

# Note on the Odd Perfect Numbers

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## Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . In 2011, Solé and Planat stated that the Riemann Hypothesis is true if and only if the inequality  $\frac{\pi^2}{6} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$  is satisfied for all primes  $q_n > 3$ , where  $\theta(x)$  is the Chebyshev function and  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant. Under the assumption of the Riemann Hypothesis is true and the inequality  $\frac{\pi^2}{6.4} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$  is satisfied for infinitely many prime numbers  $q_n$ , then we prove that there is not any odd perfect number at all.

*Keywords:* Riemann Hypothesis, Prime numbers, Odd perfect numbers, Superabundant numbers, Sum-of-divisors function

*2000 MSC:* 11M26, 11A41, 11A25

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## 1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . As usual  $\sigma(n)$  is the sum-of-divisors function of  $n$ :

$$\sum_{d|n} d$$

where  $d | n$  means the integer  $d$  divides  $n$ ,  $d \nmid n$  means the integer  $d$  does not divide  $n$  and  $d^k \parallel n$  means  $d^k | n$  and  $d^{k+1} \nmid n$ . Define  $f(n)$  and  $G(n)$  to be  $\frac{\sigma(n)}{n}$  and  $\frac{f(n)}{\log \log n}$  respectively, such that  $\log$  is the natural logarithm. We know these properties from these functions:

**Proposition 1.1.** [1]. Let  $\prod_{i=1}^r q_i^{a_i}$  be the representation of  $n$  as a product of primes  $q_1 < \dots < q_r$  with natural numbers as exponents  $a_1, \dots, a_r$ . Then,

$$f(n) = \left( \prod_{i=1}^r \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^r \left( 1 - \frac{1}{q_i^{a_i+1}} \right).$$

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**Proposition 1.2.** For every prime power  $q^a$ , we have that  $f(q^a) = \frac{q^{a+1}-1}{q^a \times (q-1)}$  [2]. If  $m, n \geq 2$  are natural numbers, then  $f(m \times n) \leq f(m) \times f(n)$  [2]. Moreover, if  $p$  is a prime number, and  $a, b$  two positive integers, then [2]:

$$f(p^{a+b}) - f(p^a) \times f(p^b) = -\frac{(p^a - 1) \times (p^b - 1)}{p^{a+b-1} \times (p - 1)^2}.$$

Say Robins( $n$ ) holds provided

$$G(n) < e^\gamma$$

where the constant  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant. The importance of this property is:

**Proposition 1.3.** Robins( $n$ ) holds for all natural numbers  $n > 5040$  if and only if the Riemann Hypothesis is true [3].

The Chebyshev function  $\theta(x)$  is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers  $p$  that are less than or equal to  $x$  [4]. We state the following property about this function:

**Proposition 1.4.** [4]. For  $x \geq 89909$ :

$$\theta(x) > \left(1 - \frac{0.068}{\log(x)}\right) \times x.$$

In mathematics,  $\Psi = n \times \prod_{q|n} \left(1 + \frac{1}{q}\right)$  is called the Dedekind  $\Psi$  function. Say Dedekinds( $q_n$ ) holds provided

$$\frac{\pi^2}{6} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$$

where  $q_n$  is the  $n$ th prime number. The importance of this inequality is:

**Proposition 1.5.** Dedekinds( $q_n$ ) holds for all prime numbers  $q_n > 3$  if and only if the Riemann Hypothesis is true [5].

Let  $q_1 = 2, q_2 = 3, \dots, q_k$  denote the first  $k$  consecutive primes, then an integer of the form  $\prod_{i=1}^k q_i^{a_i}$  with  $a_1 \geq a_2 \geq \dots \geq a_k \geq 0$  is called an Hardy-Ramanujan integer [6]. A natural number  $n$  is called superabundant precisely when, for all natural numbers  $m < n$

$$f(m) < f(n).$$

**Proposition 1.6.** If  $n$  is superabundant, then  $n$  is an Hardy-Ramanujan integer [7]. Let  $n$  be a superabundant number, then  $p \parallel n$  where  $p$  is the largest prime factor of  $n$  [7]. For large enough superabundant number  $n$ , we have that  $q^{a_q} < 2^{a_2}$  for  $q > 11$  where  $q^{a_q} \parallel n$  and  $2^{a_2} \parallel n$  [7]. For large enough superabundant number  $n$ , we obtain that  $\log n < \left(1 + \frac{0.5}{\log p}\right) \times p$  where  $p$  is the largest prime factor of  $n$  [4]. Let  $n$  be a superabundant number, then  $f(n) > (1 - \varepsilon(p)) \times \prod_{q|n} \frac{q}{q-1}$  where  $\varepsilon(p) = \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)$  and  $p$  is the largest prime factor of  $n$  [4].

In addition, we will use these properties:

**Proposition 1.7.** [5], [6]. For  $n \geq 2$ :

$$\prod_{q > q_n} \frac{q^2}{q^2 - 1} \leq e^{\frac{2}{q_n}}.$$

**Proposition 1.8.** It is known that [8]:

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

In number theory, a perfect number is a positive integer  $n$  such that  $f(n) = 2$ . Euclid proved that every even perfect number is of the form  $2^{s-1} \times (2^s - 1)$  whenever  $2^s - 1$  is prime. It is unknown whether any odd perfect numbers exist, though various results have been obtained:

**Proposition 1.9.** Any odd perfect number  $N$  must satisfy the following conditions:  $N > 10^{1500}$  and the largest prime factor of  $N$  is greater than  $10^8$  [9], [10].

Now, we state the following conjecture:

**Conjecture 1.10.** We assume that the Riemann Hypothesis is true and the inequality

$$\frac{\pi^2}{6.4} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$$

is satisfied for infinitely many prime numbers  $q_n$ .

Under the assumption of the Conjecture 1.10, we prove that there is not any odd perfect number at all.

## 2. Main Theorem

**Theorem 2.1.** Under the assumption of the Conjecture 1.10, we prove that there is not any odd perfect number at all.

*Proof.* Suppose that  $N$  is the smallest odd perfect number, then we will show its existence implies that the Conjecture 1.10 is false. There is always a large enough superabundant number  $n$  such that  $n$  is a multiple of  $N$ . We would have

$$f(n) \leq f(N) \times f\left(\frac{n}{N}\right)$$

according to the Proposition 1.2. That is the same as

$$f(n) \leq 2 \times f\left(\frac{n}{N}\right)$$

since  $f(N) = 2$ , because  $N$  is a perfect number. Hence,

$$\begin{aligned}\frac{f(n)}{2} &= \frac{(2 - \frac{1}{2^{a_2}}) \times f(\frac{n}{2^{a_2}})}{2} \\ &= f(\frac{n}{2^{a_2}}) \times \frac{(2 - \frac{1}{2^{a_2}})}{2} \\ &= f(\frac{n}{2^{a_2}}) \times \frac{2^{a_2+1} - 1}{2^{a_2+1}}\end{aligned}$$

when  $2^{a_2} \parallel n$  due to the Proposition 1.2. In this way, we have

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq \frac{2^{a_2+1}}{2^{a_2+1} - 1}.$$

However, we know that  $p < 2^{a_2}$  because of  $p > 10^8 > 11$  and the Propositions 1.6 and 1.9, where  $p$  is the largest prime factor of  $n$ . Consequently,

$$\frac{2^{a_2+1}}{2^{a_2+1} - 1} \leq \frac{2 \times p}{2 \times p - 1}$$

since  $\frac{x}{x-1}$  decreases when  $x \geq 2$  increases. In addition, we know that

$$\frac{2 \times p}{2 \times p - 1} \leq f(p)$$

where we know that  $f(p) = \frac{p+1}{p}$  from the Proposition 1.2. Certainly,

$$\begin{aligned}2 \times p^2 &\leq (p+1) \times (2 \times p - 1) \\ &= 2 \times p^2 + 2 \times p - p - 1 \\ &= 2 \times p^2 + p - 1\end{aligned}$$

where this inequality is satisfied for every prime number  $p$ . So,

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq f(p)$$

where we know that  $p \parallel n$  from the Proposition 1.6. Using the Conjecture 1.10, we have that

$$\begin{aligned}e^\gamma &> G(n) \\ &= \frac{f(\frac{n}{p}) \times f(p)}{\log \log n} \\ &\geq \frac{f(\frac{n}{p}) \times f(\frac{n}{2^{a_2}})}{f(\frac{n}{N}) \times \log \log n}\end{aligned}$$

since  $f(\dots)$  is multiplicative and as a consequence of the Proposition 1.3. This is equivalent to

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} < \frac{e^\gamma}{4} \times \log \log n.$$

Under the assumption of the Conjecture 1.10, we deduce that:

$$\frac{\pi^2}{6.4} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(p)$$

which is the same as

$$\frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log((\theta(p))^{0.8}).$$

From the Propositions 1.1 and 1.6, we know that

$$f\left(\frac{n}{2^{a_2}}\right) = \left(\prod_{i=2}^k \frac{q_i}{q_i - 1}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

where  $q_k = p$  and  $q_1 = 2$ . We know that

$$\frac{q_i}{q_i - 1} = \left(1 + \frac{1}{q_i}\right) \times \frac{q_i^2}{q_i^2 - 1}.$$

Using the previous inequality and the Conjecture 1.10, we obtain that

$$\begin{aligned} e^\gamma \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \log((\theta(p))^{0.8}) &< \frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= f\left(\frac{n}{2^{a_2}}\right) \times \frac{3}{2} \times \prod_{q > p} \frac{q^2}{q^2 - 1} \\ &\leq f\left(\frac{n}{2^{a_2}}\right) \times \frac{3}{2} \times e^{\frac{2}{p}} \end{aligned}$$

according to the Proposition 1.7. Taking into account that  $p > 10^8 > 3$  and  $n$  is superabundant:

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} > \frac{e^\gamma}{f\left(\frac{n}{2^{a_2}}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

We use the previous inequality to show that

$$\frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < \frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log \log n.$$

For large enough superabundant number  $n$  and  $p > 10^8$ , then

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log \log n \leq \frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(\left(1 - \frac{0.068}{\log 10^8}\right) \times 10^8\right)^{0.8}} \times \log\left(\left(1 + \frac{0.5}{\log 10^8}\right) \times 10^8\right)$$

because of the Propositions 1.4 and 1.6. We obtain that

$$\frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(\left(1 - \frac{0.068}{\log 10^8}\right) \times 10^8\right)^{0.8}} \times \log\left(\left(1 + \frac{0.5}{\log 10^8}\right) \times 10^8\right) < 1.87811.$$

Thus,

$$\frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.87811.$$

For every prime  $p_j$  that divides  $N$  such that  $p_j^{a_j} \parallel N$  and  $p_j^{a_j+b_j} \parallel n$  for  $a_j, b_j$  two natural numbers, we have that

$$f(p_j^{a_j+b_j}) - f(p_j^{a_j}) \times f(p_j^{b_j}) = -\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

in the Proposition 1.2. This is equal to

$$\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})} = f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}.$$

Hence,

$$\begin{aligned} \frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) &= \prod_j \left(\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= \prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &> 1.999 \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &> 1.999 \times \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) \times \frac{1}{\left(1 - \frac{1}{2^{a_2+1}}\right)} \\ &> 1.999 \times \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) \\ &> 1.999 \times \left(1 - \frac{1}{\log 10^8} \times \left(1 + \frac{1.5}{\log 10^8}\right)\right) \\ &> 1.88 \\ &> 1.87811 \end{aligned}$$

using the Propositions 1.6 and 1.1 since we know that the expression

$$\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

tends to 0 as  $b_j$  tends to infinity for every odd prime  $p_j$  where

$$\begin{aligned} \prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) &\approx \prod_j (f(p_j^{a_j})) \\ &= f(N) \\ &= 2. \end{aligned}$$

Certainly, the fraction  $\frac{f(\frac{n}{p})}{f(\frac{n}{N})}$  gets closer to 2 as long as we take  $n$  bigger and bigger. In addition, we note that

$$\begin{aligned} \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) &< \prod_{i=1}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \left(1 - \frac{1}{2^{a_2+1}}\right) \end{aligned}$$

after taking into account the Proposition 1.6. However,

$$1.87811 < \frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.87811$$

is a contradiction. By contraposition, the number  $N$  does not exist under the assumption of the Conjecture 1.10. The smallest counterexample  $N$  must comply that  $N > 10^{1500}$  and therefore, we will always be capable of obtaining a large enough superabundant number  $n$  that is multiple of  $N$ . Note that, this proof fails for even perfect numbers or for some other odd numbers  $N$  such that  $f(N) \neq 2$  (precisely when we may consider a large enough superabundant number  $n$ ).  $\square$

## Acknowledgments

The author would like to thank his mother, maternal brother and his friend Sonia for their support.

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