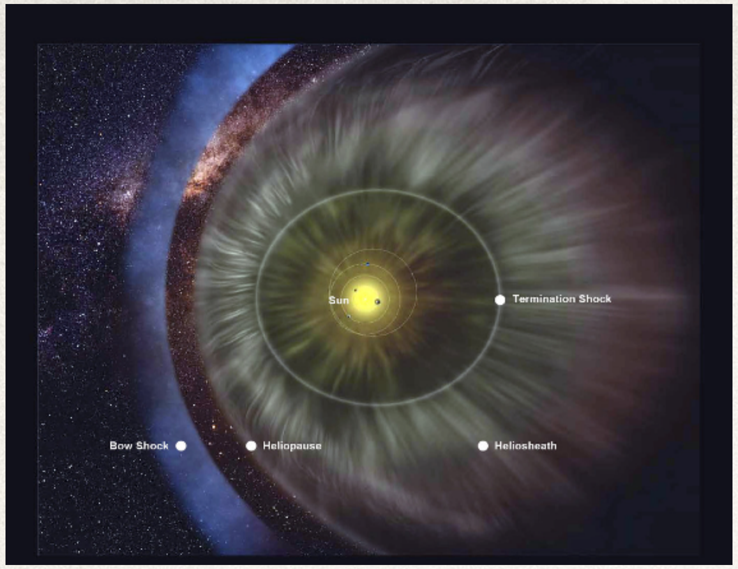
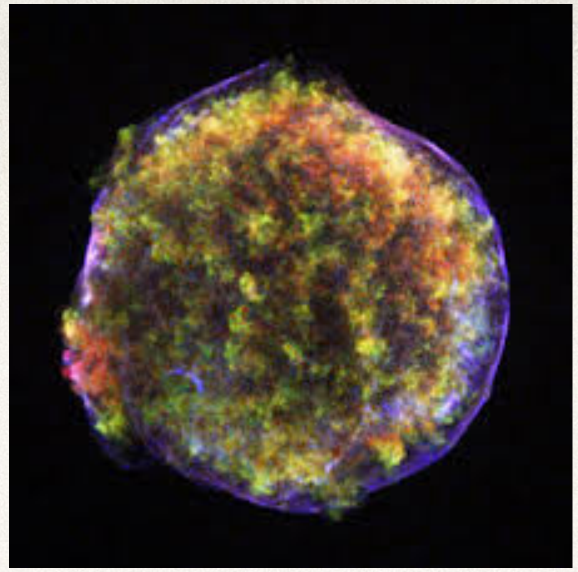
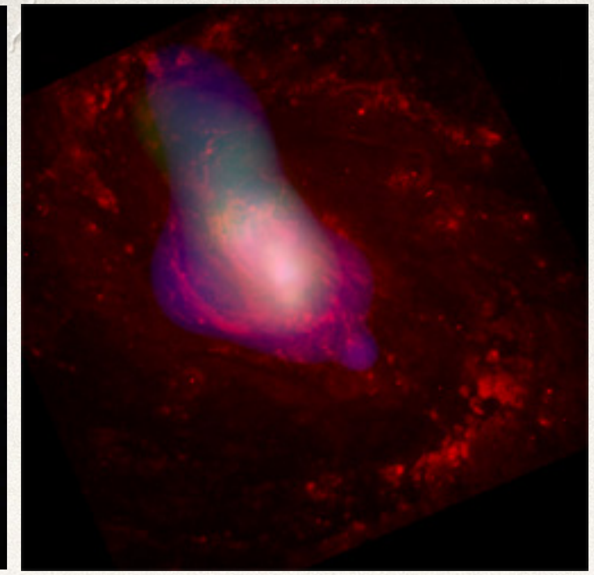
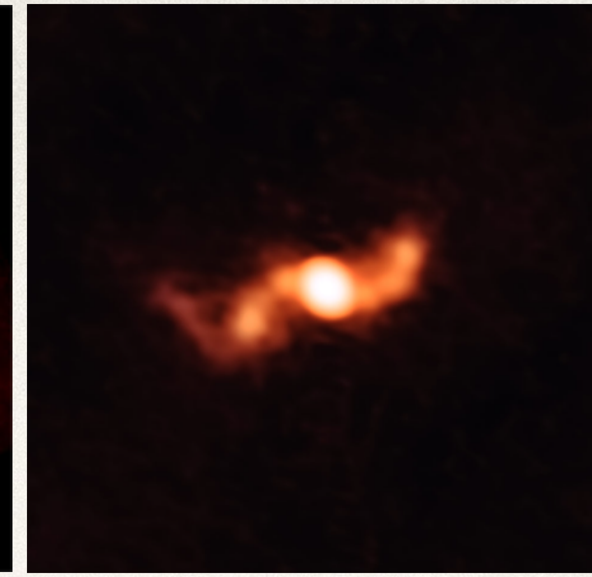
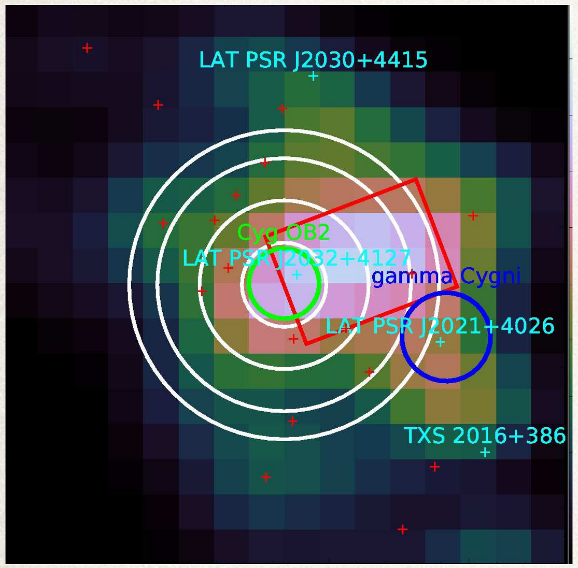
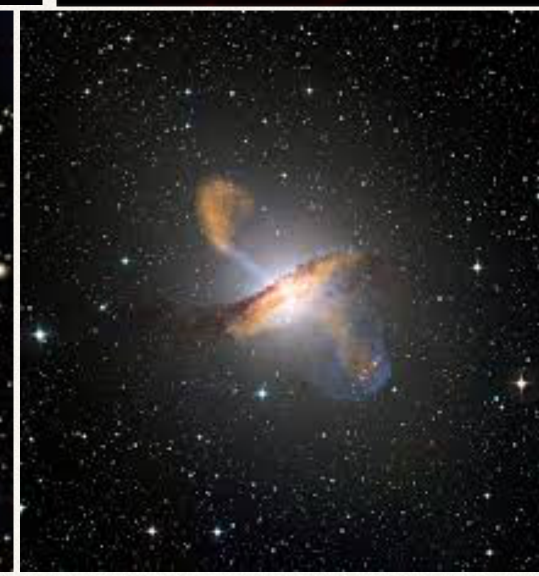



CHARGED PARTICLES IN MAGNETIC FIELDS AND COSMIC RAY TRANSPORT

LECTURES 1&2

Pasquale Blasi - Gran Sasso Science Institute

Dates: 24-25 June 2022

GRAN SASSO SCIENCE INSTITUTE
SCHOOL OF ADVANCED STUDIES
Scuola Universitaria Superiore

What is this course for?

THE NON-THERMAL ACTIVITY OF THE UNIVERSE WE LIVE IN CAN ALL BE REDUCED TO THREE QUESTIONS, THE BIG Q's:

- How does Nature accelerate particles, typically from the thermal pool?
- How do these non thermal particles propagate in the complex environment inside the source or/and from the source to us?
- How do such particles radiate?

THE FIRST TWO Q'S DEAL WITH THE FUNDAMENTAL ISSUE OF PARTICLE TRANSPORT IN E-M FIELDS - THE TOPIC OF THESE LECTURES

THESE ISSUES APPLY IN THE SAME WAY TO THE SOLAR SYSTEM, TO SUPERNOVAE, TO AGN, GRBs, CLUSTERS OF GALAXIES, …

What is this course for?

The school is focused, on purpose, on **THE FOUNDATIONS**… the pillars on which the field is built

Some such foundations were laid down a long time ago, some are being laid down right now…

It is crucial to understand the solid and the weaker points of these foundations, so as to improve on them or revise them

Some of the things we are doing now (I will talk about some of them) might become foundational later or perhaps destroy some of the current foundations, or perhaps will be forgotten…

Plan of the First lecture

- Why is it so important to get transport right?
- Derivation of the diffusive motion (simple)
- Derivation of the diffusive motion from Vlasov equation
- Transport equation in pitch angle

Plan of the Second lecture

- From pitch angle to spatial diffusion
- Applications (DSA, Galaxy transport) - short
- Elements of perpendicular transport
- simulations of transport in synthetic and MHD turbulence
- Elements of Self-Generation of perturbations (*see course by Marcowith*)
- *Advanced topic 1: self-generation in DSA*
- *Advanced topic 2: self-generation around sources*
- *Advanced topic 3: self-generation in and around the galaxy*

How do we know that something nontrivial must be going on?

- **Most of the universe is in a plasma state: the only electric fields you get are induced by plasma motion: δE~(V/c)B**

- **Only electric fields can change the particle energy**

$$\frac{dp}{dt} = q\frac{V}{c}B \rightarrow \frac{dE}{dt} \approx qVB$$

If the electric field could stay coherent over a scale R and the particles were moving at c then

$$E_{max} \approx q\frac{V}{c}BR$$

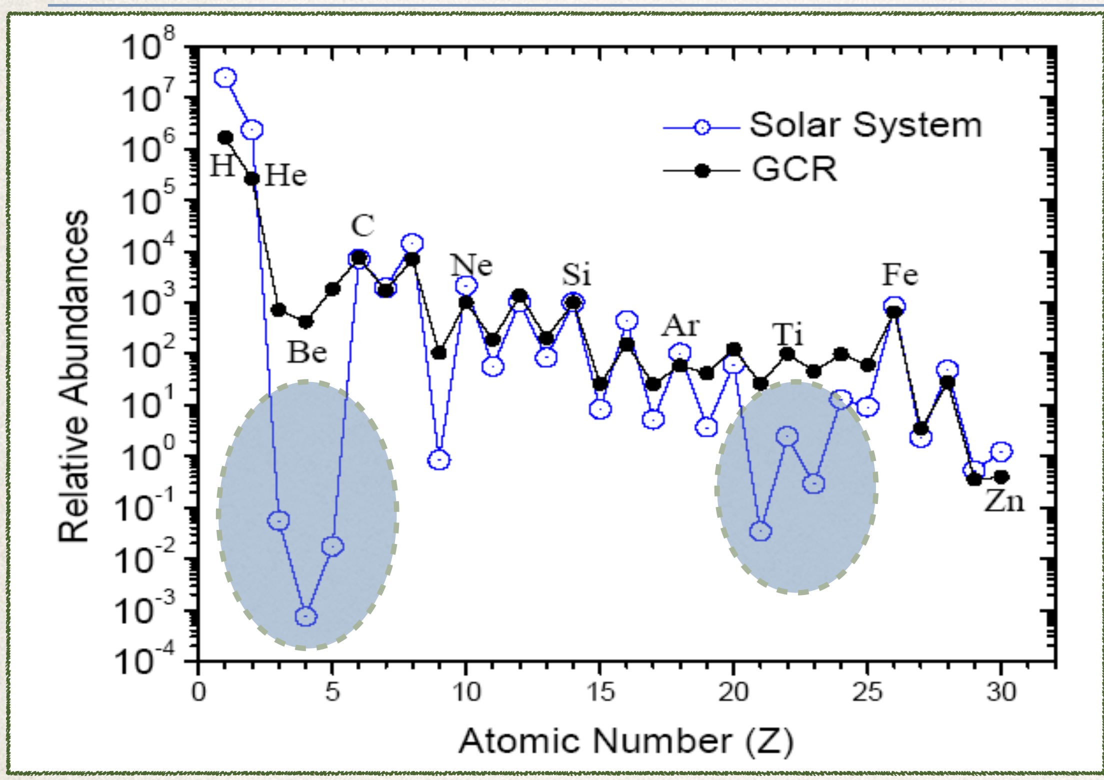
Hillas Criterion

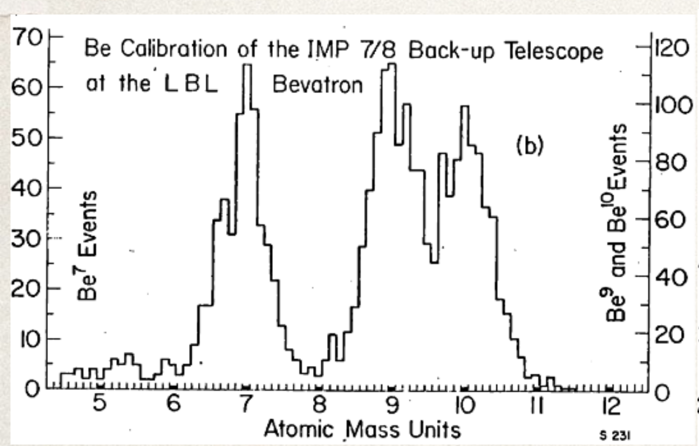
But in a plasma this does not usually happen (unless some specific conditions are fulfilled)

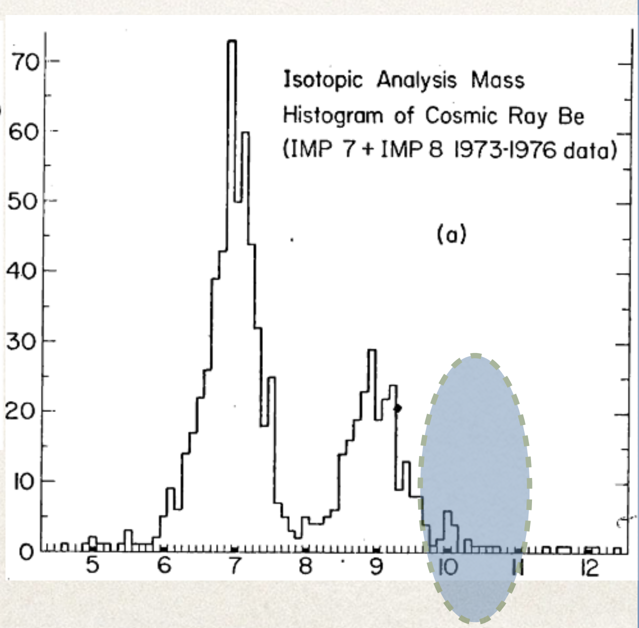
Hence: **NEED TO STAY IN THE ACCELERATION REGION MUCH LONGER THAN R/c**

HOW DO WE KNOW THAT SOMETHING NONTRIVIAL MUST BE GOING ON?

See Course by C. Evoli







Elements such as B, Be, Li are not copiously produced in the Big Bang: the universe became cold and not dense enough too quickly for these elements to be synthesised

They are formed but equally well destroyed in stellar nucleosynthesis

Hence in the ISM they are exceedingly rare…yet not rare at all in the cosmic radiation!

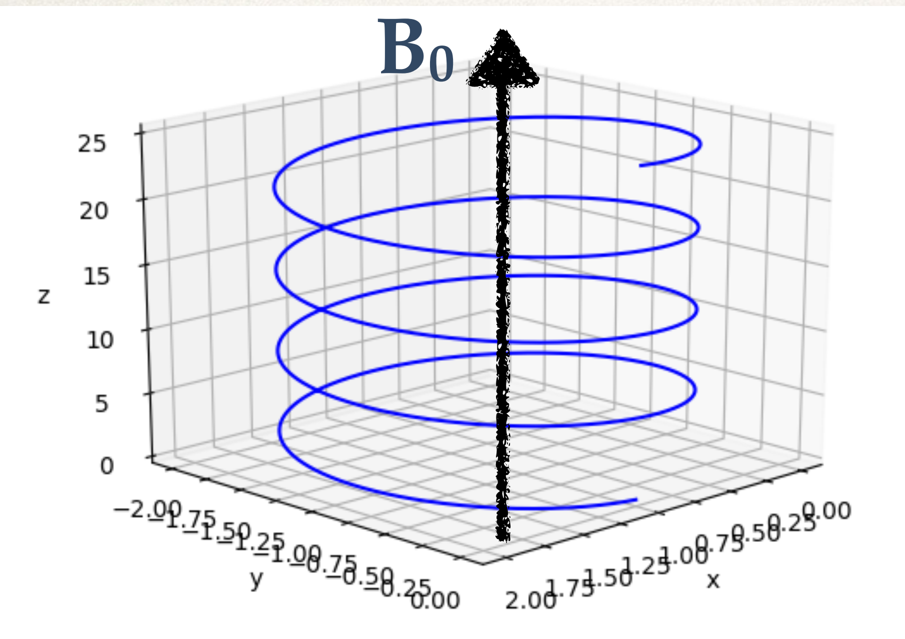
***SPALLATION**: nuclear fission of heavier elements through collisions:*

$\sigma_{sp}(A) \approx 45 A^{0.7}$ mb $\quad \tau_{sp} \approx [n_d(h/H)c\sigma_{sp}]^{-1} \approx 80 H_4 A_{12}^{-0.7}$ Myr

^{10}Be is unstable with half time of 1.4 Myr. Its abundance is related to the time that cosmic rays spend in the Galaxy before escape

BOTH TIME SCALES SUGGEST CONFINEMENT TIME THAT EXCEED THE BALLISTIC TIME BY MANY ORDERS OF MAGNITUDE

Charged particles in an ordered B-field



In general the equation of motion of a particle in an electromagnetic field is

$$\frac{d\vec{p}}{dt} = q\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$$

Given the absence of regular electric fields we will limit ourselves first to the case where only B is present

*If only B is present **the particle energy cannot change!***

$$m\gamma \frac{dv_x}{dt} = q\frac{v_y}{c}B_0$$

$$m\gamma \frac{dv_y}{dt} = -q\frac{v_x}{c}B_0$$

$$\rightarrow \quad m\gamma \frac{d^2 v_{x,y}}{dt^2} = -\left(\frac{qB_0}{mc\gamma}\right)^2 v_{x,y} \equiv -\Omega^2 v_{x,y} \qquad \Omega = \frac{qB_0}{mc\gamma}$$

Gyration Frequency

$$\frac{dv_z}{dt} = 0 \rightarrow p_z = p_{||} = constant = p\mu \qquad \mu = cos(\theta)$$

Charged particle in an ordered B-field

The solution can be written as:

$$v_x(t) = A\ cos(\Omega t) + B\ sin(\Omega t) \qquad v_y(t) = -A\ sin(\Omega t) + B\ cos(\Omega t)$$

where A and B satisfy the initial conditions that

$$v_x(t=0) = A \equiv v_\perp\ cos(\phi) \qquad v_y(t=0) = B \equiv v_\perp\ sin(\phi)$$

Hence:

$$v_x(t) = v_\perp\ [cos(\phi)cos(\Omega t) + sin(\phi)sin(\Omega t)] \ = \ v_\perp cos(\phi - \Omega t)$$

$$v_y(t) = v_\perp\ [-cos(\phi)sin(\Omega t) + sin(\phi)cos(\Omega t)] \ = \ v_\perp sin(\phi - \Omega t)$$

The unperturbed motion of the particle is period in the xy plane and rectilinear uniform in the z direction with

$$v_z = v_{||} = v\mu \rightarrow \mu = constant$$

Motion of a Charged particle in B=B$_0$+δB

For simplicity let us consider the case of a perturbation that only propagates along the ordered magnetic field $\vec{B}_0 = B_0\hat{z}$ and only having components along x and y axes.

It remains true that in the absence of an electric field the energy of the particle remains constant. In fact the perturbations can also carry an electric field, but its effect is subdominant and in first approximation the particle energy can be assumed to be constant (see discussion later).

The equation of motion of the particle is:

$$m\gamma \frac{d\vec{v}}{dt} = \frac{q}{c}\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ \delta B_x & \delta B_y & B_0 \end{pmatrix} \approx \frac{q}{c}\begin{pmatrix} v_y B_0 \\ -v_x B_0 \\ v_x \delta B_y - v_y \delta B_x \end{pmatrix}$$

$\delta B \ll B_0$

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Motion of a Charged particle in B=B$_0$+δB

$$m\gamma\frac{dv_z}{dt} = \frac{q}{c}\left[v_x(t)\delta B_y - v_y(t)\delta B_x\right] \Rightarrow m\gamma v\frac{d\mu}{dt} = \frac{q}{c}v_\perp\left[cos(\phi-\Omega t)\delta B_y - sin(\phi-\Omega t)\delta B_x\right]$$

Let us assume that the perturbed field is circularly polarized: **$\delta B_y = \pm i\delta B_x$**

$$\delta B_y = \exp\left[i(kz-\omega t+\psi)\right] = cos(kz-\omega t+\psi) + i\ sin(kz-\omega t+\psi)$$

$$\delta B_x = \mp\delta B_y = \mp i\ cos(kz-\omega t+\psi) \pm sin(kz-\omega t+\psi)$$

→ Take the Real Part →

$$\delta B_x = \pm\delta B sin(kz-\omega t+\psi)$$

$$\delta B_y = \delta B cos(kz-\omega t+\psi)$$

Clearly the mean value of the fluctuations over realisations is zero!

$$m\gamma v\frac{d\mu}{dt} = \frac{q}{c}v_\perp\delta B\left[cos(\phi-\Omega t)cos(kz-\omega t+\psi) \mp sin(\phi-\Omega t)sin(kz-\omega t+\psi)\right]$$

⇓

$$m\gamma v\frac{d\mu}{dt} = \frac{q}{c}v_\perp\delta B\ cos(\phi-\Omega t \pm kz \mp \omega t \pm \psi)$$

Motion of a Charged particle in B=B$_0$+δB

Having in mind that the perturbations are something similar to Alfven waves, for which the dispersion relation is $\omega = k\, v_A$ one can compare the two terms kz and ωt

$$\frac{kz}{\omega t} \approx \frac{kv\mu t}{kv_A t} \sim \frac{v\mu}{v_A} \gg 1$$

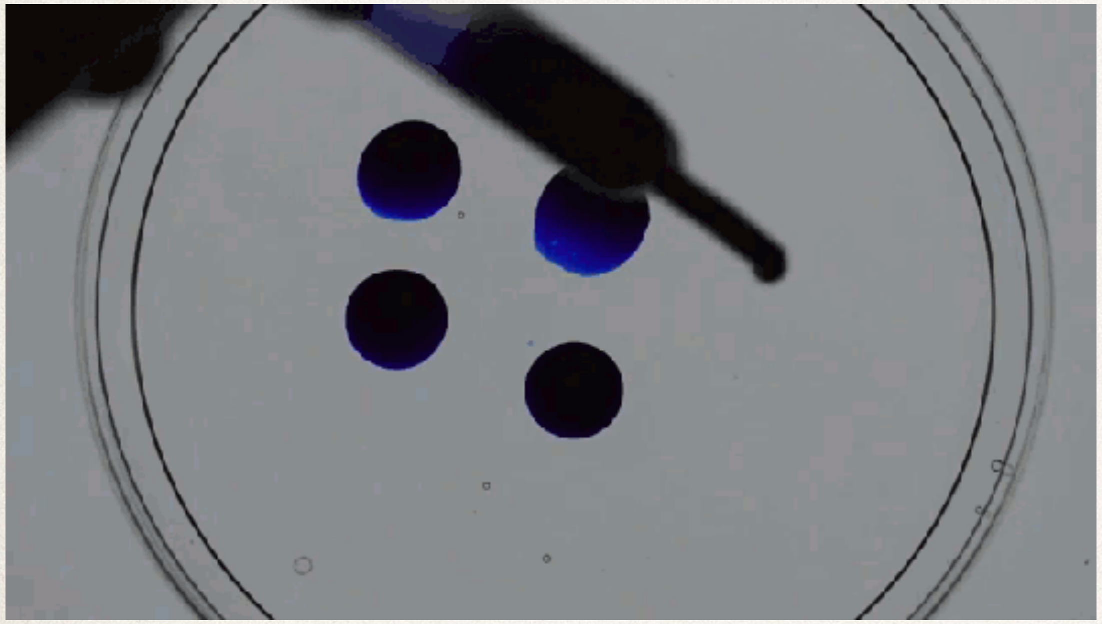
unless $\mu \ll v_A/v$

Notice that $v_A = \frac{B}{\sqrt{4\pi\rho}} = 2 \times 10^5 n_1^{-1/2} B_\mu \text{ cm/s}$

Neglecting the term ωt with respect to kz is equivalent to assume that we are sitting in the reference frame in which the waves are stationary. In turn this implies that there is no electric field carried by the perturbations. We will comment later on the implications of this assumption for particle transport.

$$\frac{d\mu}{dt} = \frac{qB_0}{mc\gamma}(1-\mu^2)^{1/2}\frac{\delta B}{B_0}\, cos\left[\phi \pm \psi \pm (kv\mu \mp \Omega)t\right] \rightarrow \langle \Delta\mu \rangle = 0$$

Motion of a Charged particle in B=B$_0$+δB



In diffusion processes it is the case that there are quantities that have zero mean value and yet the mean value of the square of the same value is not zero

For instance this is the case for ink in water… the mean value of the position of molecules is zero (if symmetric) but the stain of ink gets larger… proportionally to time

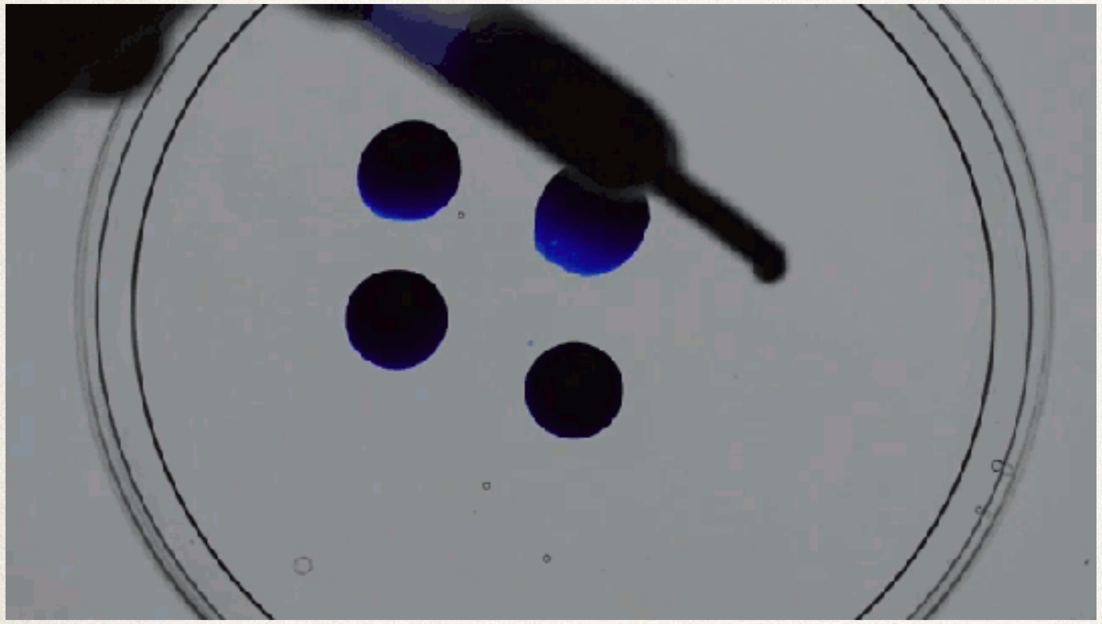
Let us explore whether this may be the case for particles in perturbed magnetic fields…

$$\langle \Delta\mu\Delta\mu \rangle = \left(\frac{qB_0}{mc\gamma}\right)^2 (1-\mu^2)\left(\frac{\delta B}{B_0}\right)^2 \int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^T dt \int_0^T dt' \cos\left[\phi \pm \psi \pm (kv\mu \mp \Omega)t\right] \times$$
$$\times \cos\left[\phi \pm \psi \pm (kv\mu \mp \Omega)t'\right]$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi cos(\phi - a) cos(\phi - b) = \frac{1}{2} cos(a-b)$$

$$\langle \Delta\mu\Delta\mu \rangle = \frac{1}{2}\Omega^2(1-\mu^2)\left(\frac{\delta B}{B_0}\right)^2 \int_0^T dt \int_0^T dt' \cos\left[(kv\mu \mp \Omega)(t-t')\right]$$

Motion of a Charged particle in B=B$_0$+δB



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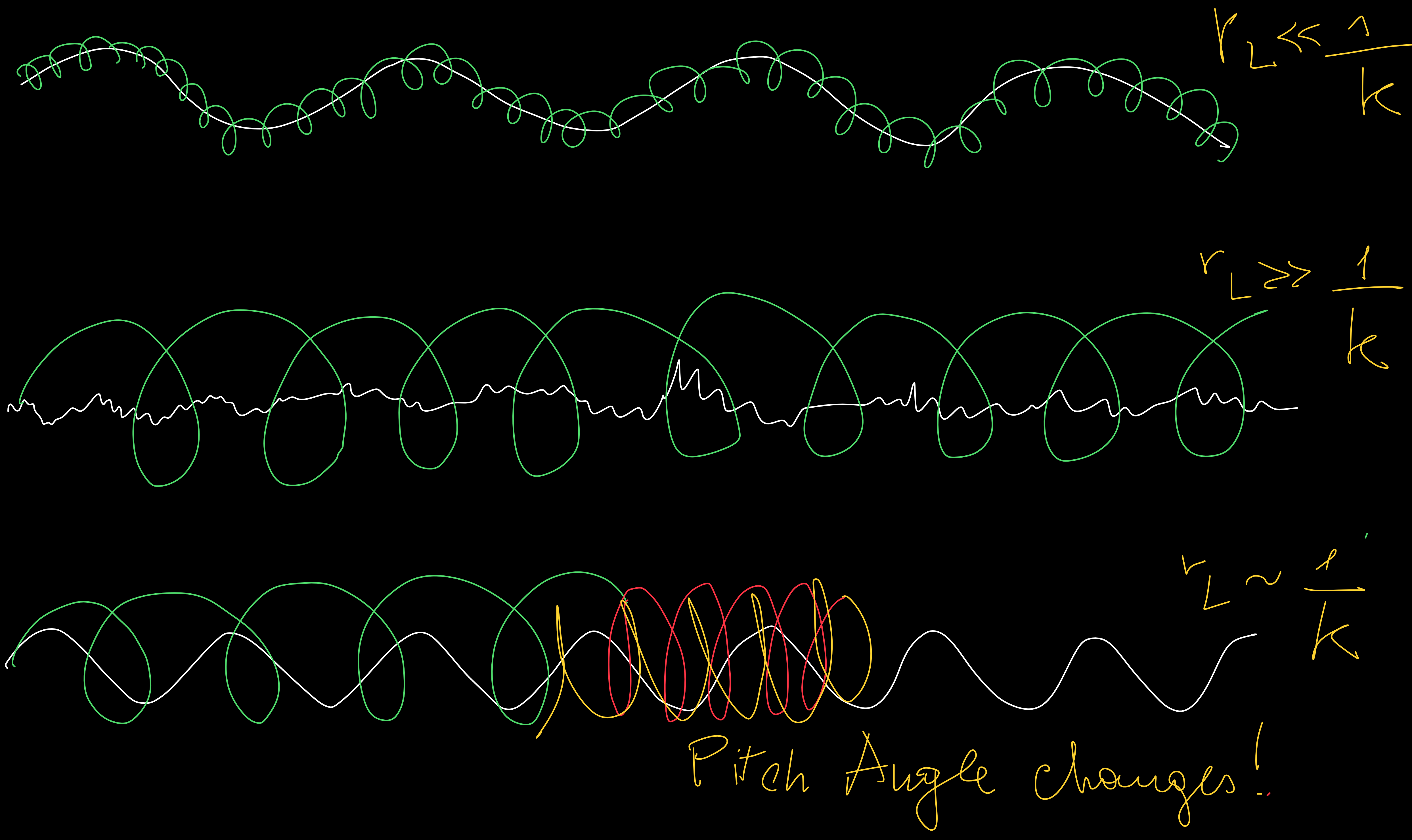
Motion of a Charged particle in B=B$_0$+δB

$$\langle \Delta\mu\Delta\mu \rangle = \pi\Omega^2(1-\mu^2)\left(\frac{\delta B}{B_0}\right)^2 T\delta(kv\mu - \Omega) =$$

$$= \pi\Omega(1-\mu^2)\left(\frac{\delta B}{B_0}\right)^2 Tk_{res}\delta\left(k - k_{res}\right)$$

- The mean value of the square of the pitch angle variation is proportional to the time lapse (Diffusion)
- This is only true when the resonance condition is fulfilled: **$k=k_{res}=\Omega/v\mu$**
- Notice that when the pitch angle is close to 90^o the wavenumber tends to infinity (possibly no waves to cause scattering)
- The scattering depends on the power available at the resonant scale and is proportional to the gyration frequency


$r_L \ll \frac{1}{k}$
$r_L \gg \frac{1}{k}$
$r_L \sim \frac{1}{k}$
Pitch Angle changes!

DIFFUSION COEFFICIENT IN THE PRESENCE OF A SPECTRUM OF WAVES

WE CAN NOW DEFINE A DIFFUSION COEFFICIENT IN THE FORM:

$$D_{\mu\mu} = \frac{1}{2}\langle\frac{\Delta\mu\Delta\mu}{\Delta t}\rangle = \frac{\pi}{2}\Omega(1-\mu^2)\left(\frac{\delta B}{B_0}\right)^2 k_{res}\delta\left(k - k_{res}\right)$$

OR IN THE CASE THAT A SPECTRUM OF WAVES IS PRESENT:

$$D_{\mu\mu} = \frac{1}{2}\langle\frac{\Delta\mu\Delta\mu}{\Delta t}\rangle = \frac{\pi}{2}\Omega(1-\mu^2)\int dk\left(\frac{\delta B(k)}{B_0}\right)^2 k_{res}\delta\left(k - k_{res}\right) =$$

$$= \frac{\pi}{2}\Omega(1-\mu^2)\mathcal{F}(k_{res}) \qquad \mathcal{F}(k_{res}) = k_{res}\left(\frac{\delta B(k_{res})}{B_0}\right)^2$$

THE QUANTITY F REPRESENTS THE DIMENSIONLESS POWER IN PERTURBATIONS AT THE RESONANT WAVENUMBER AND DETERMINES THE EFFECTIVENESS OF THE DIFFUSION PROCESS

Phenomenological considerations

Given the definition of the diffusion coefficient in pitch angle we can try to elaborate some considerations on the implications of diffusion

First, one can also define a diffusion coefficient in angle:

$$D_{\theta\theta} = \frac{1}{2}\langle\frac{\Delta\theta\Delta\theta}{\Delta t}\rangle = \frac{\pi}{2}\Omega\mathcal{F}(k_{res})$$

so that the time necessary for deflection by ~90 degrees may be estimated as:

$$\tau_{90} \sim \frac{1}{D_{\theta\theta}} \approx \frac{1}{\Omega\mathcal{F}(k_{res})}$$

Since Ω is the gyration frequency of the particle in the unperturbed magnetic field, this quantity measures how many gyrations the particle must carry out before suffering a deflection by order unity

Since by definition F<<1, typically the deflection is a slow process

PHENOMENOLOGICAL CONSIDERATIONS

DURING THIS TIME THE PARTICLE TRAVELS IN THE Z DIRECTION FOR A DISTANCE

$$\lambda_D(p) = v\tau_{90} \sim \frac{v}{D_{\theta\theta}} \approx \frac{v}{\Omega\mathcal{F}(k_{res})} = \frac{pc}{qB_0}\frac{1}{\mathcal{F}} = r_L(p)\frac{1}{\mathcal{F}(k_{res}(p))}$$

THAT PLAYS THE ROLE OF DIFFUSION PATH LENGTH OF THE PARTICLES! IT IS A MULTIPLE 1/F>>1 OF THE LARMOR RADIUS OF THE PARTICLES IN THE UNPERTURBED MAGNETIC FIELD

THIS ALLOWS US TO ESTIMATE THE DIFFUSION COEFFICIENT OF THE PARTICLES IN SPACE RATHER THAN IN ANGLE, FOLLOWING THE GENERAL DEFINITION OF A DIFFUSION COEFFICIENT:

$$D_{zz}(p) = \frac{1}{3}v\lambda_D(p) \approx \frac{1}{3}r_L(p)v\frac{1}{\mathcal{F}(k_{res}(p))} = D_{Bohm}(p)\frac{1}{\mathcal{F}(k_{res}(p))} \quad >> D_{Bohm}$$

GIVEN THE PERTURBATIVE APPROACH ADOPTED HERE, $\delta B << B_0$, THE DIFFUSION COEFFICIENT IS BOUND TO BE LARGER THAN THE BOHM DIFFUSION COEFFICIENT, WHICH IS OFTEN QUOTED AS THE LOWEST D(E) ONE CAN GET (ONE LARGE SCATTERING PER LARMOR GYRATION)

Phenomenological considerations

From other courses in this school you will learn that observations of the secondary/primary ratios and abundance of unstable isotopes, such as ^{10}Be, require that particles with energy of ~10 GeV stay in the Galaxy for about 100 Million years and the halo size is bound to be H~5 kpc

$$100Myr \approx \frac{H^2}{D(E)} \to D(E = 10GeV) \approx 7 \times 10^{28} \mathrm{cm}^2/\mathrm{s}$$

Using the expression for the diffusion coefficient derived earlier we can estimate the extent that the field needs to be perturbed to get the required diffusion

$$D_{zz}(p) \approx \frac{1}{3} r_L(p) v \frac{1}{\mathcal{F}(k_{res}(p))} \to \mathcal{F}(k_{res}(10GeV)) \approx 10^{-6} \qquad B_0 \simeq 3\mu G$$

Tiny amounts of perturbations are sufficient to have a huge impact on the motion of charged particles in magnetic fields and transform their motion from **BALLISTIC** to **DIFFUSIVE**

PHENOMENOLOGICAL CONSIDERATIONS

THIS RAISES THE CRUCIAL QUESTION OF THE ORIGIN OF THESE PERTURBATIONS, WHICH IN TURN REFLECTS ON THE ORIGIN OF THE SCATTERING OF PARTICLES

THERE ARE AT LEAST TWO ORIGINS THAT WE CAN SPECULATE UPON:

- *SELF GENERATED PERTURBATIONS (SEE COURSE BY ALEXANDRE MARCOWITH)*
- *PRE-EXISTING TURBULENCE (TYPICALLY INJECTED AT SOME LARGE SCALE AND CASCADING TOWARDS SMALLER SCALES*

ONE OR THE OTHER MAY BE THE MOST IMPORTANT DEPENDING ON WHETHER THE TRANSPORT WE ARE INTERESTED IS INSIDE AN ACCELERATOR OR IN THE GALAXY OR A GALAXY CLUSTER OR EVEN DEPENDING ON THE ENERGY OF THE PARTICLES (AT A GIVEN ENERGY ONE PROCESS MAY PREVAIL UPON THE OTHER)

FOR THE CASCADING IT IS OFTEN ASSUMED THAT THE POWER SPECTRUM RESULTING FROM THE CASCADE PROCESS IS A POWER LAW ON SCALES SMALLER THAN THE INJECTION SCALE:

$$P(k) = P_0 \left(\frac{k}{k_0}\right)^{-5/3}$$ KOLMOGOROV PHENOMENOLOGY

$$P(k) = P_0 \left(\frac{k}{k_0}\right)^{-3/2}$$ KRAICHNAN PHENOMENOLOGY

BOTH APPROACHES ARE FUNDAMENTALLY FLAWED IN THE CASE OF MHD ALFVENIC TURBULENCE IN THAT THEY NEGLECT THE ANISOTROPY OF THE CASCADE PROCESS (SEE LATER)

Phenomenological considerations

Let us specialise these considerations to the case of transport of cosmic rays in the Galaxy for which we can assume that the injection scale is $k_0 \approx \frac{1}{10pc}$

then we can estimate how much power needs to be injected at the injection scale so that the power at 10 GeV is preserved

The power spectrum is normalized as:

$$\int_{k_0}^{\infty} dk P_0 \left(\frac{k}{k_0}\right)^{-\alpha} = \frac{P_0 k_0}{\alpha - 1} = \left(\frac{\delta B_{tot}}{B_0}\right)^2 \quad \rightarrow \quad P_0 = \frac{\alpha - 1}{k_0}\left(\frac{\delta B_{tot}}{B_0}\right)^2$$

The power at the resonant scale of particles of E=10 GeV is then:

$$\mathcal{F}(k_{res}) = k_{res} P(k_{res}) = (\alpha - 1)\left(\frac{k_{res}}{k_0}\right)^{1-\alpha}\left(\frac{\delta B_{tot}}{B_0}\right)^2 \sim 10^{-6} \rightarrow \left(\frac{\delta B_{tot}}{B_0}\right)^2 \sim \frac{10^{-6}}{\alpha - 1}\left(\frac{k_{res}}{k_0}\right)^{\alpha-1} \sim$$

$\sim 3 \times 10^{-2}$ **for α=5/3 Kolmogorov phenomenology**

$\sim 3.5 \times 10^{-3}$ **for α=3/2 Kraichnan phenomenology**

Good news: the perturbative approach seems to remain valid on all scales, even at injection

One Last Point…

We have consciously made the assumptions to neglect the electric fields associated with the perturbations…clearly we need to check what we missed in doing so…

The electric fields can change the momentum of the particles… one can easily expect that the change in momentum may be of order:

$$\Delta p \sim p\frac{v_A}{c}$$

being it positive or negative depending on the relative orientation of the particle velocity and the perturbation electric field

We are again in the situation in which the motion of the particle in momentum space is diffusive: the mean value of the momentum change vanishes but not its square

$$D_{pp} = \langle \frac{\Delta p \Delta p}{T} \rangle \approx \frac{p^2}{T}\left(\frac{v_A}{c}\right)^2 \rightarrow \tau_{pp} = \frac{p^2}{D_{pp}} \simeq \left(\frac{c}{v_A}\right)^2 \tau_{90} \gg \tau_{90}$$

Diffusion in momentum space is what we call second order Fermi acceleration, but it occurs on time scales >> than diffusion in pitch angle

IS PARTICLE MOTION REALLY DIFFUSIVE?

EARLIER WE HAVE PRESENTED A SIMPLE ARGUMENT TO SHOW THAT CHARGED PARTICLES IN THE PRESENCE OF PERTURBATIONS SHOULD ACQUIRE A DIFFUSIVE MOTION. **HOW CAN WE BE SURE OF THAT? AND ANYWAY WHAT IS THE EQUATION THAT DESCRIBES SUCH A DIFFUSIVE MOTION?**

THE DYNAMICS OF AN ENSEMBLE OF CHARGED PARTICLES UNDER THE ACTION OF THE ELECTRO-MAGNETIC FIELDS PRODUCED BY THE SAME PARTICLES (PLUS THE PRE-EXISTING FIELDS) IS DESCRIBED BY THE **VLASOV EQUATION**

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{q}{c} \left(\vec{v} \times \vec{B} \right) \cdot \frac{\partial f}{\partial \vec{p}} = 0$$

WHERE f(p, x, t) IS THE PHASE SPACE DENSITY OF PARTICLES, v IS THE PARTICLE VELOCITY:

$$n(\vec{x}, t) = \int d^3\vec{p} f(\vec{p}, \vec{x}, t)$$

DENSITY OF PARTICLES AT LOCATION x AT TIME t

HERE WE ARE ASSUMING A PRIORI THAT THERE ARE NO LARGE SCALE ELECTRIC FIELDS AND THAT EVEN THE EFFECTS OF THE SMALL SCALE PERTURBED ELECTRIC FIELDS ARE NEGLIGIBLE (WE WILL COMMENT ON THAT LATER)

VLASOV EQUATION AND CR TRANSPORT

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{q}{c}\left(\vec{v} \times \vec{B}\right) \cdot \frac{\partial f}{\partial \vec{p}} = 0$$

LET US INVESTIGATE HOW THE VLASOV EQUATION REACTS TO PERTURBATIONS:

$$f = f_0 + \delta f \qquad \langle \delta f \rangle = 0$$

$$B = B_0 + \delta B \qquad \langle \delta B \rangle = 0$$

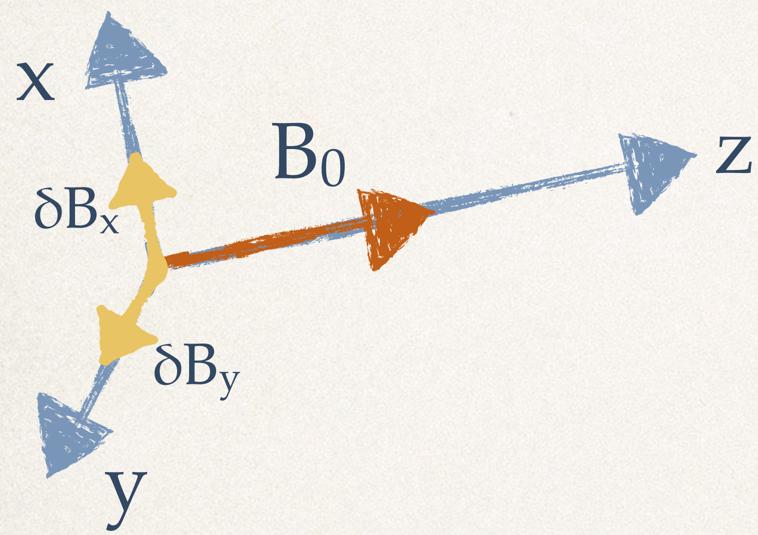
LIMITING OURSELVES TO FIRST ORDER TERMS THE VLASOV EQUATION BECOMES:

$$\frac{\partial \delta f}{\partial t} + \vec{v} \cdot \vec{\nabla} \delta f + \frac{q}{c}\left(\vec{v} \times \vec{B}_0\right) \cdot \frac{\partial \delta f}{\partial \vec{p}} + \frac{q}{c}\left(\vec{v} \times \vec{\delta} B\right) \cdot \frac{\partial f_0}{\partial \vec{p}} = 0$$

FOR THE SAKE OF KEEPING THINGS SIMPLER LET US FOCUS ON WAVES THAT PROPAGATE ALONG THE PRE-EXISTING MAGNETIC FIELD $\vec{B}_0 = B_0 \hat{z}$. IN TERMS OF THE FOURIER MODES OF THE PERTURBATIONS THE MAXWELL EQUATION $\vec{\nabla} \cdot \vec{B} = 0$ IMPLIES THAT $\vec{k} \cdot \delta \vec{B} = 0$

HENCE IF THE WAVES PROPAGATE ALONG B_0 THEN THE PERTURBATIONS ONLY HAVE THAT X AND Y COMPONENTS

Vlasov Equation and CR transport



$$\vec{B}_0 = (0, 0, B_0) \qquad \delta\vec{B} = (\delta B_x, \delta B_y, 0)$$

$$\frac{q}{c}\left(\vec{v} \times \vec{B}_0\right) \cdot \frac{\partial \delta f}{\partial \vec{p}} = \frac{q}{c}\left(v_y B_0 \frac{\partial \delta f}{\partial p_x} - v_x B_0 \frac{\partial \delta f}{\partial p_y}\right)$$

It is convenient to work in cylindrical coordinates:

$$v_x = v_\perp cos\phi \qquad v_y = v_\perp sin\phi \qquad v_z = v_\parallel$$

$$dp_\perp = cos\phi \; dp_x + sin\phi \; dp_y$$

$$d\phi = -\frac{dp_x}{p_\perp} \; sin\phi + \frac{dp_y}{p_\perp} \; cos\phi$$

$$\frac{q}{c}\left(v_y B_0 \frac{\partial \delta f}{\partial p_x} - v_x B_0 \frac{\partial \delta f}{\partial p_y}\right) = -\Omega \frac{\partial \delta f}{\partial \phi}$$

Vlasov Equation and CR transport

$$\frac{q}{c}\left(\vec{v}\times\vec{\delta B}\right)\cdot\frac{\partial f_0}{\partial\vec{p}} = \frac{q}{c}\begin{pmatrix}\hat{x} & \hat{y} & \hat{z}\\ v_x & v_y & v_z\\ \delta B_x & \delta B_y & 0\end{pmatrix}\cdot\frac{\partial f_0}{\partial\vec{p}} =$$

$$= \frac{q}{c}\left\{-v_\parallel\delta B_y\frac{\partial f_0}{\partial p_x}+v_\parallel\delta B_x\frac{\partial f_0}{\partial p_y}+\left(v_x\delta B_y - v_y\delta B_x\right)\frac{\partial f_0}{\partial p_\parallel}\right\} =$$

Back to cylindrical coordinates and assuming $df_0/d\phi=0$

$$= \frac{q}{c}\left\{-v_\parallel\delta B_y\cos\varphi\frac{\partial f_0}{\partial p_\perp}+v_\parallel\delta B_x\sin\varphi\frac{\partial f_0}{\partial p_\perp}+v_\perp\left(\cos\varphi\,\delta B_y-\sin\varphi\,\delta B_x\right)\frac{\partial f_0}{\partial p_\parallel}\right\}$$

It is useful to introduce the two field superpositions:

$$\delta B_+ = \delta B_x + i\delta B_y$$
$$\delta B_- = \delta B_x - i\delta B_y$$

$$\Rightarrow\quad \delta B_x = \frac{\delta B_+ + \delta B_-}{2}\qquad \delta B_y = \frac{\delta B_+ - \delta B_-}{2i}$$

Vlasov Equation and CR transport

Replacing this into the previous expression leads to the simple result that:

$$\frac{q}{c}\left(\vec{v}\times\vec{\delta}B\right)\cdot\frac{\partial f_0}{\partial\vec{p}} = i\frac{q}{2c}\left(v_\parallel\frac{\partial f_0}{\partial p_\perp} - v_\perp\frac{\partial f_0}{\partial p_\parallel}\right)\delta\tilde{B} \qquad \delta\tilde{B} = \delta B_+ e^{-i\phi} - \delta B_- e^{i\phi}$$

It is customary to introduce the operator H defined as: $\hat{H}\equiv\left(v_\parallel\frac{\partial f_0}{\partial p_\perp} - v_\perp\frac{\partial f_0}{\partial p_\parallel}\right)$

so that the perturbed vlasov equation reads:

$$\frac{\partial\delta f}{\partial t} + \vec{v}\cdot\vec{\nabla}\delta f - \Omega\frac{\partial\delta f}{\partial\phi} = -i\frac{q}{2c}\hat{H}f_0\delta\tilde{B}$$

Introducing the fourier transform of the perturbations:

$$\delta B(z,t) = \int dk\int d\omega\,\delta B(k,\omega)\exp\left[ikz - i\omega t\right]$$

$$\delta f(z,t) = \int dk\int d\omega\,\delta f(k,\omega)\exp\left[ikz - i\omega t\right]$$

Vlasov Equation and CR transport

$$\frac{\partial \delta f}{\partial t} + \vec{v} \cdot \vec{\nabla} \delta f - \Omega \frac{\partial \delta f}{\partial \phi} = -i \frac{q}{2c} \hat{H} f_0 \delta \tilde{B}$$

$$-i\omega \delta f + i k v_{\parallel} \delta f - \Omega \frac{\partial \delta f}{\partial \phi} = -i \frac{q}{2c} \hat{H} f_0 \left(\delta B_+ e^{-i\phi} - \delta B_- e^{i\phi} \right)$$

Since the only dependence of ϕ is in the exponentials it is natural to look for a solution in the form:

$$\delta f = A e^{-i\phi} + B e^{i\phi} \quad \Rightarrow \quad \delta f = A \frac{q}{2c} \hat{H} f_0 \left[\frac{\delta B_+}{\omega - k v_{\parallel} - \Omega} e^{-i\phi} - \frac{\delta B_-}{\omega - k v_{\parallel} + \Omega} e^{i\phi} \right]$$

- *Look for similarities between this calculation and those discussed in the course on instabilities… there will be many!… and with physical meaning that will become clear…*
- *Recall that we have assumed the independence of f_0 on ϕ. This makes sense given the symmetry, but will prevent us from calculating the perpendicular diffusion coefficient!!!*
- *What we just devised is an extremely powerful tool…let us see how we can use it*

Vlasov Equation and CR transport

Take a look at the vlasov equation again and calculate its ensemble average recalling that all the means of first order quantities are zero:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{q}{c}\left(\vec{v} \times \vec{B}\right) \cdot \frac{\partial f}{\partial \vec{p}} = 0 \quad \Rightarrow \quad \underbrace{\frac{\partial f_0}{\partial t} + \vec{v} \cdot \vec{\nabla} f_0}_{\text{Evolution of the average CR distribution}} = \underbrace{-\frac{q}{c}\langle\left(\vec{v} \times \delta\vec{B}\right) \cdot \frac{\partial \delta f}{\partial \vec{p}}\rangle}_{\text{Second order term that contains all the information previously derived}}$$

Where we can recall that we have deduced the Fourier components of the perturbations and that we need to average over the coordinate ϕ:

$$\underbrace{\frac{\partial f_0}{\partial t} + v_\parallel \frac{\partial f_0}{\partial z}}_{\text{Evolution of the average CR distribution}} = -\frac{q}{c}\int_0^{2\pi}\frac{d\phi}{2\pi}\int dk \int d\omega \int dk' \int d\omega' \langle\left(\vec{v} \times \delta B(\vec{k'}, \omega')\right) \cdot \frac{\partial \delta f(k,\omega)}{\partial \vec{p}}\rangle$$

Vlasov Equation and CR transport

The calculation is very similar to the ones we already carried out, so you will only need a few steps:

$$\left(\vec{v} \times \delta\vec{B}\right) \cdot \frac{\partial \delta f}{\partial \vec{p}} = v_{\|} \frac{\partial \delta f}{\partial p_{\perp}} \left[-\delta B_y cos\phi + \delta B_x sin\phi\right] + \\ + \frac{v_{\|}}{p_{\perp}} \frac{\partial \delta f}{\partial \phi} \left[\delta B_y sin\phi + \delta B_x cos\phi\right] + \\ + v_{\perp} \frac{\partial \delta f}{\partial p_{\|}} \left[\delta B_y cos\phi - \delta B_x sin\phi\right]$$

and introducing again the two superposition states δB_+ and δB_-:

$$\left(\vec{v} \times \delta\vec{B}\right) \cdot \frac{\partial \delta f}{\partial \vec{p}} = \frac{i}{2} \left(v_{\|} \frac{\partial \delta f}{\partial p_{\perp}} - v_{\perp} \frac{\partial \delta f}{\partial p_{\|}}\right) \left[\delta B_+ e^{-i\phi} - \delta B_- e^{i\phi}\right] + \\ + \frac{1}{2} \frac{v_{\|}}{p_{\perp}} \frac{\partial \delta f}{\partial \phi} \left[\delta B_+ e^{-i\phi} + \delta B_- e^{i\phi}\right]$$

Vlasov Equation and CR transport

Recall that we have already calculated the dependence of δf on ϕ

$$\delta f = A\frac{q}{2c}\hat{H}f_0\left[\frac{\delta B_+}{\omega - kv_\parallel - \Omega}e^{-i\phi} - \frac{\delta B_-}{\omega - kv_\parallel + \Omega}e^{i\phi}\right]$$

hence it is easy to put things together, introducing again δB_+ and δB_- and obtain this important result:

$$\frac{\partial f_0}{\partial t} + v_\parallel\frac{\partial f_0}{\partial z} = -i\left(\frac{q}{2c}\right)^2\int_0^{2\pi}\frac{d\phi}{2\pi}\int dk\int d\omega\int dk'\int d\omega'$$
$$\langle\left\{\left[\left(\delta B_+(k',\omega')e^{-i\phi} - \delta B_-(k',\omega')e^{i\phi}\right)\hat{H}f_0 - \frac{v_\parallel}{p_\perp}\left(\delta B_+(k',\omega')e^{-i\phi} + \delta B_-(k',\omega')e^{i\phi}\right)\right]\right.$$
$$\left.\left(\frac{\delta B_+ e^{-i\phi}}{\omega - k_\parallel v_\parallel - \Omega} + \frac{\delta B_- e^{i\phi}}{\omega - k_\parallel v_\parallel + \Omega}\right)\hat{H}f_0\right\}\rangle$$

At this point we introduce the correlation function of the fields:

$$\langle\delta B_i(k,\omega)\delta B_j(k',\omega')\rangle = \frac{(2\pi)^4}{V\,T}\langle\delta B_i(k,\omega)\delta B_j^*(k,\omega)\rangle\delta(k+k')\delta(\omega+\omega')$$

Reality of the Fields → $\delta B_i(-k,-\omega) = \delta B_i^*(k,\omega)$ →

$$\delta B_+(-k,-\omega) = \delta B_-^*(k,\omega)$$
$$\delta B_-(-k,-\omega) = \delta B_+^*(k,\omega)$$

Vlasov Equation and CR transport

After replacing these in the previous expression and after a trivial integration over the phase Φ we obtain:

$$\frac{\partial f_0}{\partial t}+v_\parallel\frac{\partial f_0}{\partial z}=i\left(\frac{q}{2c}\right)^2\frac{(2\pi)^4}{VT}\int dk\int d\omega\left(\hat{H}+\frac{v_\parallel}{p_\perp}\right)\left[\frac{\langle\delta B_-^*\delta B_-\rangle\hat{H}f_0}{\omega-kv_\parallel+\Omega}+\frac{\langle\delta B_+^*\delta B_+\rangle\hat{H}f_0}{\omega-kv_\parallel-\Omega}\right]$$

Now you have to remember that the frequency is a complex number $\omega=\omega_R+i\omega_I$ but the RHS of the equation above is real:

$$\frac{i}{\omega_R+i\omega_I-kv_\parallel\pm\Omega}=\frac{i(\omega_R-kv_\parallel\pm\Omega)}{(\omega_R-kv_\parallel\pm\Omega)^2+\omega_I^2}+\frac{\omega_I}{(\omega_R-kv_\parallel\pm\Omega)^2+\omega_I^2}\xrightarrow[\omega_I\to 0]{\text{Lim}}\pi\delta(\omega_R-kv_\parallel\pm\Omega)$$

Mathematically, this is the origin of resonant scattering!

Exercise: Show that for a given function g: $\left(\hat{H}+\frac{v_\parallel}{p_\perp}\right)g=\frac{1}{p_\perp}\hat{H}\left(p_\perp g\right)$

$$\frac{\partial f_0}{\partial t}+v_\parallel\frac{\partial f_0}{\partial z}=$$
$$\left(\frac{q}{2c}\right)^2\frac{(2\pi)^4}{VT}\frac{1}{p_\perp}\int dk\int d\omega\hat{H}\left\{p_\perp\left[\langle\delta B_-^*\delta B_-\rangle\delta(\omega-kv_\parallel+\Omega)+\langle\delta B_+^*\delta B_+\rangle\delta(\omega-kv_\parallel+\Omega)\right]\hat{H}f_0\right\}$$

Vlasov Equation and CR transport

We can introduce the zero frequency limit of the spectrum

$$\lim_{V,T\to\infty} \frac{(2\pi)^4}{VT} \langle \delta B^*_\pm \delta B_\pm \rangle = \delta(\omega) P_\pm(k)$$

In this limit, the energy of the particles cannot change and we can neglect the derivatives with respect to the modulus of the momentum. Hence:

$$\frac{1}{p_\perp}\hat{H} = -\frac{v}{p^2}\frac{\partial}{\partial\mu} \qquad \text{with } \mathrm{p}_\parallel = \mathrm{p}\mu$$

(Please show this as an exercise, it's trivial)

And that is where we wanted to get:

$$\frac{\partial f_0}{\partial t} + v_\parallel \frac{\partial f_0}{\partial z} = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f_0}{\partial \mu}\right]$$

Diffusion Equation

$$D_{\mu\mu} = \frac{\pi}{4}\Omega^2(1-\mu)^2 \int dk \left[\delta(kv_\parallel - \Omega)\frac{P_-(k)}{B_0^2} + \delta(kv_\parallel + \Omega)\frac{P_+(k)}{B_0^2}\right]$$

Diffusion Coefficient

FROM PITCH ANGLE DIFFUSION TO SPATIAL DIFFUSION

IT IS INTUITIVELY CLEAR HOW A PARTICLE THAT IS DIFFUSING IN ITS PITCH ANGLE MUST BE ALSO DIFFUSING IN SPACE. LET US SEE HOW THE TWO ARE RELATED TO EACH OTHER BY INTEGRATING THE BOLTZMANN EQUATION IN PITCH ANGLE:

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f}{\partial \mu}\right]$$

$$f_0(p,t,z) = \frac{1}{2}\int_{-1}^{1} d\mu f(p,t,\mu,z)$$

ISOTROPIC PART OF THE PARTICLE DISTRIBUTION FUNCTION. FOR MOST PROBLEMS THIS IS ALSO VERY CLOSE TO THE ACTUAL DISTRIBUTION FUNCTION

$$\frac{\partial f_0}{\partial t} + \frac{1}{2} v \int_{-1}^{1} d\mu \mu \frac{\partial f}{\partial z} \equiv 0$$

ONE CAN SEE THAT THE QUANTITY

$$J = \frac{1}{2} v \int_{-1}^{1} d\mu \mu f$$

BEHAVES AS A PARTICLE CURRENT, AND THE BOLTMANN EQUATION BECOMES:

$$\frac{\partial f_0}{\partial t} = -\frac{\partial J}{\partial z}$$

NOTICE THAT YOU CAN ALWAYS WRITE:

$$\mu = -\frac{1}{2} \frac{\partial}{\partial \mu} \left(1 - \mu^2\right)$$

WITH THIS TRICK:

$$J = \frac{1}{2} v \int_{-1}^{1} d\mu \mu f = \frac{v}{4} \int_{-1}^{1} d\mu (1-\mu^2) \frac{\partial f}{\partial \mu}$$

RECONSIDER THE INITIAL EQUATION

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left[D_{\mu\mu} \frac{\partial f}{\partial \mu} \right]$$

AND INTEGRATE IT AGAIN FROM -1 TO μ:

$$\frac{\partial}{\partial t} \int_{-1}^{\mu} d\mu' f + \int_{-1}^{\mu} d\mu' v \mu' \frac{\partial f}{\partial z} = D_{\mu\mu} \frac{\partial f}{\partial \mu}$$

AND MULTIPLYING BY

$$(1-\mu^2)/D_{\mu\mu}$$

$$(1-\mu^2)\frac{\partial f}{\partial \mu} = \frac{1-\mu^2}{D_{\mu\mu}}\frac{\partial}{\partial t}\int_{-1}^{\mu} d\mu' f + \frac{1-\mu^2}{D_{\mu\mu}}\int_{-1}^{\mu} d\mu' v\mu' \frac{\partial f}{\partial z}$$

NOW RECALL THAT THE DISTRIBUTION FUNCTION TENDS TO ISOTROPY, SO THAT AT THE LOWEST ORDER IN THE ANISOTROPY ONE HAS:

$$(1-\mu^2)\frac{\partial f}{\partial \mu} = \frac{1-\mu^2}{D_{\mu\mu}}\frac{\partial f_0}{\partial t}(1+\mu) + \frac{1-\mu^2}{D_{\mu\mu}}\frac{1}{2}v(\mu^2-1)\frac{\partial f_0}{\partial z}$$

AND RECALLING THE DEFINITION OF CURRENT:

$$J = \frac{v}{4}\frac{\partial f_0}{\partial t}\int_{-1}^{1} d\mu \frac{1-\mu^2}{D_{\mu\mu}}(1+\mu) - \frac{v^2}{8}\frac{\partial f_0}{\partial z}\int_{-1}^{1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}} = \kappa_t \frac{\partial f_0}{\partial t} - \kappa_z \frac{\partial f_0}{\partial z}$$

USING THE TRANSPORT EQ IN TERMS OF CURRENT:

$$J = -\kappa_t \frac{\partial J}{\partial z} - \kappa_z \frac{\partial f_0}{\partial z}$$

NOW WE RECALL THE TRANSPORT EQUATION IN CONSERVATIVE FORM:

$$\frac{\partial f_0}{\partial t} = -\frac{\partial J}{\partial z}$$

AND PUTTING THINGS TOGETHER:

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial z}\left[-\kappa_t \frac{\partial J}{\partial z} - \kappa_z \frac{\partial f_0}{\partial z}\right]$$

BUT IT IS EASY TO SHOW THAT THE FIRST TERM MUST BE NEGLIGIBLE:

$$J = \frac{v}{2}\int_{-1}^{1} d\mu \mu f_0 (1+\delta\mu) = \frac{1}{3} v \delta f_0 \ll v f_0 \qquad \delta \ll 1$$

IT FOLLOWS THAT THE ISOTROPIC PART OF THE DISTRIBUTION FUNCTION MUST SATISFY THE DIFFUSION EQUATION:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial z}\left[\kappa_z \frac{\partial f_0}{\partial z}\right]$$

DIFFUSION EQUATION

$$\kappa_z = \frac{v^2}{8}\int_{-1}^{1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}} = \frac{1}{3} v \lambda_{\parallel}$$

SPATIAL DIFFUSION COEFFICIENT

The effect of inhomogeneous motion of the background plasma

What happens when the particles diffuse in a medium which is itself in motion?

Physically it is easy to expect that the particles will be advected with the plasma (the derivative in time becomes the total time derivative) but is there more than this?

One should expect that as long as the plasma velocity is uniform in space, nothing exceptional should be expected: in fact one could always move to a reference frame where the plasma is at rest and recover the previous result

But what happens if the plasma velocity is ***NOT UNIFORM****? In this case there is no unique frame in which the plasma is at rest everywhere*

An obvious application of this situation is that in which a shock front forms in the plasma, so that there is a discontinuity in the velocity of the background plasma

THE EFFECT OF INHOMOGENEOUS MOTION OF THE BACKGROUND PLASMA

LET US ASSUME THAT THE BACKGROUND PLASMA MOVES WITH A NON-RELATIVISTIC SPEED u ALONG THE z DIRECTION

THEN THE TOTAL VELOCITY OF THE PARTICLE IN THE LAB FRAME IS SIMPLY **u+vμ** AND THE MOMENTUM ALONG THE Z DIRECTION BECOMES $p_z=-(u/c^2)E+p\mu$ WHILE THE MOMENTUM COMPONENTS IN THE X AND Y DIRECTIONS ARE UNCHANGED:

$$\frac{\partial f}{\partial z} \to \frac{\partial f}{\partial z} + \frac{\partial f}{\partial p_z}\frac{dp_z}{dz} = \frac{\partial f}{\partial z} - \frac{E}{c^2}\frac{du}{dz}\frac{\partial f}{\partial p_z}$$

RECALLING THAT $p_z = p\mu$ $\quad p_\perp = p(1-\mu^2)^{1/2}$ AND TRANSFORMING TO SPHERICAL COORDINATES:

$$\frac{\partial f}{\partial p_z} = \frac{\partial f}{\partial p}\mu + \frac{\partial f}{\partial \mu}\frac{1-\mu^2}{p}$$

IN CONCLUSION THE VLASOV EQUATION GETS MODIFIED AS FOLLOWS:

$$\frac{\partial f}{\partial t} + (u+v\mu)\frac{\partial f}{\partial z} - \frac{E}{c^2}\frac{du}{dz}(u+v\mu)\left[\frac{\partial f}{\partial p}\mu + \frac{\partial f}{\partial \mu}\frac{1-\mu^2}{p}\right] = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f}{\partial \mu}\right]$$

The effect of inhomogeneous motion of the background plasma

Let us proceed in a similar way as done before: $f_0(p,t,z) = \frac{1}{2}\int_{-1}^{1} d\mu f(p,t,\mu,z)$

$$\frac{\partial f}{\partial t} + (u+v\mu)\frac{\partial f}{\partial z} - \frac{E}{c^2}\frac{du}{dz}(u+v\mu)\left[\frac{\partial f}{\partial p}\mu + \frac{\partial f}{\partial \mu}\frac{1-\mu^2}{p}\right] = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}\frac{\partial f}{\partial \mu}\right]$$

This returns zero when integrated between -1 and 1

Order $(u/c)^2$

This is negligible since at zero order f_0 is isotropic

This returns the same result as before upon integration

$$\frac{1}{2}\int_{-1}^{1} d\mu \frac{E}{c^2}\frac{du}{dz}\frac{\partial f}{\partial p}v\mu^2 = \frac{1}{3}\frac{E}{c^2}\frac{du}{dz}v\frac{\partial f_0}{\partial p} \equiv \frac{1}{3}\frac{du}{dz}p\frac{\partial f_0}{\partial p}$$

Hence the equation in the general case becomes:

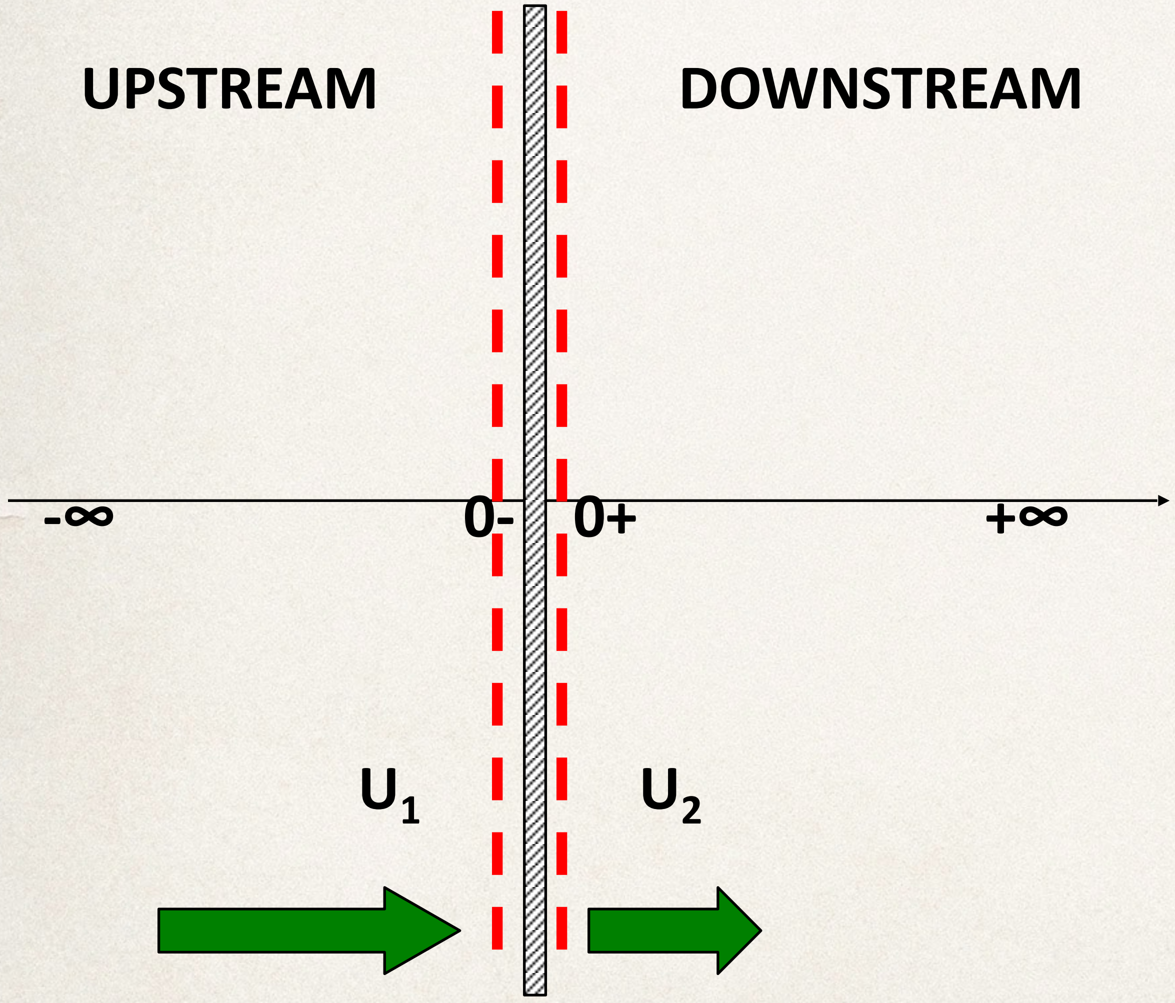
$$\frac{\partial f_0}{\partial t} + u\frac{\partial f_0}{\partial z} - \frac{1}{3}\left(\frac{du}{dz}\right)p\frac{\partial f_0}{\partial p} = \frac{\partial}{\partial z}\left[\kappa_{zz}\frac{\partial f_0}{\partial z}\right]$$

We confirm that no energy change occurs if the plasma velocity is uniform!

A SIMPLE APPLICATION:
THE TRANSPORT EQUATION APPROACH TO DSA

$$\frac{\partial f_0}{\partial t} + u\frac{\partial f_0}{\partial z} - \frac{1}{3}\left(\frac{du}{dz}\right)p\frac{\partial f_0}{\partial p} = \frac{\partial}{\partial z}\left[D\frac{\partial f_0}{\partial z}\right]$$

Advection Compression Diffusion



THE EQUATION WE JUST FOUND DESCRIBES MANY PROCESSES, SUCH AS THE DIFFUSION OF COSMIC RAYS IN THE GALAXY, BUT ALSO THE PARTICLE ACCELERATION AT SHOCKS!!!

FOR A PLANE PARALLEL SHOCK:

$$\frac{du}{dz} = (u_2 - u_1)\delta(z)$$

LET US ALSO ASSUME **STATIONARITY**!

UPSTREAM solution

LET US ASSUME STATIONARITY (LATER WE SHALL DISCUSS IMPLICATIONS)

IN THE UPSTREAM THE EQUATION READS

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] - u\frac{\partial f}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z} - uf\right] = 0$$

FLUX IS CONSERVED!

THE SOLUTION THAT HAS VANISHING f AND VANISHING DERIVATIVE AT UPSTREAM INFINITY IS

$$f(x,p) = f_0 \exp\left[\frac{u_1 z}{D}\right] \quad \rightarrow \quad D\frac{\partial f}{\partial z}|_{z\rightarrow 0^-} = u_1 f_0(p)$$

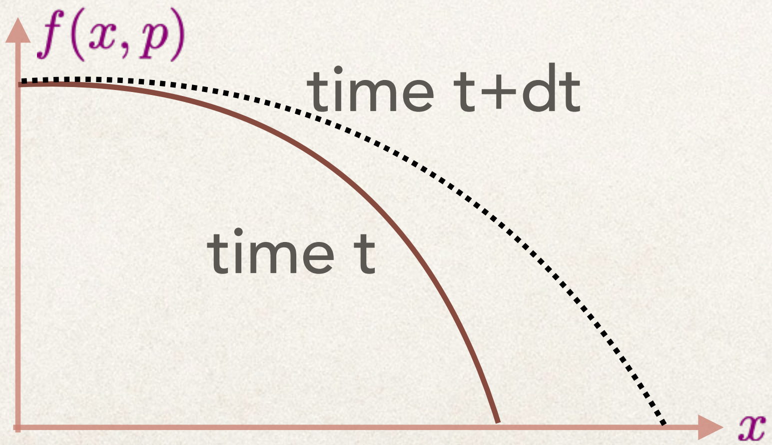
DOWNSTREAM solution

IN THE DOWNSTREAM THE EQUATION READS

$$\frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z}\right] - u\frac{\partial f}{\partial z} = 0 \quad \rightarrow \quad \frac{\partial}{\partial z}\left[D\frac{\partial f}{\partial z} - uf\right] = 0$$

FLUX IS CONSERVED!

NOTICE THAT WE HAVE REQUIRED STATIONARITY AND OBVIOUSLY THE ONLY SOLUTION THAT IS CONSISTENT WITH THAT ASSUMPTION IS



$$f(x,p) = \text{constant} = \mathrm{f_0(p)}$$

$$D\frac{\partial f}{\partial z}|_{z\to 0^+} = 0$$

Around the shock

INTEGRATING THE TRANSPORT EQUATION IN A NARROW NEIGHBORHOOD OF THE SHOCK WE GET

$$D\frac{\partial f}{\partial z}|_{z\to 0^+} - D\frac{\partial f}{\partial z}|_{z\to 0^-} + \frac{1}{3}(u_2 - u_1)p\frac{df_0}{dp} = 0$$

WHERE WE USED $du/dz=(u_2-u_1)\boldsymbol{\delta}(z)$

REPLACING THE EXPRESSIONS FOR THE DERIVATIVES DERIVED BEFORE:

$$-u_1 f_0(p) + \frac{1}{3}(u_2 - u_1)p\frac{df_0}{dp} = 0 \qquad p > p_{inj}$$

WHICH HAS THE SOLUTION:

$$f_0(p) = Kp^{-\alpha} \qquad \alpha = \frac{3u_1}{u_1 - u_2} = \frac{3r}{r-1}$$

Spectrum of accelerated particles

- THE SPECTRUM OF ACCELERATED PARTICLES IS A POWER LAW IN MOMENTUM
- THE POWER LAW EXTENDS TO INFINITE MOMENTA!!!
- THE SLOPE DEPENDS UNIQUELY ON THE COMPRESSION FACTOR AND IS INDEPENDENT OF THE DIFFUSION PROPERTIES
- NO DEPENDENCE UPON DIFFUSION (MICRO-PHYSICS)

Notes on perpendicular diffusion

Whatever the spatial coordinate is, it is easy to write the diffusion coefficient starting from the definition of the spatial displacement:

$$\langle \Delta x_i \Delta x_i \rangle = \langle \int_0^t d\tau v_i(\tau) \int_0^t d\xi v_i(\xi) \rangle = \int_0^t \int_0^t \langle v_i(\tau) v_i(\xi) \rangle$$

Let us see where this leads us to:

$$\langle \Delta x_i \Delta x_i \rangle = \int_0^t d\tau \int_0^\tau d\xi \langle v_i(\tau) v_i(\xi) \rangle + \int_0^t d\tau \int_\tau^t d\xi \langle v_i(\tau) v_i(\xi) \rangle =$$

$$= \int_0^t d\tau \frac{d\tau}{d\tau} \int_0^\tau d\xi \langle v_i(\tau - \xi) v_i(0) \rangle + \int_0^t d\tau \frac{d\tau}{d\tau} \int_\tau^t d\xi \langle v_i(\xi - \tau) v_i(0) \rangle$$

where we used the fact that the correlation can only depend upon the time difference and not on absolute time (if the perturbations are homogeneous in space). Integrating by parts (see notes):

$$\langle \Delta x_i \Delta x_i \rangle = 2 \int_0^t d\tau (t - \tau) \langle v_i(\tau) v_i(0) \rangle$$

Notes on perpendicular diffusion

Hence, we can define the quantity:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \Delta x_i \Delta x_i \rangle = \int_0^t d\tau \langle v_i(\tau) v_i(0) \rangle \xrightarrow[t \to \infty]{\text{Lim}} \int_0^\infty d\tau \langle v_i(\tau) v_i(0) \rangle \equiv \kappa_{ii}$$

Let us consider the case with no E-fields and magnetic perturbations only in the x-y plane

$$\frac{dv_x}{dt} = \Omega\left(v_y - v_z \frac{\delta B_y}{B_0}\right)$$
$$\frac{dv_y}{dt} = \Omega\left(-v_x + v_z \frac{\delta B_x}{B_0}\right)$$

The solution of these equations can be written in the form:

$$v_x(t) = A(t)cos(\Omega t) + B(t)sin(\Omega t)$$
$$v_y(t) = B(t)cos(\Omega t) - A(t)sin(\Omega t)$$

By substitution one immediately gets: $v_x(t) = v_x(0)cos(\Omega t) + v_y(0)sin(\Omega t) +$

$$cos(\Omega t)\int_0^t dt'\left[-v_z\Omega\frac{\delta B_y}{B_0}cos(\Omega t') - v_z\Omega\frac{\delta B_x}{B_0}sin(\Omega t')\right] +$$
$$sin(\Omega t)\int_0^t dt'\left[-v_z\Omega\frac{\delta B_y}{B_0}sin(\Omega t') + v_z\Omega\frac{\delta B_x}{B_0}cos(\Omega t')\right]$$

and a similar expression for $v_y(t)$

NOTES ON PERPENDICULAR DIFFUSION

AT THIS POINT WE DEFINE THE VELOCITY OF THE GUIDING CENTER

$$\tilde{v}_{x,y}(t) = \frac{1}{T}\int_t^{t+T} dt' v_{x,y}(t') \qquad T \equiv gyroperiod$$

AND AFTER SOME ALGEBRAIC STEPS WE SHOW THAT:

$$\tilde{v}_{x,y}(t) \approx v_z \frac{\delta B_{x,y}}{B_0}$$

ASSUMING THAT THE PERTURBED FIELD CHANGES SLOWLY UPON ONE GYRATION

$$\kappa_{xx} = \kappa_{yy} = \int_0^\infty d\tau \langle \tilde{v}_x(\tau)\tilde{v}_x(0)\rangle = \left(\frac{1}{B_0}\right)^2 \int_0^\infty d\tau \langle v_z(\tau) v_z(0) \delta B_x(\tau) \delta B_x(0)\rangle$$

NOT THAT EASY AN OBJECT!!! IT IS AN HIGHER ORDER CORRELATOR…

AT ZERO ORDER ONE COULD SPECULATE THAT $v_z \approx v\mu$ and $dz = v\mu\, dt$

$$\kappa_{xx} = \frac{v|\mu|}{B_0^2}\int_0^\infty dz \langle \delta B_x(z)\delta B_x(0)\rangle \sim v|\mu| \left(\frac{\delta B_x}{B_0}\right)^2 L$$

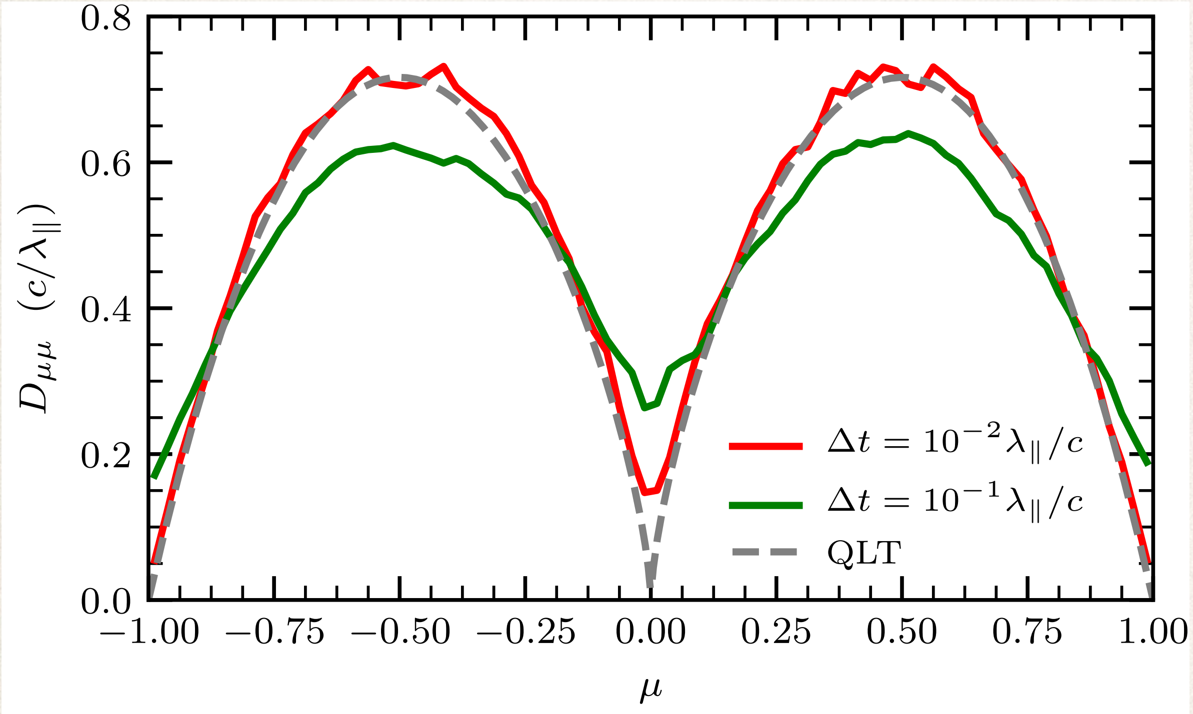
WHERE L IS SOME SCALE OF ORDER OF THE PERTURBATION… THIS SCALING (**FIELD LINE RANDOM WALK**) IS TYPICALLY NOT RIGHT, AS SHOWN IN SIMULATIONS, BUT IT REPRESENTS A FAIR TRY…

Why would you want more?

- The whole structure we built applies to small perturbations—not always the case
- Turbulence is complicated, many subtle aspects to take into account (anisotropy, intermittency, coherent structures, dissipation, …)
- Perpendicular diffusion is still subject to much debate, semi-analytic approaches often do not compare well with simulations
- there are situations in which one or the other of the assumptions fails
- Different types of problems are treated with different approaches: PIC/Hybrid sims., synthetic turbulence, MHD turbulence, …) which themselves have different caveats

Numerical simulations of particle diffusion: Synthetic turbulence

You can fix the spectrum and morphology of the turbulence and generate synthetic maps on a grid…on top of that turbulence you can propagate test particles to investigate their transport properties



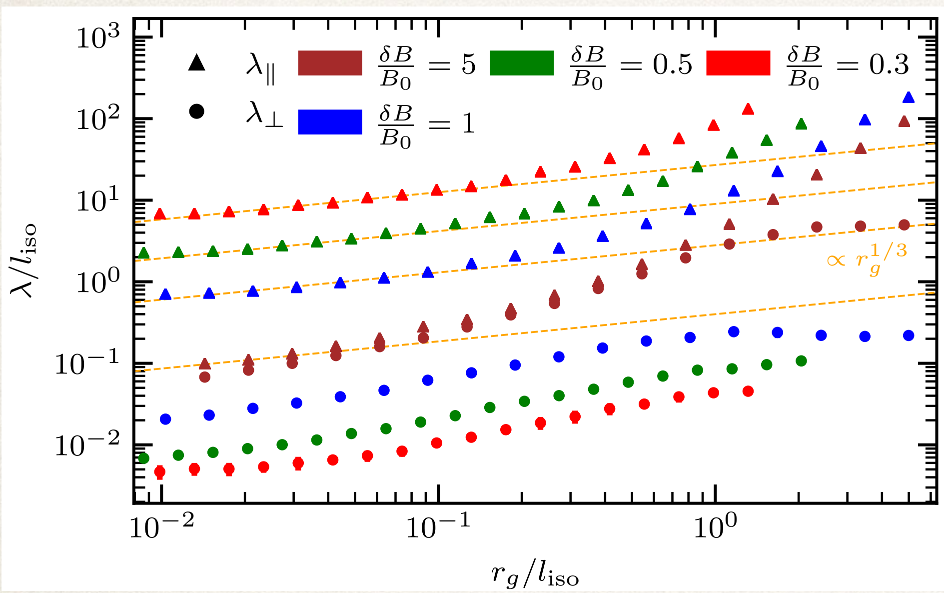
Dundovic+2021

For instance you can study the pitch angle diffusion coefficient and compare it with the Quasi-Linear-Theory we investigated

the results are in excellent agreement for small values of $\delta B/B_0$ and away from μ=0

at μ=0 the simulations show that the resonance is broad enough for particles to cross over… this can also be understood in a physical manner (for finite values of $\delta B/B_0$)

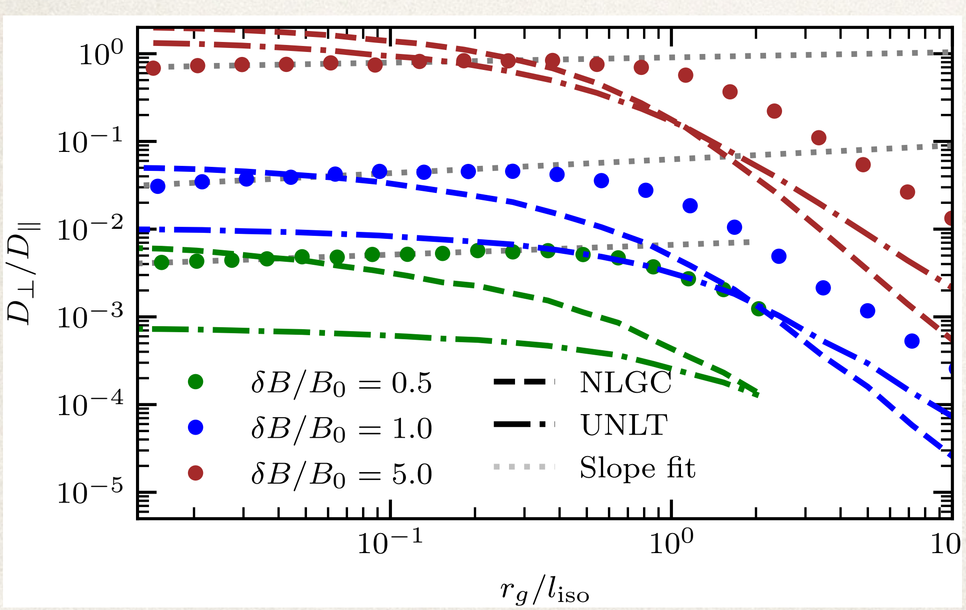
Numerical simulations of particle diffusion: Synthetic turbulence



Dundovic+2021

One can also calculate the path length for parallel and perpendicular diffusion for different turbulent spectra (this is for Kolmogorov):

Notice that these simulations are extremely challenging numerically, because you need to have in the same box the large scales (>>coherence), important for perpendicular transport and the small scales (<<Larmor radius) to identify the resonances



Both $D_\parallel$ and $D_\perp$ grow with energy but there is evidence that the energy dependence is different

we tried to understand this in the context of the so-called non-linear Guiding center theory, but we failed. There is at present no explanation for this behaviour

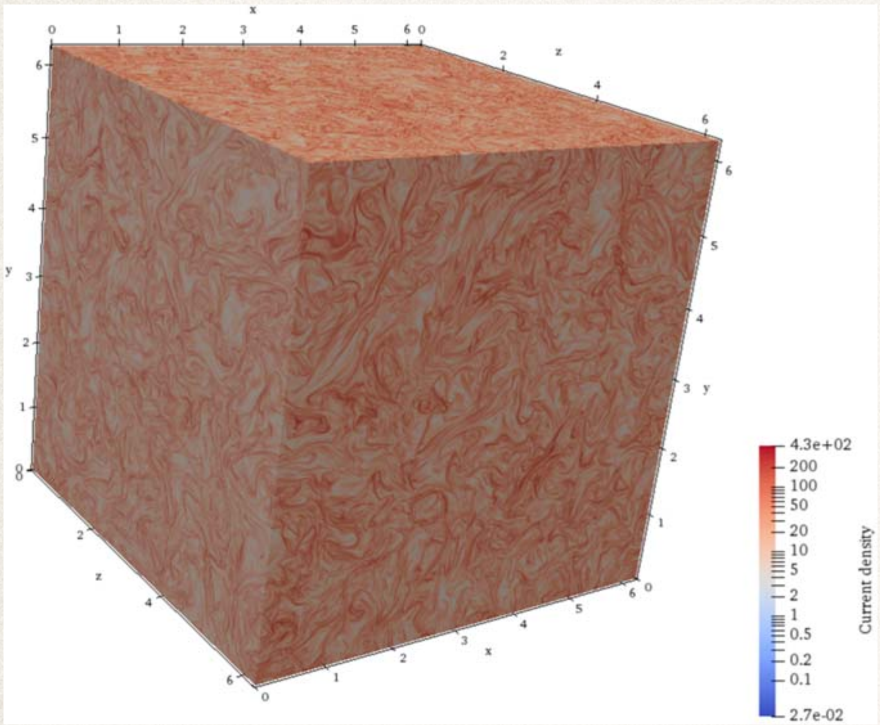
Numerical simulations of particle diffusion: MHD Turbulence

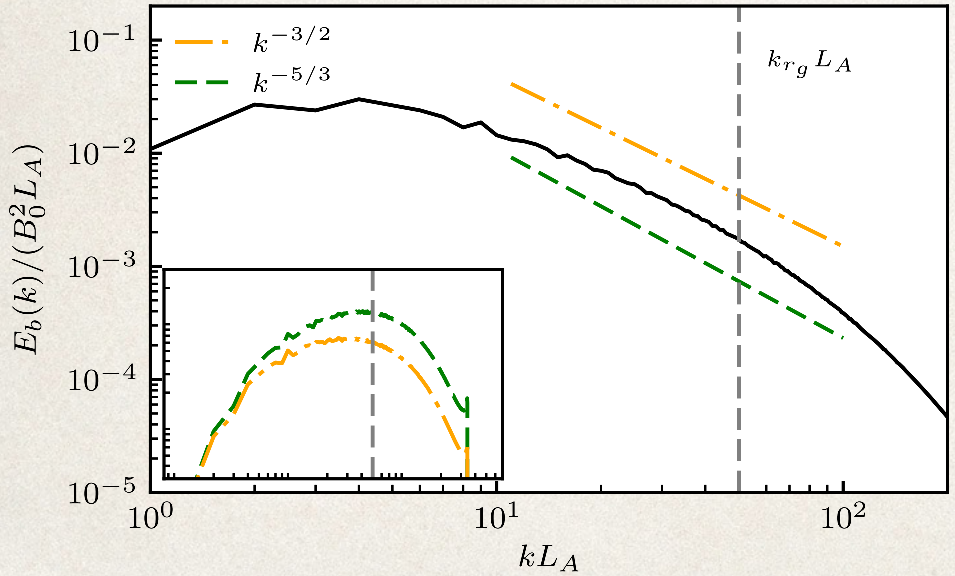
Instead of generating the turbulence in a synthetic manner, one can use MHD simulations of a plasma and stir the box until turbulence develops

the turbulence that results from this procedure often depends on the way you stir, but some general considerations can be identified:

- In the presence of an ordered field, Alfvenic turbulence develops anisotropically, with perp prevailing over kparalle (this phenomenon is known as Goldreich-Shridhar spectrum) and the power in parallel modes quickly becomes to small to be helpful for scattering (lack of resonances). Deep implications for CR (e.g. Yan & Lazarian 2004)
- The spectrum of fast magnetosonic modes seem to cascade isotropically, but damping???
- At some small scales (that in the simulations are dictated by numerics) dissipation occurs
- On these scales, in nature, one can expect reconnection events (see lectures by Sironi)

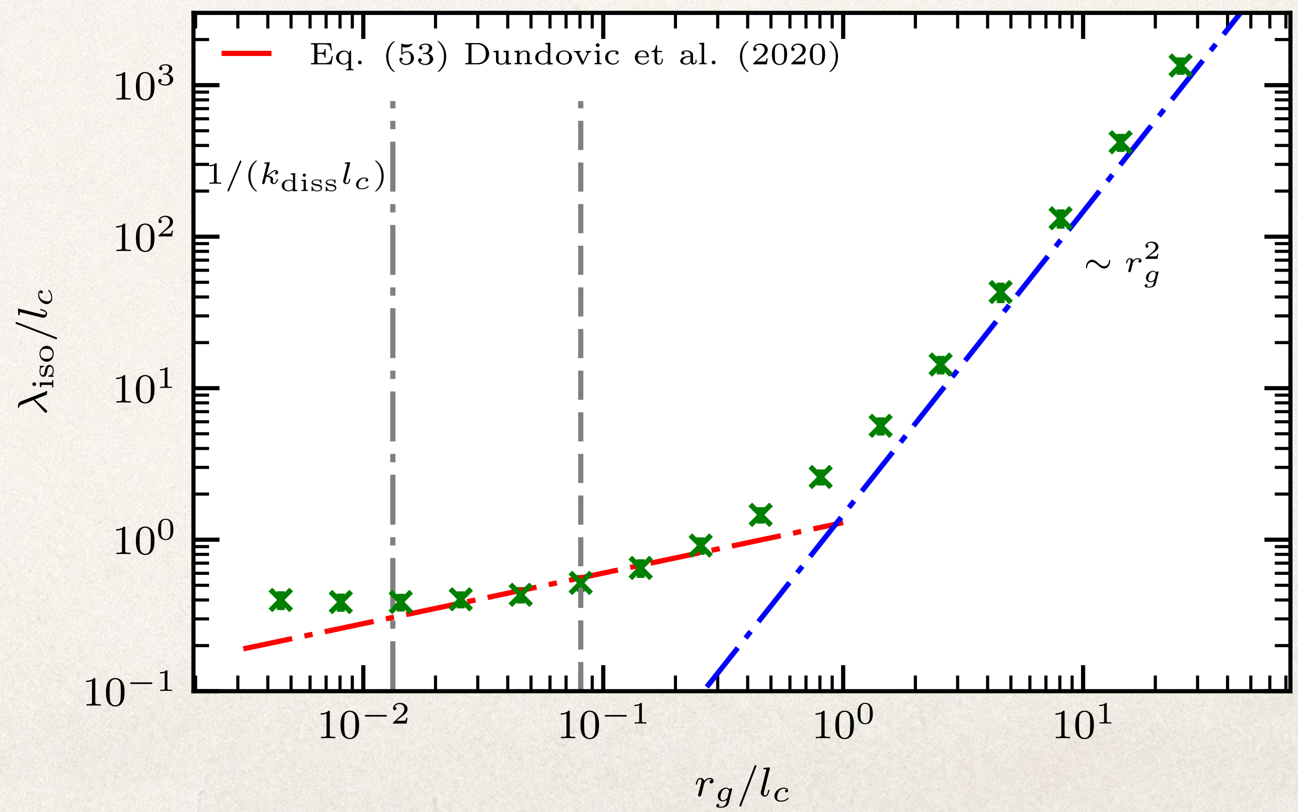
NUMERICAL SIMULATIONS OF PARTICLE DIFFUSION: MHD TURBULENCE





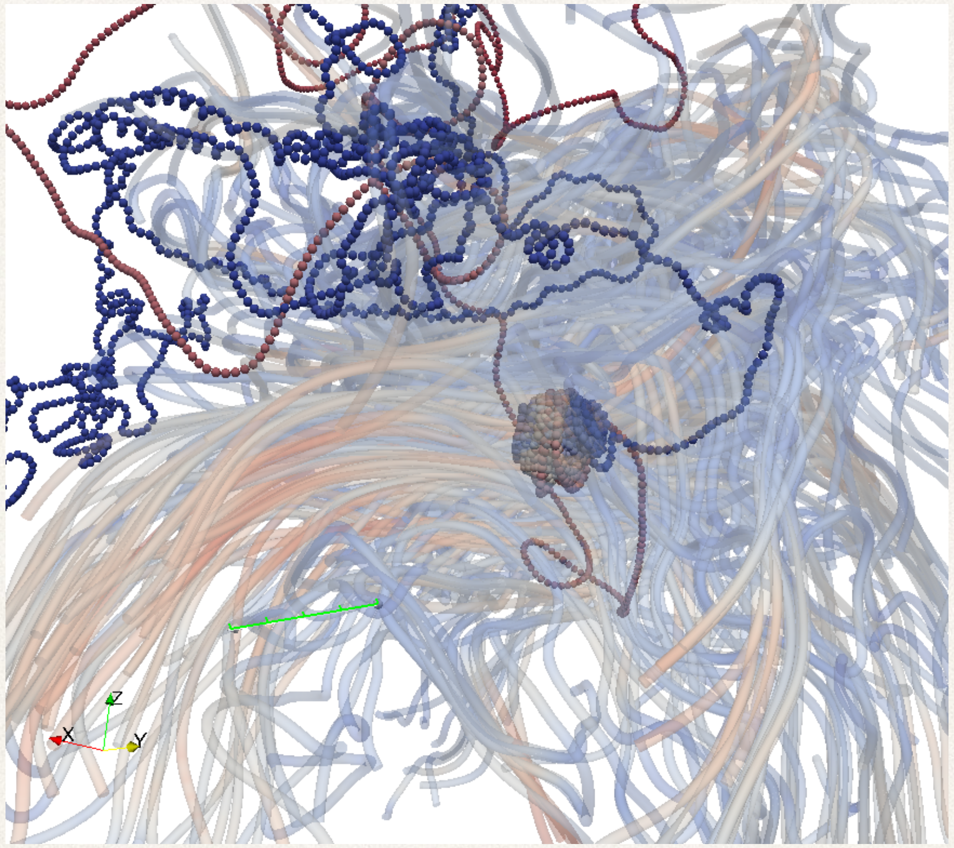
MANY GROUPS HAVE CARRIED OUT THE EXERCISE OF GENERATING MHD TURBULENCE AND DESCRIBE PARTICLE TRANSPORT

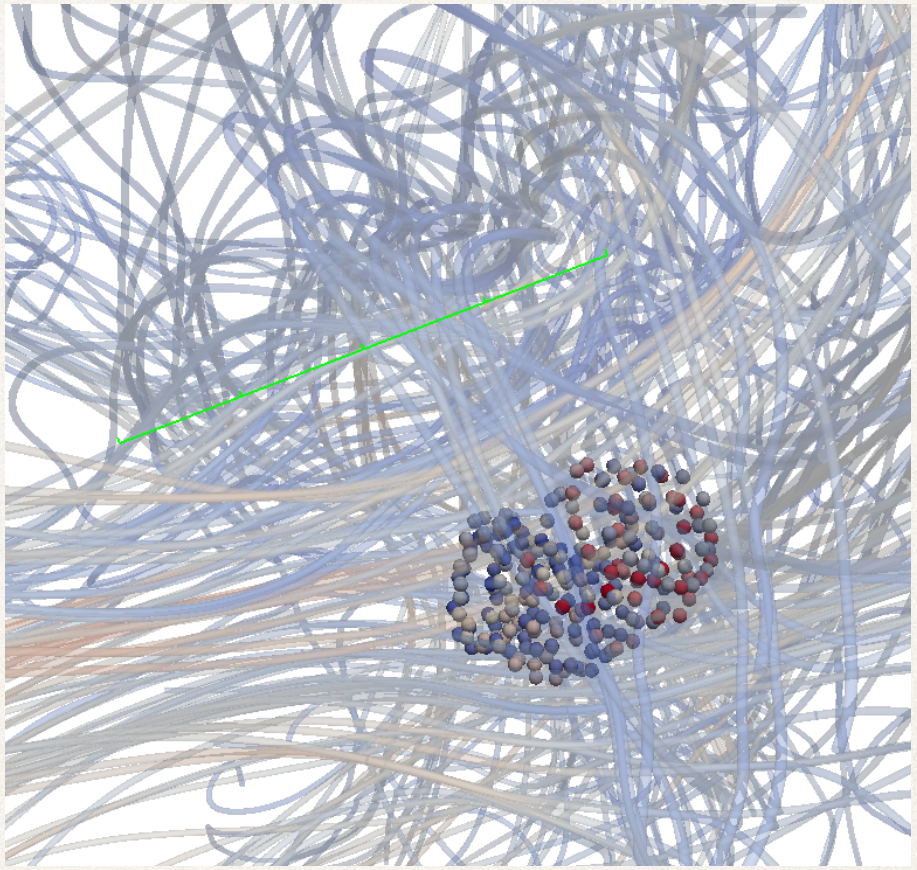
THE DYNAMIC RANGE IS TYPICALLY VERY SMALL AND IT IS DIFFICULT TO INFER VERY DEEP CONCLUSIONS ON CR TRANSPORT IN THE RANGE WE ARE INTERESTED IN



Pezzi, PB & Matthaeus 2022

SO FAR SO GOOD…BUT SOMETHING NEW POPPED OUT — PARTICLE TRAPPING



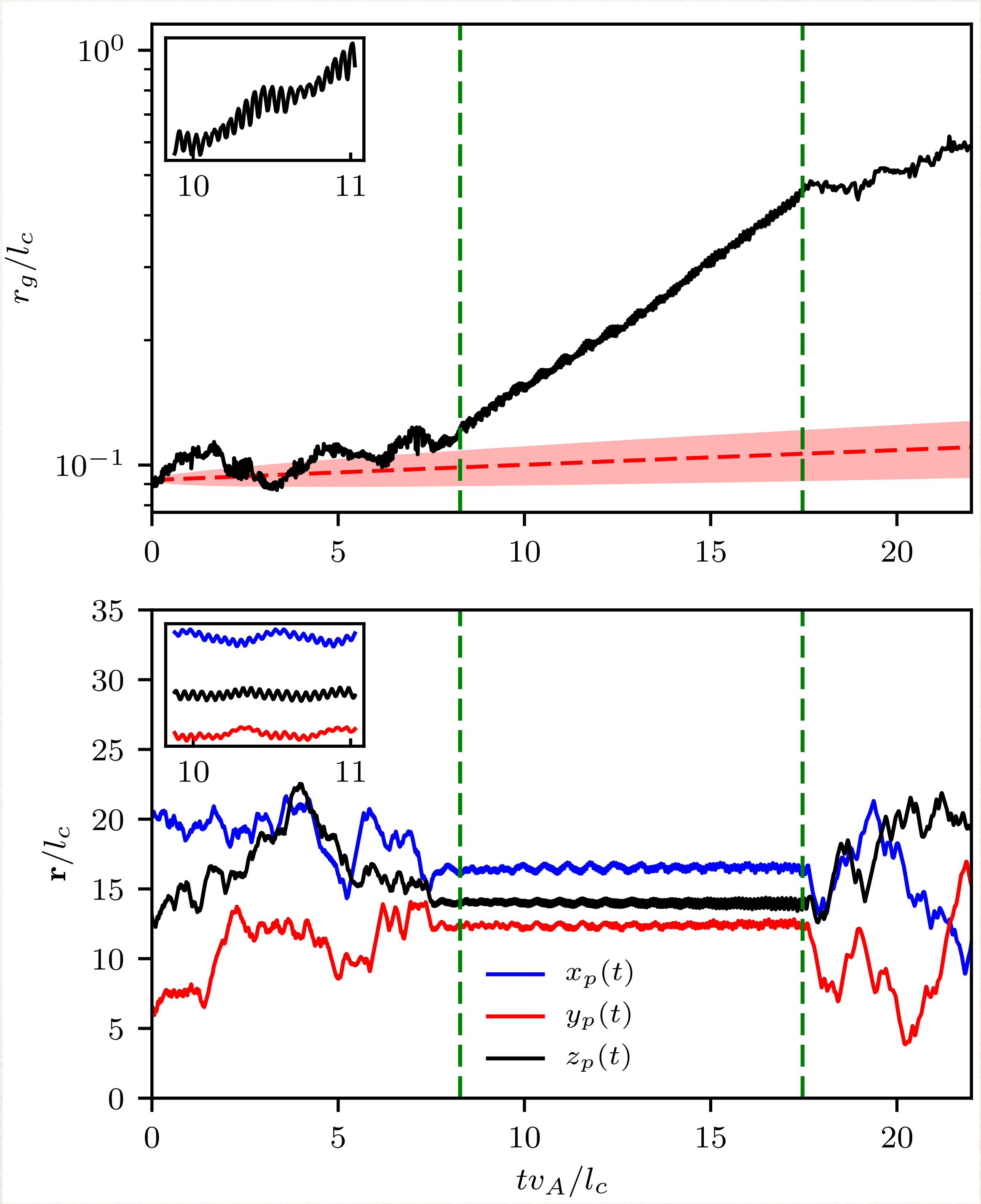


Pezzi, PB & Matthaeus 2022

A FEW OUT OF 100,000 PARTICLES SEEM TO EXPERIENCE THIS PHENOMENON— BUT THOSE FEW PARTICLES BEHAVE IN VERY PECULIAR MANNER

PARTICLE TRAPPING — EXPONENTIAL ENERGY INCREASE

- FOR THE LONGEST TIME PARTICLES SIMPLY DIFFUSE IN SPACE (AND ENERGY)
- THEN EVENTUALLY A FEW OF THEM GET TRAPPED SOMEWHERE
- DURING THOSE PERIODS THE ENERGY GROWS EXPONENTIALLY
- …UNTIL THEY EVENTUALLY ESCAPE THE TRAPPING REGION



Pezzi, PB & Matthaeus 2022

Self-Generation of perturbations

Charged particles moving in a plasma can generate unstable alfven modes which in turn lead to an enhancement of particle scattering: we call this phenomenon self-generation, and will be covered in detail in the lectures of A. Marcowith

the formal derivation relies on an analysis not very different from the one discussed earlier, but there is a more physical way to estimate the rate of growth of these perturbations, as suggested by Kulsrud

diffusion in the z direction leads to slowing down of the CR beam: the mean momentum decreases while particles try to slow down from their drift speed to the Alfven speed

This "lost" momentum needs to be transferred to the other actors on the scene (thermal momentum and momentum of the waves)

all this should happen on the time scale scale for deflection by 90 degrees

$$\tau_{90} \sim \frac{1}{D_{\theta\theta}} \approx \frac{1}{\Omega \mathcal{F}(k_{res})}$$

Self-Generation of perturbations

$$n_{CR} m v_D \to n_{CR} m V_A \Rightarrow \frac{dP_{CR}}{dt} = \frac{n_{CR} m (v_D - V_A)}{\tau}$$

Rate of momentum lost by CR beam

$$\frac{dP_w}{dt} = \gamma_W \frac{\delta B^2}{8\pi} \frac{1}{V_A}$$

Rate of momentum gained by waves

By requiring a sort of balance between the two rates one can estimate the rate of growth of the waves:

$$\gamma_W = \sqrt{2}\,\frac{n_{CR}}{n_{gas}} \frac{v_D - V_A}{V_A} \Omega_{cyc}$$

within a factor of order 2 this is the correct growth rate

$$\gamma_W(k) = \frac{n_{CR}(>p)}{n_{gas}} \frac{v_D - v_A}{v_A} \Omega_c \approx \frac{4\pi p^3 f(p)}{n_{gas}} \frac{v_D - v_A}{v_A} \Omega_c \qquad k = k_{res}(p)$$

MANIFESTATIONS IN DSA

WE HAVE SEEN THAT THE SPECTRUM OF PARTICLES ACCELERATED AT A SHOCK IS A POWER LAW IN MOMENTUM. IN THE LECTURES OF D. CAPRIOLI YOU WILL SEE THAT FOR A STRONG SHOCK THE SLOPE IN MOMENTUM TENDS TO 4. IF PARTICLES CARRY AWAY A FRACTION ξ_{CR} OF THE RAM PRESSURE, IT IS EASY TO SEE THAT

$$f(p) = \frac{3\xi_{CR}\rho v_s^2}{4\pi p_0^4 c\Lambda}\left(\frac{p}{p_0}\right)^{-4} \qquad \Lambda = \ln\left(\frac{p_{max}}{p_0}\right)$$

HENCE, REPLACING IN THE PREVIOUS EXPRESSION AND REARRANGING THINGS:

$$\gamma_W(k) \approx \frac{3\xi_{CR}\rho v_s^2}{c\Lambda p_0}\left(\frac{p}{p_0}\right)^{-1}\frac{1}{n_{gas}}\frac{v_s}{v_A}\frac{qB_0}{m_p c} = \frac{3\xi_{CR}\rho v_s^2}{\Lambda U_B}\frac{v_s}{c}\frac{v_A}{r_L(p)} \qquad U_B = \frac{B_0^2}{4\pi}$$

UPON INTRODUCING THE ALFVEN MACH NUMBER $M_A = v_s/v_A$

$$\gamma_W(k) = 3\frac{\xi_{CR}}{\Lambda}M_A^2\frac{v_s}{c}\frac{v_A}{r_L(p)}$$

→ FOR TYPICAL PARAMETERS

$$\gamma_W^{-1}(1GeV) \simeq 3\times 10^5 s$$

$$\gamma_W^{-1}(1PeV) \simeq 10^4 yr$$

MANIFESTATIONS IN DSA:E_{MAX}

YOU WILL SEE IN THE COURSES OF MARCOWITH AND CAPRIOLI THAT

1) OTHER INSTABILITIES ARE GENERATED BY THE CR STREAMING
2) THE EFFECTS OF ONE OF THOSE INSTABILITIES (NON-RESONANT CR STREAMING INSTABILITY) IS MORE IMPORTANT THAN THIS ONE FOR REACHING HIGH VALUES OF THE MAXIMUM ENERGY

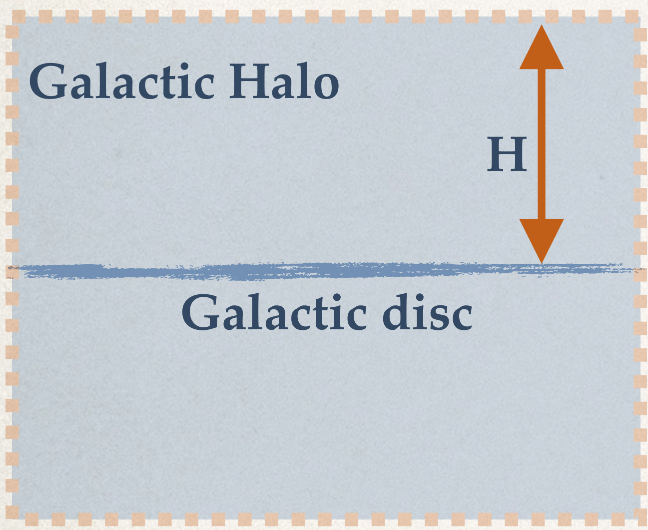
BUT IT IS INSTRUCTIVE TO SEE THAT SINCE MOST OF THE ACCELERATION AT THE HIGHEST ENERGIES IN SUPERNOVA REMNANTS OCCURS AT THE BEGINNING OF THE SEDOV-TAYLOR PHASE (TYPICALLY A FEW HUNDRED YEARS AFTER EXPLOSION) THE MAX ENERGY ***CANNOT BE HIGHER THAN A FEW TENS OF TeV***

IT GETS WORSE, SINCE THE ALFVENIC MACH NUMBER RAPIDLY DROPS WITH TIME AND WE NEGLECTED DAMPING OF THE PERTURBATIONS, WHICH ARE CRUCIAL IN THIS PROBLEM

YOU JUST TOUCHED BY HAND WHY IT IS SO DIFFICULT TO ACCELERATE PARTICLES IN ASTROPHYSICAL SOURCES

…AND IN THE ABSENCE OF THESE INSTABILITIES IT IS BASICALLY IMPOSSIBLE TO REACH ENERGIES LARGER THAN A FEW GEV

MANIFESTATIONS IN GALACTIC CR TRANSPORT



THIS IS ANOTHER APPLICATION OF OUR SIMPLE TRANSPORT EQUATION: OUTSIDE THE DISC OF THE GALAXY THE SIMPLEST VERSION OF THE TRANSPORT EQUATION READS:

$$D\frac{\partial f}{\partial z} = \text{constant}$$

TOGETHER WITH THE BOUNDARY CONDITION THAT F(Z=|H|)=0 (FREE ESCAPE BOUNDARY CONDITION) LEADS TO

$$f(z) = f(z=0)\left(1 - \frac{z}{H}\right) \qquad \left|\frac{\partial f}{\partial z}\right| = \frac{f_0}{H}$$

$$\gamma_W(k) = \frac{n_{CR}(>p)}{n_{gas}}\frac{v_D - v_A}{v_A}\Omega_c \approx \frac{1}{n_{gas}}\frac{\sqrt{4\pi\rho}}{B_0}\frac{qB_0}{m_p c}D\frac{\partial f}{\partial z} \approx \frac{P_{CR}(>p)}{U_B}\frac{v_A}{H}\frac{1}{\mathcal{F}(k_{res})}$$

$$D(p) \approx \frac{1}{3}r_L(p)v\frac{1}{\mathcal{F}(k_{res})}$$

MANIFESTATIONS IN GALACTIC CR TRANSPORT

IF ONE TAKES THE REFERENCE VALUES OF THE PARAMETERS FOR THE HALO, AND RECALLS THAT FOR PARTICLES IN THE GEV ENERGY RANGE THE PRESSURE IS SIMILAR TO THAT OF THE MAGNETIC FIELD, THE GROWTH TIME CAN BE ESTIMATED AS ~(H/VA)F ~ 500 YRS.

IN THE CASE OF THE GALAXY THE GROWTH OF UNSTABLE ALFVEN WAVES IN THE HALO IS MAINLY LIMITED BY NON-LINEAR LANDAU DAMPING, FOR WHICH

$$\Gamma_D = k c_s \mathcal{F}$$

THE CR PRESSURE DROPS WITH ENERGY AS $P_{CR} \sim E^{-0.7}$ (PURE OBSERVATION!) HENCE:

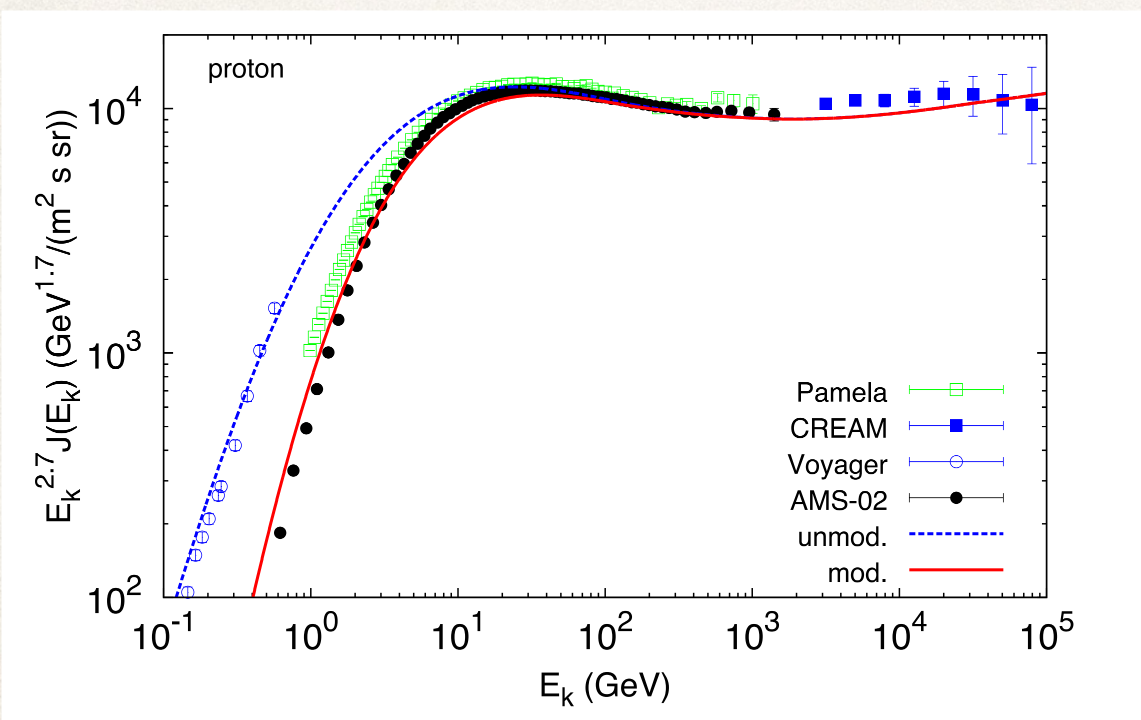
$$\Gamma_D = \gamma_W \rightarrow \mathcal{F}(k_{res}) \approx \left(\frac{v_A}{H} \frac{r_L(E)}{c_s} E_{GeV}^{0.7} \right)^{1/2} \sim 10^{-5} E_{GeV}^{0.15}$$

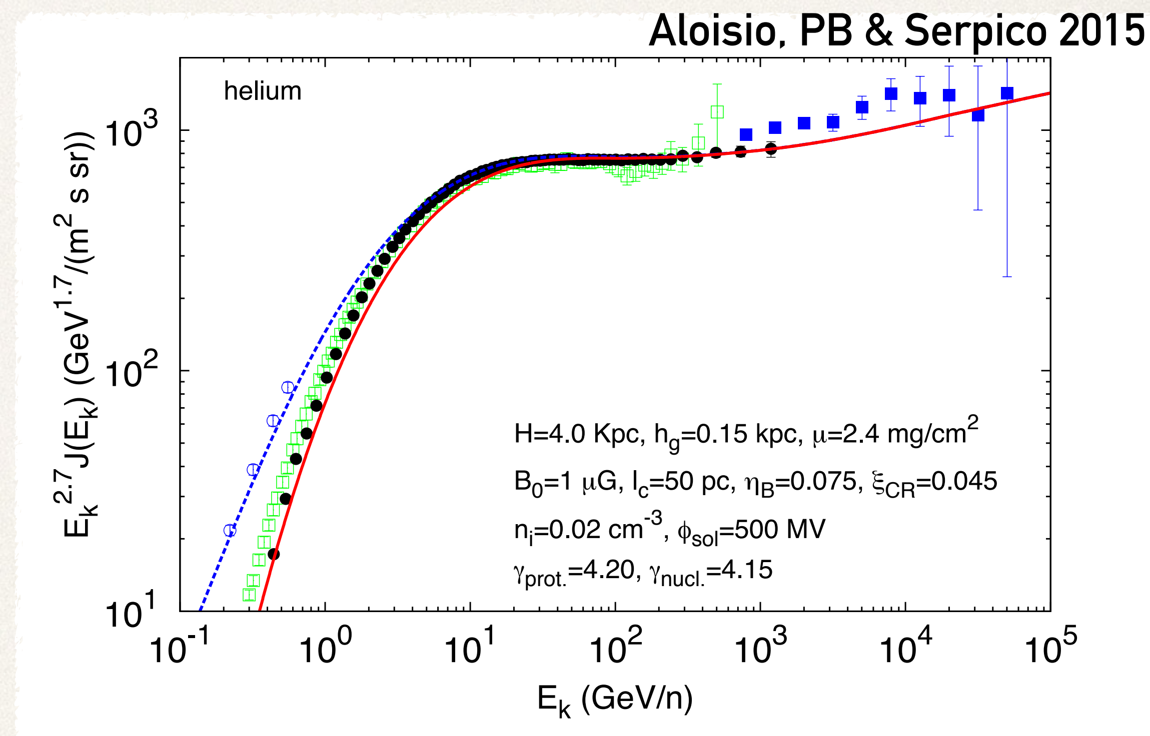
$$D(p) \approx \frac{1}{3} r_L(p) v \frac{1}{\mathcal{F}(k_{res})} \sim E^{0.85}$$

NOTE HOW IN THESE APPROACHES THE DIFFUSION COEFFICIENT IS AN OUTPUT OF THE PROBLEM

Manifestations in Galactic CR transport

The combination of scattering off pre-existing turbulence and self-generated turbulence leads to a change in D(p) at few hundred GV [*PB, Amato & Serpico 2012, Aloisio & PB 2014, Aloisio, PB & Serpico 2015, Evoli+ 2019*] — This phenomenon reflects in spectral breaks





Interestingly the real problem seems to be to understand the scattering for high energy CR, mainly due to the fact that Alfvenic turbulence becomes anisotropic and fast waves damp (Kempski&Quataert 2022)

Instabilities and Escape

The problem of how particles leave their acceleration sites or their confinement region is central and unsolved

We model it with boundary conditions, but we do not have a physical picture of why should the galactic halo have a free escape boundary or why the particles should leave the shock region of a SNR

Most likely these are linked to the development of CR induced instabilities

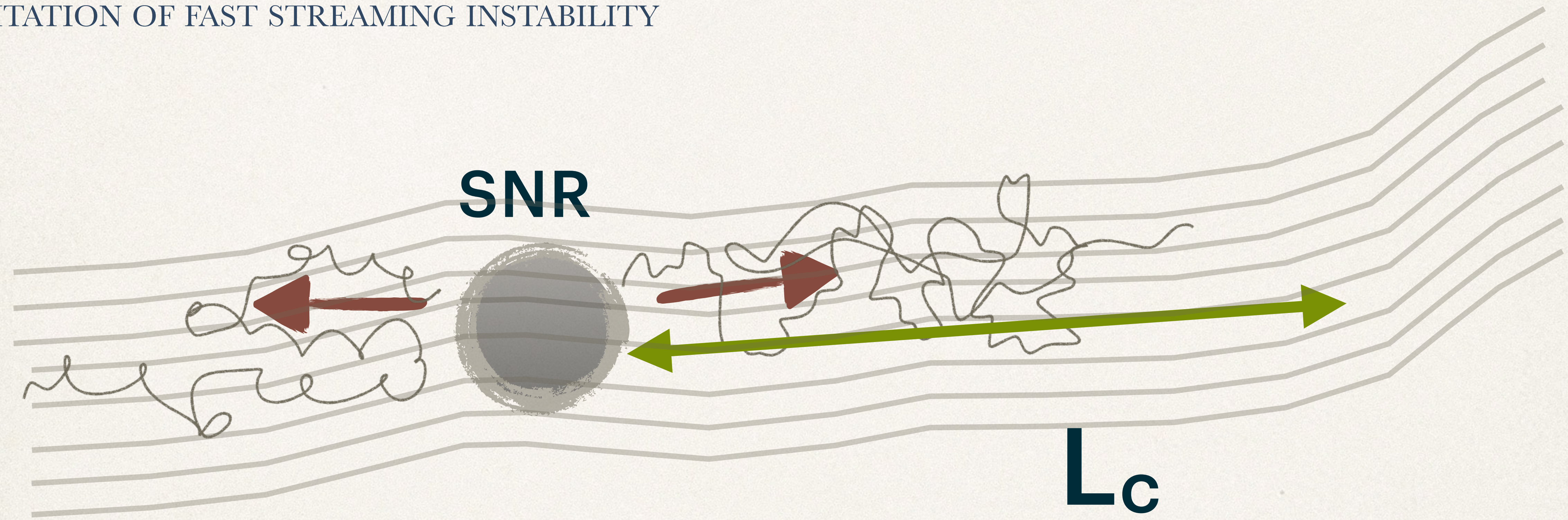
We look for manifestations of the implications of CR escape

Two cases here:

- Escape of particles from a supernova remnant
- escape of cosmic rays from our galaxy

Escape from a SNR

The gradients in the particle distribution around a source are very large and can lead to excitation of fast streaming instability

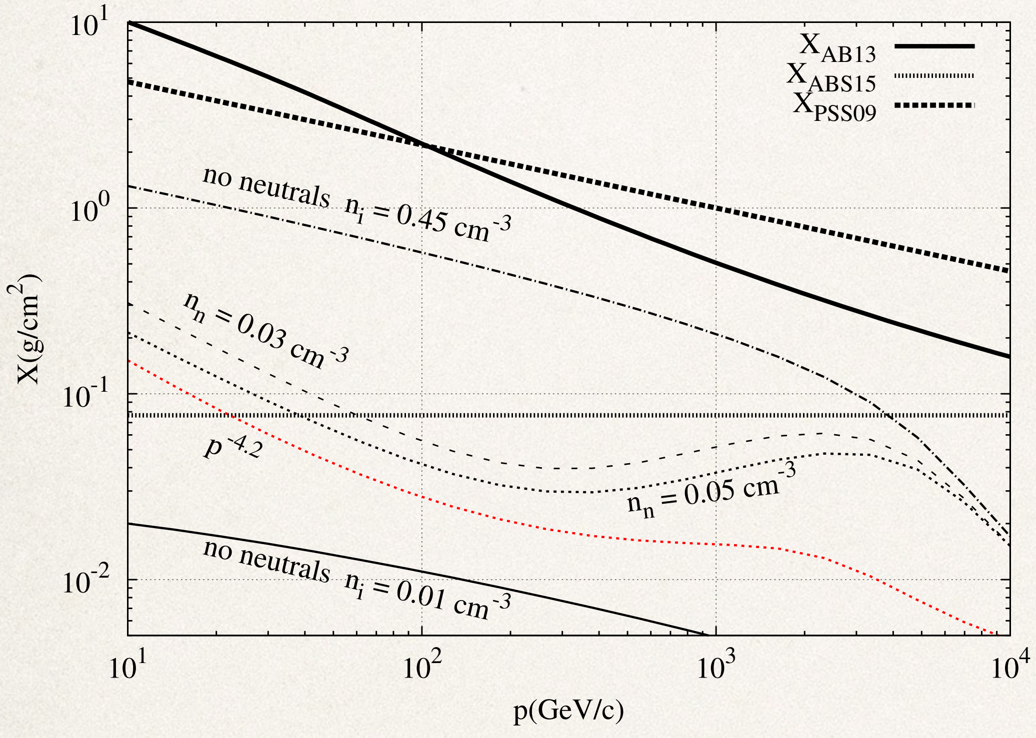


In the absence of non-linear effects the CR density inside Lc remains > than the Galactic average for a time

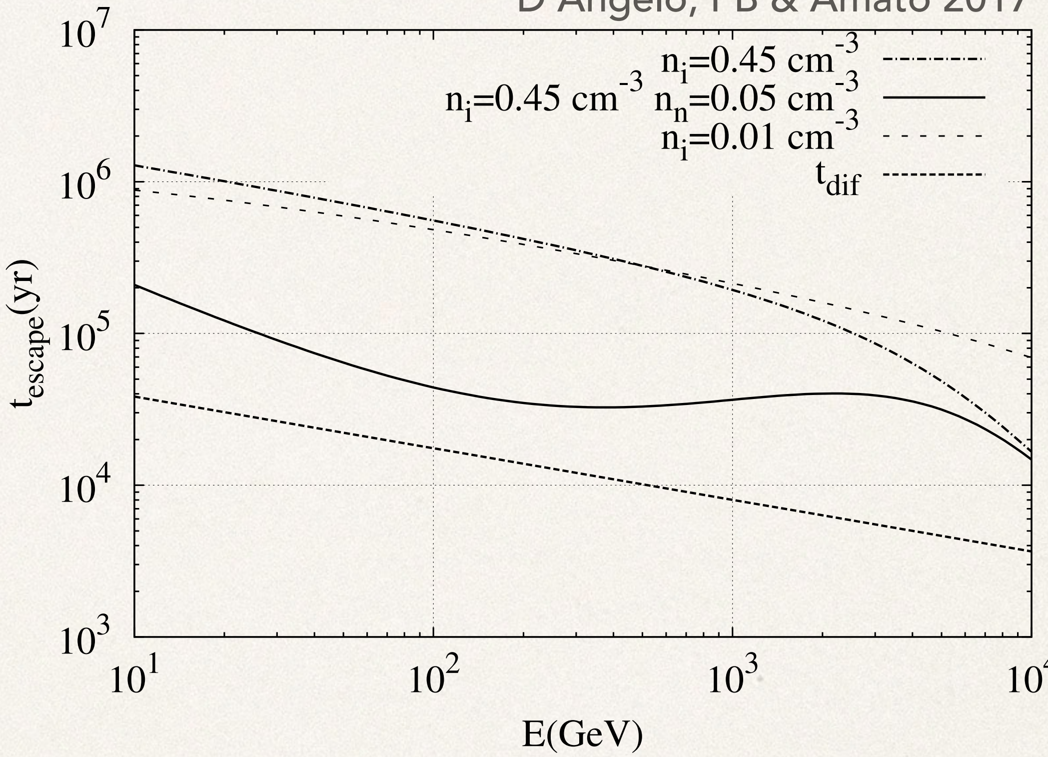
$$t_s \sim 2 \times 10^4 E_{GeV}^{-1/3} \ yr$$

After that, propagation becomes 3D and the density drops rapidly

Escape from a SNR

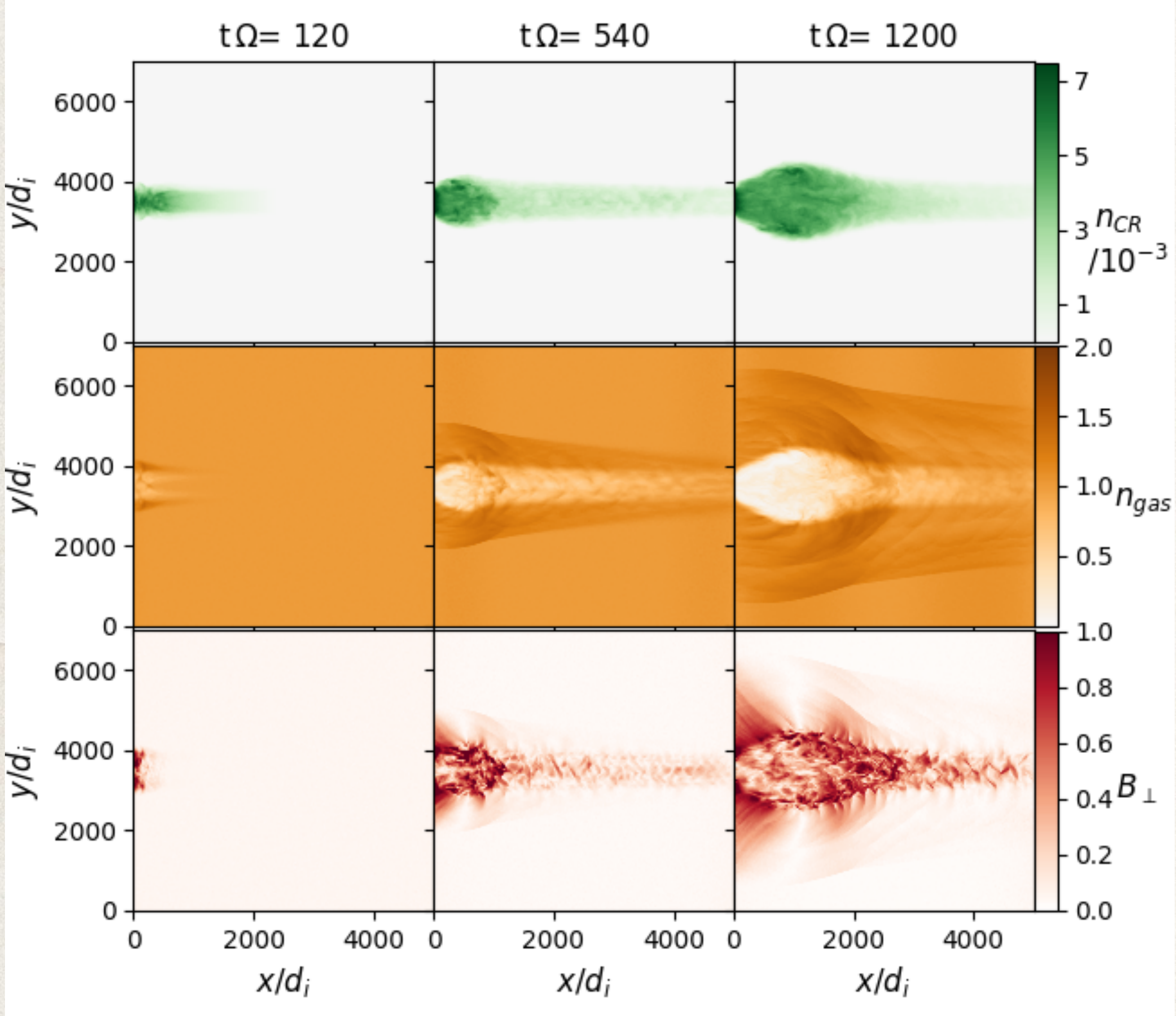


D'Angelo, PB & Amato 2017



A bulk of work already done (Malkov+2013, Nava+2016, D'Angelo+2017, Recchia+2022)… Depending on conditions (density, ionisation) the effect of CR may be more or less important and affect the grammage accumulated by CR around sources

Hybrid simulations



Schroer+, 2021 and 2022

- THE EXCITATION OF THE INSTABILITY LEADS TO STRONG PARTICLE SCATTERING, WHICH IN TURN INCREASES CR DENSITY NEAR THE SOURCE
- THE PRESSURE GRADIENT THAT DEVELOPS CREATES A FORCE THAT LEADS TO THE INFLATION OF A BUBBLE AROUND THE SOURCE
- THE SAME FORCE EVACUATES THE BUBBLE OF MOST PLASMA
- THERE IS NO FIELD IN THE PERP DIRECTION TO START WITH, BUT CR CREATE IT AT LATER TIMES (SUPPRESSED DIFFUSION, ABOUT 10 TIMES BOHM)

Non-resonant CR current driven instability: A primer

It will be shown elsewhere that, under some circumstances, CR can drive a non-resonant instability, that grows faster than the resonant one (Bell 2004)

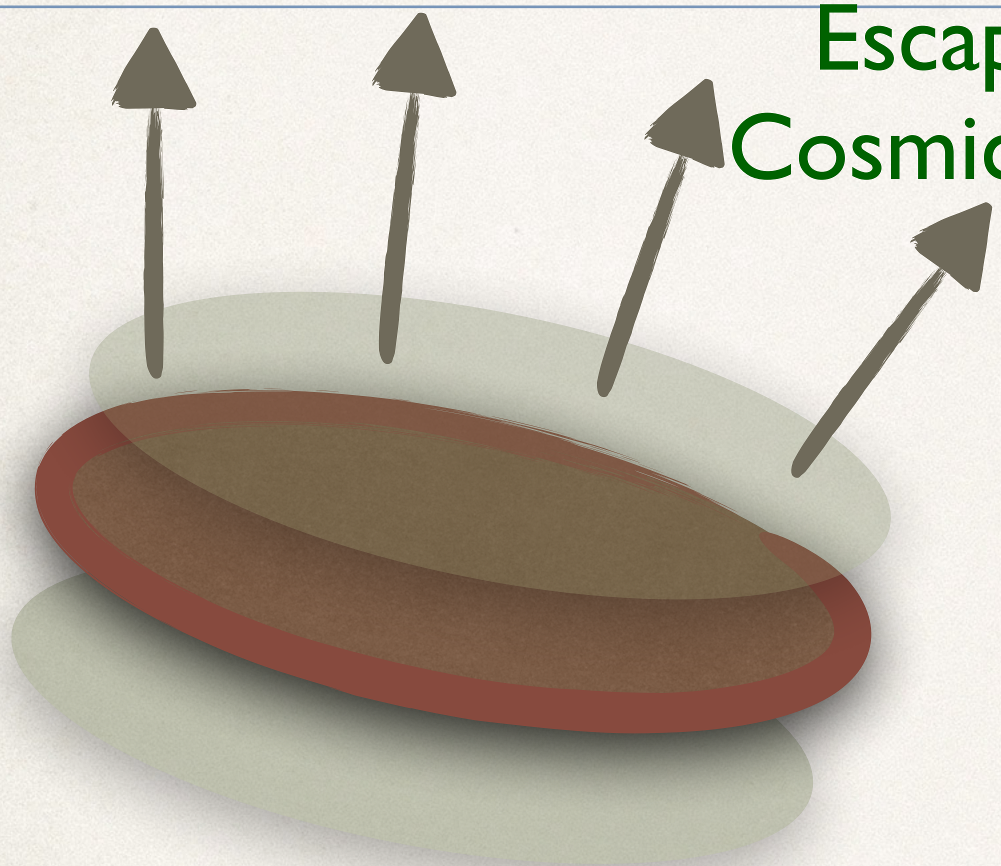
since it is non resonant, at least to start with, the current of CR is only perturbed weakly and it keeps driving the instability, on scales much smaller than R_L. The instability is excited if the energy density in the current is larger than that of the pre-existing B_0

In the case of transport in the galaxy: $D\frac{\partial f}{\partial z} = constant$

The spectrum of cosmic rays observed by an observer outside our galaxy is the same as injected by sources, not the same as we measure at the Earth!

$$\Phi_{esc}(p) = D\frac{\partial f}{\partial z}|_{z=H} = D\frac{\partial f}{\partial z}|_{z=0^+} = \frac{Q_0(p)}{2\pi R_d^2}$$

Escape of Galactic CR



As discussed above, the current of escaping CRs is very well known

$$J_{CR}(p) = eD\frac{\partial f}{\partial z}|_{z=H} = \frac{eQ_0(p)}{2\pi R_d^2}$$

Such current in the typical IGM excites a non-resonant Bell-like instability provided:

$$B_0 \leq B_{sat} \approx 2.4 \times 10^{-8} L_{41}^{1/2} R_{10}^{-1} \text{ G}$$

At a wavenumber $k_{max} = \frac{4\pi}{cB_0} J_{CR}$

and with a growth rate: $\gamma_{max} = k_{max} v_A \approx 0.5 \text{ yr}^{-1} \; \delta_G^{-1/2} E_{\text{GeV}}^{-1} L_{41} R_{10}^{-2}$

Escape of Galactic CR

Cosmic Rays should eventually slow down their free escape and accumulate at large distances from the Galaxy (~30 kpc)

Around every galaxy there should be a region with $B \sim L_{CR}^{1/2}$ due to the action of escaping CR

The interaction of CR with the locally overdense intergalactic medium leads to production of a quasi-isotropic flux of secondaries (for instance neutrinos)