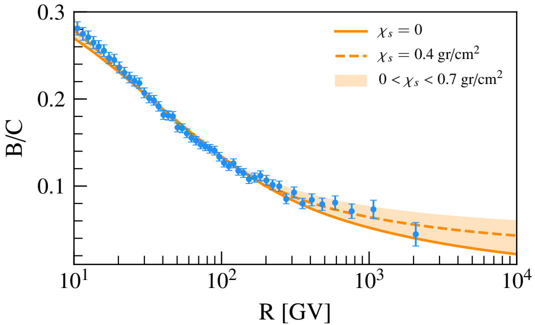
# Another example of “Little things affect Big things”

Transport ($\sim 10^{22}$ cm) $\longrightarrow$ mean free path ($\sim 10^{18}$ cm) $\longrightarrow$ waves lenght ($\sim 10^{13}$ cm)

Such a tiny perturbation at the scale of the Solar System stretches the transport time in the Galaxy from kyrs’ to 100 Million of years!

# Additional effects not included in this picture

Evoli et al., PRD 99 (2019)




- Second-order Fermi acceleration in the ISM [Ptuskin et al., 2006, ApJ 642; Drury & Strong, 2017, A&A 597]
- Shock re-acceleration of secondary nuclei [Blasi, 2017, MNRAS 471; Bresci et al., 2019, MNRAS 488]
- Grammage at the sources [D'Angelo et al., 2016, PRD 94; Nava et al., 2016, MNRAS 461; Jacobs et al., 2022, JCAP 05]
- Secondary production at the sources [Blasi, 2009, PRL 103; Mertsch & Sarkar, 2014, PRD 90]
- ...

# Lecture I (extra)

# An old friend: the leaky-box model

Seo & Ptuskin, ApJ 431 (1994)

- An extremely simplified approximation of the diffusion model is the so called leaky-box model.
- Widely popular to infer on the nail relevant properties of the galactic transport from the data but watch at the caveats!
- In this case we ignore any spatial dependence and express the diffusion term as a leakage rate:

$$\nabla(D\nabla I_\alpha) \longrightarrow -\frac{I_\alpha}{\tau_\alpha^{\text{esc}}}$$

where $\tau_{\text{esc}}$ is the usual $H^2/D$.

- The steady-state transport equation then reads

$$0 = q_\alpha - \frac{I_\alpha}{\tau_\alpha^{\text{esc}}} - \left(\frac{1}{\tau_\alpha^{\text{in}}} + \frac{1}{\tau_\alpha^{\text{d}}}\right) I_\alpha + \sum_{\alpha'>\alpha} \left(\frac{1}{\tau_{\alpha'\to\alpha}^{\text{in}}} + \frac{1}{\tau_{\alpha'}^{\text{d}}}\right) I_{\alpha'}$$

where $\tau_\alpha^{\text{in}} = (v\bar{n}\sigma_\alpha)^{-1}$

# The leaky-box model: secondary-over-primary ratio

- Considering only stable species $\tau^{\mathrm{d}} \to \infty$ the solution becomes

$$I_\alpha = \left( q_\alpha + \frac{I_{\alpha'}}{\tau^{\mathrm{in}}_{\alpha' \to \alpha}} \right) \left( \frac{1}{\tau^{\mathrm{esc}}_\alpha} + \frac{1}{\tau^{\mathrm{in}}_\alpha} \right)^{-1}$$

- In terms of the grammage:

$$I_\alpha = \left( \frac{q_\alpha}{\bar{n} m_p v} + \frac{I_{\alpha'}}{\hat{\chi}_{\alpha' \to \alpha}} \right) \frac{\chi_\alpha}{1 + \frac{\chi_\alpha}{\hat{\chi}_\alpha}}$$

- The solution we just derived is precisely the same as the one we obtained using the full approach if the source term is taken as:

$$q_\alpha = A p^2 q_{0,\alpha}(p) \frac{h}{H}$$

- which reflects in the following B/C ratio:

$$\frac{\mathrm{B}}{\mathrm{C}} = \frac{\frac{\chi(T)}{\hat{\chi}_{\mathrm{C} \to \mathrm{B}}}}{1 + \frac{\chi(T)}{\hat{\chi}_{\mathrm{B}}}}$$

# The leaky-box model: unstable-over-stable ratio

- Considering only secondary species the solution becomes

$$I_\alpha = \left( \frac{I_{\alpha'}}{\tau^{\rm in}_{\alpha' \to \alpha}} \right) \left( \frac{1}{\tau^{\rm esc}_\alpha} + \frac{1}{\tau^{\rm in}_\alpha} + \frac{1}{\tau^{\rm d}_\alpha} \right)^{-1}$$

- assuming $\sigma_{10} \simeq \sigma_9$, the ratio:

$$\frac{\mathrm{Be}^{10}}{\mathrm{Be}^9} = \frac{(\tau^{\rm esc})^{-1} + (\tau^{\rm in})^{-1}}{(\tau^{\rm esc})^{-1} + (\tau^{\rm in})^{-1} + (\tau^{\rm d}_{10})^{-1}} \xrightarrow{\tau^{\rm esc} \ll \tau^{\rm in}} \frac{1}{1 + \frac{\tau^{\rm esc}}{\tau^{\rm d}_{10}}}$$

- To be compared against the solution we derived for the thin disk case

$$\frac{\mathrm{Be}^{10}}{\mathrm{Be}^9} \simeq \sqrt{\frac{\tau^{\rm d}_{10}}{\tau^{\rm esc}}}$$

- At $\gamma \sim 1$, measurements points to a ratio $\frac{\mathrm{Be}^{10}}{\mathrm{Be}^9} \sim 0.3$ which would corresponds to:

$$\tau_{\rm esc} = \begin{cases} \sim 6 \text{ Myr} & \text{in the Leaky-box approximation} \\ \sim 20 \text{ Myr} & \text{in the thin disc model} \end{cases}$$

# Lecture II: Leptons

# A quick look to lepton energy losses

- The main difference with respect to the nuclei case is that leptons are very prone to radiative energy losses
- High-energy leptons lose energy predominantly for

  synchrotron emission on Galactic magnetic field
  inverse Compton scattering on Galactic radiation fields (CMB, IR, optical...)

- We limit our model to the Thomson limit, ignoring the corrections to the $\gamma$-e$^-$ cross-section due to the Klein-Nishina effect (= ignoring the electron recoil)
- the energy loss rate reads

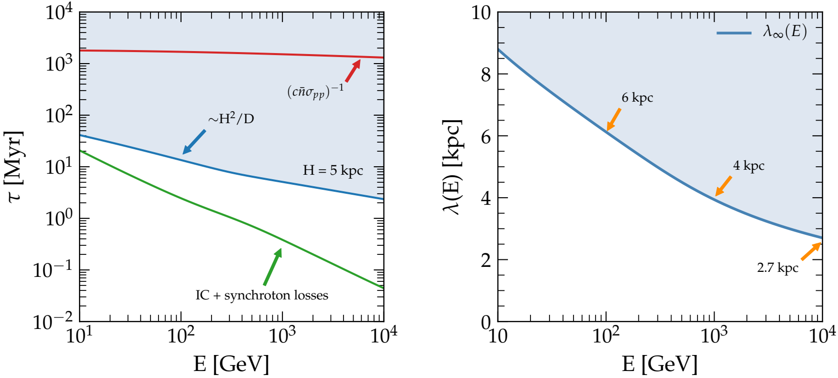
$$\left|\frac{dE}{dt}\right| = \frac{4}{3}\sigma_{\rm T} c\gamma^2\beta^2(\mathcal{U}_\gamma + \mathcal{U}_{\rm B}) \equiv b_0 \left(\frac{E}{10\,\text{GeV}}\right)^2$$

  where $\mathcal{U}_\gamma$ is the energy density in soft photons (IC) and $\mathcal{U}_{\rm B} = \frac{B^2}{8\pi}$ in magnetic field (synchrotron)

- in Galactic environments $\mathcal{U}_i \sim \mathcal{O}(0.1 - 1\ \text{eV/cm}^3) \to b_0 \sim 10^{-14} \left(\frac{\mathcal{U}_\gamma + \mathcal{U}_{\rm B}}{\text{eV/cm}^3}\right)\left(\frac{E}{10\ \text{GeV}}\right)^2$ GeV/s
- the energy loss time is a decreasing function with energy:

$$\tau_{\rm loss} \simeq \frac{E}{-\frac{dE}{dt}} \sim 3\,\text{Myr} \left(\frac{E}{10\,\text{GeV}}\right)^{-1}$$

# How does it compare with the CR escape time in Galaxy?




- Leptons lose their energy mainly by IC with the interstellar radiation fields (ISRFs) or synchrotron emission
- Milky Way is a very inefficient calorimeter for nuclei and a perfect calorimeter for leptons
- Translate losses into propagation scale: $\lambda \sim \sqrt{D(E)\tau_{\rm loss}} \to$ horizon

# Transport of leptons

- The transport equation to deal with is

$$-\frac{\partial}{\partial z}\left[D\frac{\partial f_e}{\partial z}\right] = q_e(p)\delta(z) - \frac{1}{p^2}\frac{\partial}{\partial p}\left[\dot{p}p^2 f_e\right]$$

- It is convenient to approximate the loss terms as a catastrophic loss term

$$-\frac{\partial}{\partial z}\left[D\frac{\partial f_e}{\partial z}\right] = q_e(p)\delta(z) - \frac{f_e}{\tau_{\text{loss}}}$$

  that can be solved similarly to Be since the energy losses are effective in all the propagation volume.

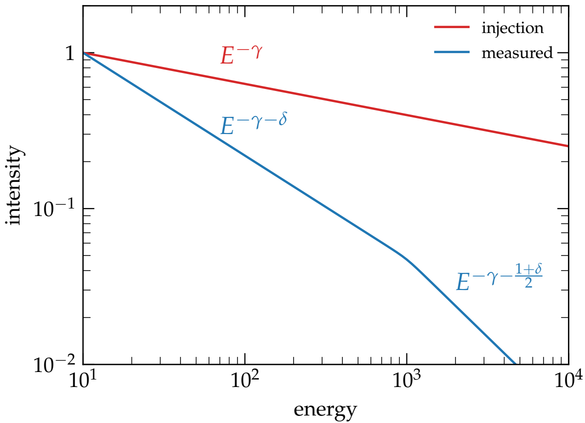
- Therefore, in the low energy limit where losses are weak, $\tau_{\text{loss}} \gg \tau_{\text{esc}}$:

$$f_{e,0}(E) = \frac{q_{e,0}(E)\mathcal{R}_{\text{SN}}}{2\pi R_d^2}\frac{\tau_{\text{esc}}}{H} \sim E^{-\gamma-\delta}$$

- In the high energy limit where losses dominate transport, $\tau_{\text{loss}} \ll \tau_{\text{esc}}$:

$$f_{e,0}(E) = \frac{q_{e,0}(E)\mathcal{R}_{\text{SN}}}{2\pi R_d^2}\frac{\tau_{\text{loss}}}{\sqrt{D\tau_{\text{loss}}}} \sim E^{-\gamma-\frac{\delta+1}{2}}$$

# Transport of leptons




For fiducial values of CR transport in the Milky Way the transition between the two regimes is at $\lesssim 10$ GeV

# A quick application to the positron fraction

- Secondary positrons are produced through $pp \to \pi^{\pm} + \ldots$ and tipically the energy of the secondary positron is a fraction $\xi \sim \mathcal{O}(10\%)$ of the parent proton energy $E_p$:

$$E_{e^+} \simeq \xi E_p \quad \leftarrow \text{inelasticity}$$

- The rate of positron $e^+$ production in the ISM is then

$$q_{e^+}(E)dE = n_p(E_p)dE_p n_d \sigma_{\rm pp} c 2 h_d \delta(z)$$

- Applying standard solutions we approach the usual limits, when losses are unimportant:

$$f_{e^+}(E) = n_p\left(\frac{E}{\xi}\right)\frac{2c\sigma_{\rm pp} n_d h_d}{\xi}\frac{H}{D(E)}$$

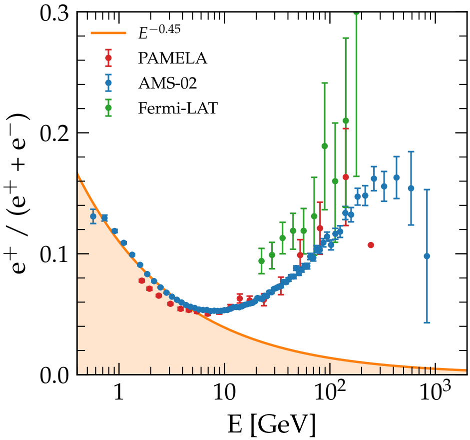
- while in the limit when losses dominate:

$$f_{e^+}(E) = n_p\left(\frac{E}{\xi}\right)\frac{2c\sigma_{\rm pp} n_d h_d}{\xi}\frac{\tau_{\rm loss}(E)}{\sqrt{\tau_{\rm loss}(E)D(E)}}$$

- as a consequence, in both cases (! homework):

$$\frac{f_{e^+}}{f_{e^-}}(E) = \frac{q_{p,0}(E/\xi)}{q_{e,0}(E)}\frac{1}{\xi}\frac{\chi(E/\xi)}{\hat{\chi}} \sim E^{-\gamma_p+\gamma_e-\delta}$$

# A quick application to the positron fraction




- Assuming $\gamma_p = \gamma_e \to$ positron fraction is a decreasing function with energy $\sim E^{-\delta}$
- To grow with energy must be $\gamma_e > \gamma_p + \delta$ unlikely!

# Solving the transport equation with the Green functions

- We aim at solving the inhomogeneous diffusion equation with spatially constant $D$:

$$\frac{\partial I(E)}{\partial t} - D(E)\nabla^2 I(E) = Q(\vec{r}, E, t)$$

- We seek a Green's function $\mathcal{G}$ such that:

$$\frac{\partial}{\partial t}\mathcal{G}(\vec{r}, t \leftarrow \vec{r}_\star, t_\star) - D\nabla^2\mathcal{G}(\vec{r}, t \leftarrow \vec{r}_\star, t_\star) = \delta^{(3)}(\vec{r} - \vec{r}_\star)\delta(t - t_\star)$$

  with the same boundary conditions that we assumed for $I(\vec{r}, t)$.

- $\mathcal{G}(\vec{r}, t \leftarrow \vec{r}_\star, t_\star)$ translates the probability for a CR injected at position $\vec{r}_\star$ and time $t_\star$ to travel through the Galaxy until it reaches an observer located at $\vec{r}$ at time $t$.

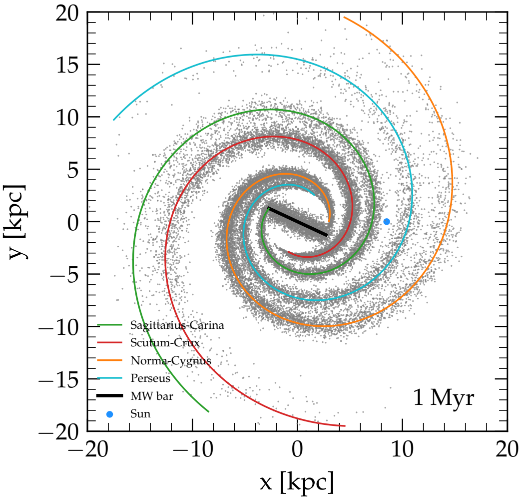
- The formal solution of the diffusion equation may be expressed as the convolution over space and time of the Green's function with the source term:

$$I(\vec{r}, t) = \int_0^\infty \int_V \mathcal{G}(\vec{r}, t \leftarrow \vec{r}_\star, t_\star)\, Q(\vec{r}_\star, t_\star)\, dt_\star\, d^3\vec{r}_\star$$

- In particular, we can replace the continuous and smooth source term, $Q(\vec{r}, t)$, with an ensemble of $N$ sources with distances $\{\vec{r}_i\}$ and ages $\{t_i\}$, such that the total flux is the sum of the fluxes from individual sources:

$$Q(\vec{r}_\star, t_\star) = \sum_{i=0}^{N} q_0 \delta^{(3)}(\vec{r}_\star - \vec{r}_i)\delta(t_\star - t_i) \;\rightarrow\; I(\vec{r}, t) = q_0 \sum_{i=0}^{N} \mathcal{G}(\vec{r}, t \leftarrow \vec{r}_i, t_i)$$

## Solving the transport equation with the Green functions



# Solving the transport equation with the Green functions: MonteCarlo approach

- As we don't know the actual distribution of the sources we must rely on a MonteCarlo approach
- In each realization $\alpha$ of the ensamble, the position and the age of each source are random variables $\longrightarrow$ the CR intensity $I_\alpha$ behaves as a stochastic variable
- The conventional CR model must be recovered in the mean field limit, i.e., by taking the average of the flux over the ensemble of all possible populations

$$I = \langle I_\alpha(\vec{r}, t) \rangle_\alpha = q_0(E) N \int_0^\infty \int_V \mathcal{G}(\vec{r}, t \leftarrow \vec{r}_\star, t_\star)\, P(\vec{r}_\star, t_\star)\, dt_\star\, d^3\vec{r}_\star$$

where the $N$ sources are distributed according to the probability distribution $P(\vec{r}, t)$, and we assumed that they all share the same injection spectrum $q_0(E)$

- Thanks to this approach, we are now able to compute the spread of the flux around its average value $\longrightarrow$ to develop a statistical model for the fluctuations $\delta I_\alpha = I_\alpha - I$

$$\langle \delta I_\alpha(\vec{r}, t) \delta I_\alpha(\vec{r}', t') \rangle_\alpha = q_0(E)^2 N \langle \mathcal{G}(\vec{r}, t \leftarrow \vec{r}_\star, t_\star) \mathcal{G}(\vec{r}', t' \leftarrow \vec{r}_\star, t_\star) \rangle - \overbrace{\cancel{\frac{\langle I \rangle \langle I' \rangle}{N}}}^{\to 0}$$

# The Green's function for the diffusion equation

- To construct the Green function, we take the Fourier transform w.r.t the spatial variable $\vec{r}$, recalling that $\mathcal{F}[\delta^{(3)}(\vec{r} - \vec{r}_\star)] = \mathrm{e}^{-i\vec{k}\cdot\vec{r}_\star}$:

$$\frac{\partial}{\partial t}\tilde{\mathcal{G}}(\vec{k}, t \leftarrow \vec{r}_\star, t_\star) + Dk^2\tilde{\mathcal{G}}(\vec{k}, t \leftarrow \vec{r}_\star, t_\star) = \mathrm{e}^{-i\vec{k}\cdot\vec{r}_\star}\delta(t - t_\star)$$

being subject to the initial condition $\tilde{\mathcal{G}}(\vec{k}, 0 \leftarrow \vec{r}_\star, t_\star) = 0$.

- After multiplying by $\mathrm{e}^{Dk^2 t}$ we obtain

$$\frac{\partial}{\partial t}\left[\mathrm{e}^{Dk^2 t}\tilde{\mathcal{G}}(\vec{k}, t \leftarrow \vec{r}_\star, t_\star)\right] = \mathrm{e}^{-i\vec{k}\cdot\vec{r}_\star + Dk^2 t}\delta(t - t_\star)$$

- This equation is easily solved, and we find that the Fourier transform of the Green's function reads

$$\tilde{\mathcal{G}}(\vec{k}, t) = \mathrm{e}^{-i\vec{k}\cdot\vec{r}_\star - Dk^2 t}\int_0^t \mathrm{e}^{Dk^2 t'}\delta(t' - t_\star)dt' = \begin{cases} 0 & t < t_\star \\ \mathrm{e}^{-i\vec{k}\cdot\vec{r}_\star - Dk^2(t - t_\star)} & t > t_\star \end{cases}$$

The upper limit $t$ of this integral express causality: the solution at time $t$ depends only on causes lying in its past $t_\star < t$

# The Green's function for the diffusion equation

- Finally, taking the inverse Fourier transfor in the $\vec{k}$ variables we recover the Green's function that we are looking for

$$\mathcal{G}(\vec{r}, t \leftarrow \vec{r}_\star, t_\star) = \frac{\Theta(t - t_\star)}{(2\pi)^3} \int \mathrm{e}^{i\vec{k}\cdot(\vec{r}-\vec{r}_\star)} \mathrm{e}^{-Dk^2(t-t_\star)} d^3\vec{k}$$

- We recognize the integral as the (inverse) Fourier transform of a Gaussian, thereby:

$$\mathcal{G}(\vec{r}, t \leftarrow \vec{r}_\star, t_\star) = \frac{\Theta(\tau)}{(4\pi D\tau)^{3/2}} \mathrm{e}^{-\frac{d^2}{4D\tau}}$$

where $\tau = t - t_\star$ and $\vec{d} = \vec{r} - \vec{r}_\star$

- Thus an initial Gaussian retains a Gaussian form, with its squared width spreading linearly with $t \to$ linear growth of variance is typical of diffusing probabilistic processes
- A striking property of this solution is that $I > 0$ everywhere for any finite $t > 0$, no matter how small $\to$ violate Special Relativity $\longrightarrow \Theta(c\Delta t - d)$

# The Green's function for the diffusion equation

- We derived the free-space Green's function, namely without imposing any boundary condition in $z$ (no $H$ in the Green's function)
- In fact, we require that the Green's function vanishes on the boundaries of our diffusion region $z = \pm H$ and for that we can apply the method of images
- We can use a set of image charges with coordinates:

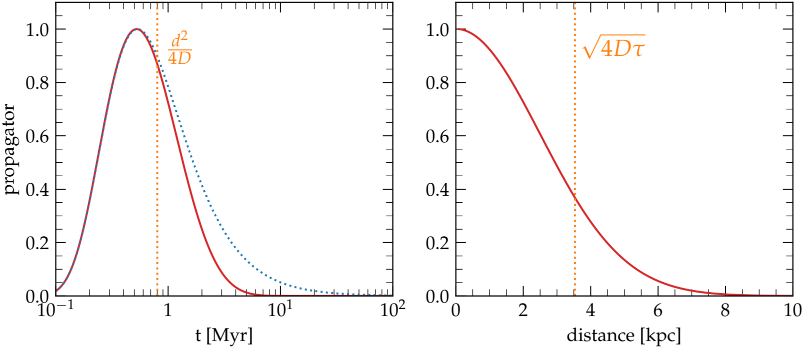
$$\vec{r}'_{\star,n} = \left( \begin{array}{c} x_\star \\ y_\star \\ 2Hn + (-1)^n z_\star \end{array} \right)$$

and compute the Green's functions as

$$\mathcal{G}_H(\vec{r}, t \leftarrow \vec{r}_\star, t_\star) = \sum_{n=-\infty}^{\infty} (-1)^n \mathcal{G}_{\text{free}}(\vec{r}, t \leftarrow \vec{r}'_{\star,n}, t_\star)$$

- It is easy to check that $\mathcal{G}_H(x, y, z = \pm H, t \leftarrow \vec{r}_\star, t_\star) = 0$

# The Green's function for the diffusion equation




- The normalized Green's function as a function of burst age (left) and distance (right) for $D = 10^{29}$ cm$^2$/s
- Higher energy CRs (larger $D$) reach us before the low-energy ones

# The mean of the Galactic intensity

- We consider the simple case in which we are sitting in the center of the disc and we limit ourselves to the proton case (diffusion only, no spallation or energy losses)
- The probability distribution for a constant rate and homogeneous injection:

$$P = \frac{1}{N} \frac{\mathcal{R}}{\pi R_d^2}$$

- We make use of this to compute the mean Galactic intensity as

$$I = \frac{q_0(E)\mathcal{R}}{\pi R_d^2} \int_0^\infty d\tau \int_0^{R_d} 2\pi r_\star dr_\star \frac{\mathrm{e}^{-\frac{r_\star^2}{4D\tau}}}{(4\pi D\tau)^{3/2}} \sum_{n=-\infty}^{+\infty} (-1)^n \mathrm{e}^{-\frac{(2nH)^2}{4D\tau}}$$

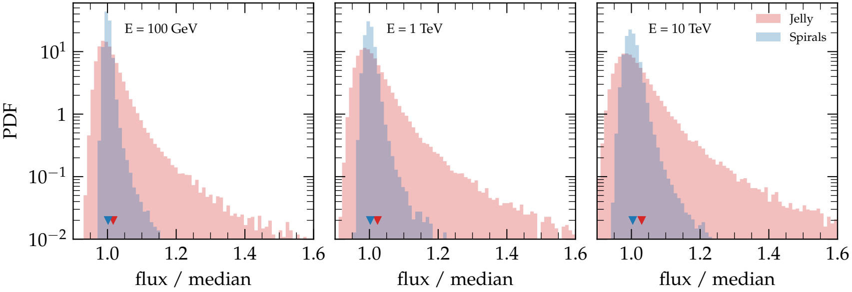
where sources are assumed to lie on disc $z_\star = 0$ and the observer sits at the Galactic Centre.

- Performing the integrals, first on $\tau$ and then on $r_\star$, one easily obtains

$$I = \frac{q_0\mathcal{R}}{2\pi D R_d} \sum_{n=-\infty}^{+\infty} (-1)^n \left[ \sqrt{1 + \left(\frac{2nH}{R_d}\right)^2} - \sqrt{\left(\frac{2nH}{R_d}\right)^2} \right] \xrightarrow{H \ll R_d} \frac{q_0(E)\mathcal{R}}{2\pi R_d^2} \frac{H}{D(E)}$$

- In the mean field limit we recover the solution of the diffusion equation

# The variance of the Galactic intensity




- The variance of the flux $\longrightarrow \infty$ (formally due to the occurrence of sources arbitrarily close: $n \to \infty$ as $r_s \to 0$)
- This remains true also in numerical simulations including the spiral structure of the Milky Way.
- Heavy-tail PDF $\to$ mean $\neq$ median (problematic for codes based on the mean field approach!)
- Heavy-tail PDF: What is the significance of any excess with respect to the mean value?

# The Green's function for leptons

- The equation of interest for leptons, assuming steady-state, is

$$\cancelto{0}{\frac{\partial I_e}{\partial t}} - D(E)\nabla^2 I_e(E) - \frac{\partial}{\partial E}\left[b(E) I_e\right] = Q(\vec{r}, E, t)$$

where $b(E) = dE/dt$ contains the energy losses.

- It is convenient to introduce a new variable:

$$\tilde{t} = 4\int_E^\infty \frac{D(E')}{b(E')} dE' \longrightarrow \frac{d}{d\tilde{t}} = -\frac{b(E)}{4D(E)}\frac{d}{dE}$$

which we utilise to obtain

$$-D(E)\nabla^2 I_e(E) + 4\frac{D(E)}{b(E)}\frac{\partial}{\partial \tilde{t}}\left[b(E) I_e(E)\right] = Q(\vec{r}, E, t)$$

and after re-ordering we obtain the usual diffusion equation for the new variable $\tilde{I} = b(E)I(E)$:

$$\frac{\partial}{\partial \tilde{t}}\left[b(E) I_e(E)\right] - \frac{1}{4}\nabla^2\left[b(E) I_e(E)\right] = \tilde{Q}(\vec{r}, E, t)$$

- thereby the Green's function in the new variables (and introducing $\lambda^2 = \tilde{t} - \tilde{t}_\star$) is

$$\tilde{\mathcal{G}}(\vec{r}, \tilde{t} \leftarrow \vec{r}_\star, \tilde{t}_\star) = \frac{\Theta(\lambda^2)}{(\pi\lambda^2)^{3/2}} \mathrm{e}^{-\frac{d^2}{\lambda^2}}$$

# The Green's function for leptons

- ▷ The Green's function for the time dependent solution can be expressed in terms of the steady-state solution as:

$$\mathcal{G}(\vec{r}, t, E \leftarrow \vec{r}_\star, t_\star, E_\star) = \delta(\Delta t - \tau) \frac{\tilde{\mathcal{G}}(\vec{r}, \tilde{t} \leftarrow \vec{r}_\star, \tilde{t}_\star)}{|b(E)|} = \frac{1}{|b(E)|} \frac{\delta(\Delta t - \tau)}{(\pi \lambda^2)^{3/2}} \mathrm{e}^{-\frac{d^2}{\lambda^2}}$$

  that contains the propagation scale (known also as Syrovatskii variable):

$$\lambda^2(E, E_\star) = 4 \int_E^{E_\star} dE' \frac{D(E')}{b(E')}$$

- ▷ the loss time is defined as

$$\tau(E, E_\star) = \int_E^{E_\star} \frac{dE'}{b(E')}$$

  which corresponds to the average time during which the energy of a particle decreases from $E_\star$ to $E$ because of losses.

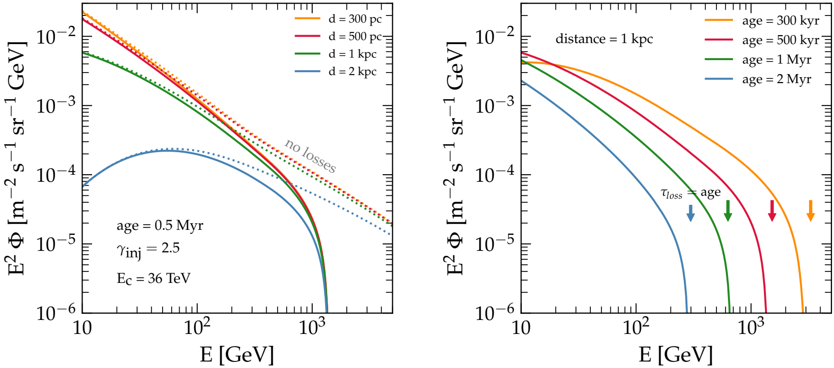
- ▷ Thus, the particles that we observe with energy $E$ have been injected with energy $\tilde{E}$ at a time $\Delta t$ such that:

$$\boxed{\tau(E, \tilde{E}) = \Delta t}$$

- ▷ This condition sets a maximum energy as $\tau(E_{\rm max}, \infty) = t_{\rm age}$ which, in the Thomson limit, reads:

$$E_{\rm max} = \frac{E_0^2}{b_0 t_{\rm age}} \simeq 400 \text{GeV} \left( \frac{t_{\rm age}}{\text{Myr}} \right)^{-1}$$

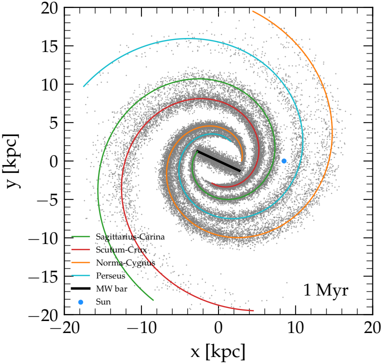
# The Green's function for the diffusion-losses equation




$$I_e(d,t,E) = \frac{1}{|b(E)|}\int dE_\star \delta[t-\tau(E,E_\star)]\,\frac{e^{-\frac{d^2}{\lambda^2}}}{(\pi\lambda^2)^{3/2}}\,q_{\rm burst}(E_\star)$$

# Galactic factories of cosmic-ray leptons

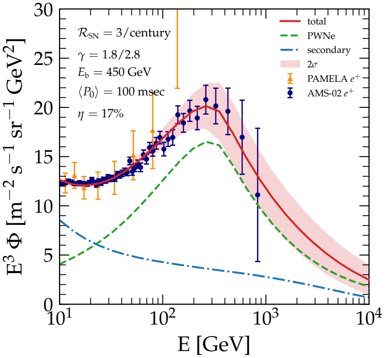
Lee, ApJ, 1979; Ptuskin+, APPh 2006; Delahaye+, A&A 2010; Mertsch, JCAP 2011; Blasi & Amato 2011; Mertsch, JCAP 2018




$$n(t_\odot, E, \vec{r}_\odot) = \iiint dt_s\, dE_s\, d^3\vec{r}_s\, \delta(\Delta t - \Delta\tau)\mathcal{G}_{\vec{r}}(E, \vec{r}_\odot \leftarrow E_s, \vec{r}_s)\mathcal{Q}(t_s, E_s, \vec{r}_s).$$

# Pulsars as positron galactic factories

Hooper+, JCAP 2009; Grasso+, APh 2009; Delahaye+, A&A 2010; Blasi & Amato 2011; Manconi+, PRD 2020; Evoli, Amato, Blasi & Aloisio, PRD 2021

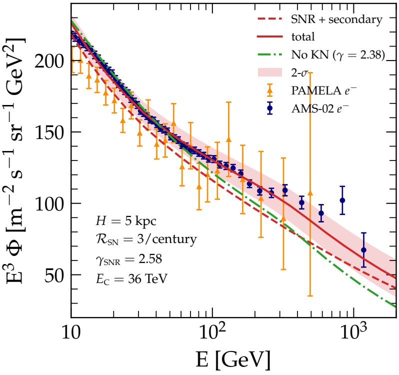


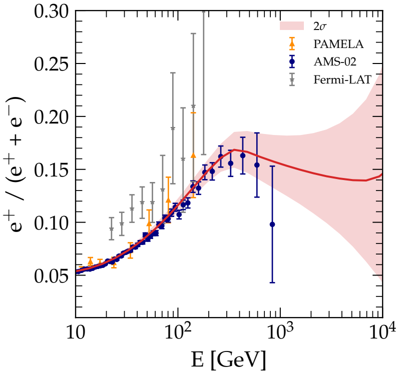

$$Q_0(t) e^{-E/E_c(t)} \times \begin{cases} (E/E_b)^{-\gamma_L} & E < E_b \\ (E/E_b)^{-\gamma_H} & E \geq E_b \end{cases}$$

Shaded areas: 2-sigma fluctuations due to cosmic variance

# The electron spectrum from SNRs

Evoli, Amato, Blasi & Aloisio, PRD 2021




- Electrons injected by SNRs with a power law with an intrinsic cutoff at $\sim 40$TeV (cooling dominated)
- Electrons require a spectrum steeper than protons by $\sim 0.3 \to$ puzzling!
- The only aspect that is different between e$^-$ and $p$ is the loss rate $\to$ negligible inside the sources unless $B$ is very strongly amplified [Diesing & Caprioli, PRL 2020; Cristofari+, A&A 2021]
- Watch at the positron fraction! [Schroer+, in preparation]

Lecture II (extra)

# Non-linear cosmic ray transport

Skilling, ApJ 1971; Kulsrud & Cesarsky, ApJL 1971; Wentzel, ARAA 1974

- Spatial diffusion tends to reduce the CR momentum forcing them to move at the wave speed $v_A$ [Kulsrud's book (2004)]:

$$\frac{dP_{\rm CR}}{dt} = -\frac{n_{\rm CR} m (v_D - v_A)}{\tau} \longrightarrow \text{Waves}$$

- If CR stream faster than the waves ($v_D > v_A$) the net effect of diffusion is to make waves grow: this process is known as self-generation of waves (notice that self-generated waves are such $k \sim r_L$)

- Waves are amplified by CRs through streaming instability:

$$\Gamma_{\rm CR} = \frac{16\pi^2}{3} \frac{v_A}{kW(k)B_0^2} \left[ v(p) p^4 \frac{\partial f}{\partial z} \right] \propto \frac{P_{\rm CR}(>p)}{P_{\rm B}} \frac{v_A}{H} \frac{1}{kW(k)}$$

- and are damped by wave-wave interactions that lead the development of a turbulent cascade (NLLD):

$$\Gamma_{\rm NLLD} = (2c_k)^{-3/2} k v_A (kW)^{1/2}$$

What is the typical scale/energy up to which self-generated turbulence is dominant?

# Non-linear cosmic ray transport

Blasi, Amato & Serpico, PRL, 2012

- Transition occurs at scale where external turbulence equals in energy density the self-generated turbulence:

$$W_{\text{ext}}(k_{\text{tr}}) = W_{\text{CR}}(k_{\text{tr}})$$

where $W_{\text{CR}}$ corresponds to $\Gamma_{\text{CR}} = \Gamma_{\text{NLLD}}$

- After normalization of $W_{\text{ext}}$ is set to reproduce the CR flux much above the break:

$$E_{\text{tr}} = 228\,\text{GeV} \left( \frac{R_{d,10}^2 H_3^{-1/3}}{\epsilon_{0.1} E_{51} \mathcal{R}_{30}} \right)^{3/2(\gamma_p - 4)} B_{0,\mu}^{(2\gamma_p - 5)/2(\gamma_p - 4)}$$

- Applying QLT it follows:

$$D_{\text{sg}}(1\,\text{GV}) \sim \frac{c r_L}{3} \frac{1}{k W_{\text{CR}}(k)} \sim 10^{28} \text{cm}^2 \text{s}^{-1}$$

# The turbulence evolution equation

Eilek, ApJ 1979

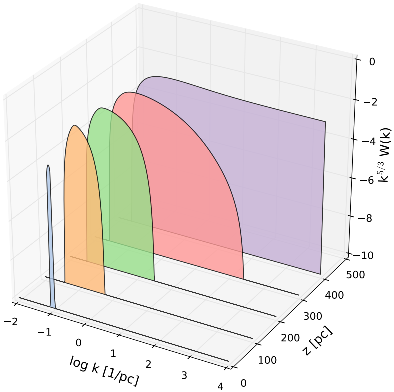
$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial k}\left[D_{kk}\frac{\partial W}{\partial k}\right] + \frac{\partial}{\partial z}\left(v_A W\right) + \Gamma_{\rm CR} W + Q(k)$$

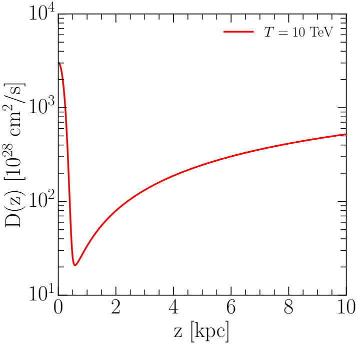
- Diffusion in $k$-space damping: $D_{kk} = c_k |v_A| k^{7/2} W^{1/2}$
- Advection of the Alfvén waves
- Waves growth due to cosmic-ray streaming: $\Gamma_{\rm CR} \propto \partial f / \partial z$
- External (e.g., SNe) source term $Q \sim \delta(z)\delta(k - k_0)$
- In the absence of the instability $\Gamma_{\rm CR} = 0$ it returns a kolmogorov spectrum: $W(k) \sim k^{-5/3}$

Non-linear evolution: turbulence and CR transport equations are now strongly coupled!

# The wave advection originates the turbulent halo

Evoli, Blasi, Morlino & Aloisio, PRL 2018

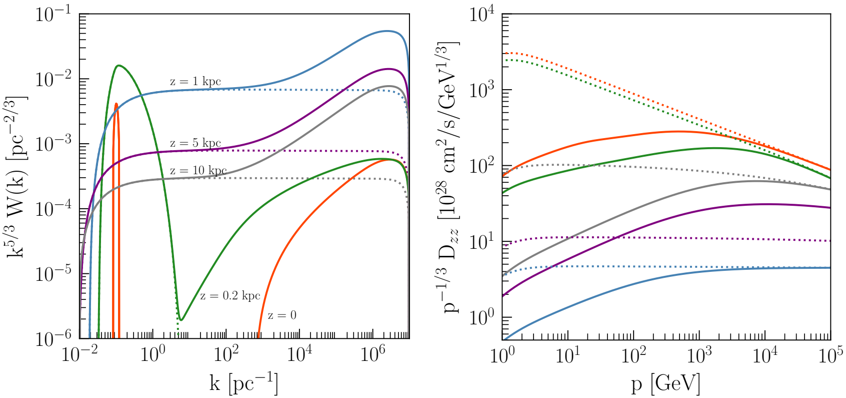



$$\tau_{\text{cascade}} = \tau_{\text{adv}} \to \frac{k_0^2}{D_{kk}} = \frac{z_{\text{peak}}}{v_A} \to z_{\text{peak}} \sim \mathcal{O}(\text{kpc})$$

# Non-linear cosmic ray transport: diffusion coefficient

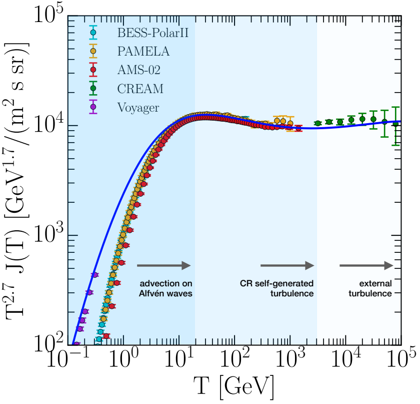
Evoli, Blasi, Morlino & Aloisio, PRL 2018




- Turbulence spectrum (left) and diffusion coefficient (right) without (dotted) and with (solid) CR self-generated waves at different distances from Galactic plane
- $D(p, z)$ is now an output of the model

# Non-linear cosmic ray transport: a global picture

Evoli, Blasi, Morlino & Aloisio, PRL 2018




Main remarks:

- Pre-existing waves (Kolmogorov) dominates above the break.
- Self-generated turbulence between 1-100 GeV.
- Voyager data are reproduced with no additional breaks (single injection slope), but due to advection with self-generated waves (+ ionization losses).
- $H$ is not predetermined here.
- None of these effects were included in the numerical simulations of CR transport before.

## Conclusions

- ▷ Cosmic Ray transport in the Galaxies is complex!
- ▷ The numerical diffusion models are certainly a big step forward, but don't forget are based on simple notions.
- ▷ The framework is still incomplete... time for new ideas!
- ▷ Need to look at all the observational constraints and model them simultaneously.

# Thank you!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