Note on the Odd Perfect Numbers

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Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.

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1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides n, $d \nmid n$ means the integer d does not divide n and $d^k \parallel n$ means $d^k \mid n$ and $d^{k+1} \nmid n$. Define $f(n)$ and $G(n)$ to be $\frac{\sigma(n)}{n}$ and $\frac{f(n)}{\log \log n}$ respectively, such that log is the natural logarithm. We know these properties from these functions:

Proposition 1.1. *[1]. Let $\prod_{i=1}^{r} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_r$ with natural numbers as exponents $a_1, \ldots, a_r$. Then,*

$$f(n) = \left(\prod_{i=1}^{r} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

Proposition 1.2. *For every prime power q^a, we have that $f(q^a) = \frac{q^{a+1}-1}{q^a \times (q-1)}$ [2]. If $m, n \geq 2$ are natural numbers, then $f(m \times n) \leq f(m) \times f(n)$ [2]. Moreover, if p is a prime number, and a, b two positive integers, then [2]:*

$$f(p^{a+b}) - f(p^a) \times f(p^b) = -\frac{(p^a - 1) \times (p^b - 1)}{p^{a+b-1} \times (p-1)^2}.$$

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Say Robins(n) holds provided

$$G(n) < e^{\gamma}$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The importance of this property is:

Proposition 1.3. Robins(n) *holds for all natural numbers* $n > 5040$ *if and only if the Riemann Hypothesis is true [3].*

The Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [4]. We state the following properties about this function:

Proposition 1.4. *[4]. For* $x \geq 89909$*:*

$$\theta(x) > (1 - \frac{0.068}{\log(x)}) \times x.$$

Proposition 1.5. *[5]. For every* $x > 1$*:*

$$\theta(x) < (1 + \frac{0.15}{\log^3 x}) \times x.$$

Proposition 1.6. *[5]. Under the assumption of the Riemann Hypothesis, then there are infinitely many prime numbers* q_n *for which*

$$0 < \theta(q_n) - q_n < \frac{1}{8 \times \pi} \times \sqrt{q_n} \times \log^2 q_n.$$

In mathematics, $\Psi = n \times \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function. Say Dedekinds(q_n) holds provided

$$\frac{\pi^2}{6} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log \theta(q_n)$$

where q_n is the nth prime number. The importance of this inequality is:

Proposition 1.7. Dedekinds(q_n) *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann Hypothesis is true [6].*

Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ is called an Hardy-Ramanujan integer [7]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 1.8. *If n is superabundant, then n is an Hardy-Ramanujan integer [8]. Let n be a superabundant number, then* $p \parallel n$ *where p is the largest prime factor of n [8]. For large enough superabundant number n, we have that* $q^{a_q} < 2^{a_2}$ *for* $q > 11$ *where* $q^{a_q} \parallel n$ *and* $2^{a_2} \parallel n$ *[8]. For large enough superabundant number n, we obtain that* $\log n < (1 + \frac{0.5}{\log p}) \times p$ *where p is the largest prime factor of n [4]. Let n be a superabundant number, then* $f(n) > (1 - \varepsilon(p)) \times \prod_{q|n} \frac{q}{q-1}$ *where* $\varepsilon(p) = \frac{1}{\log p} \times (1 + \frac{1.5}{\log p})$ *and p is the largest prime factor of n [4].*

On the sum of the reciprocals of power prime numbers not exceeding x, we have these results:

Proposition 1.9. *[5]. For $x \geq 2278383$:*

$$\sum_{p \leq x} \frac{1}{p} \geq \log \log x + B - \frac{1}{5 \times \log^3 x}$$

where $B \approx 0.261497212847642$ is the Meissel-Mertens constant [9].

Proposition 1.10. *[10]. For $y \geq 10^8$:*

$$\sum_{p \geq x} \frac{1}{p^2} \leq \frac{1}{y \times \log y} - \frac{1}{y \times \log^2 y} + \frac{2}{y \times \log^3 y} - \frac{2.07}{y \times \log^4 y}.$$

In addition, we will use these properties:

Proposition 1.11. *It is known that [11]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

Proposition 1.12. *[6], [7]. For $n \geq 2$:*

$$\prod_{q > q_n} \frac{q^2}{q^2 - 1} \leq e^{\frac{2}{q_n}}.$$

Proposition 1.13. *[12]. For $x \geq 1$:*

$$\frac{1}{x + 0.5} < \log(1 + \frac{1}{x}).$$

In number theory, a perfect number is a positive integer n such that $f(n) = 2$. Euclid proved that every even perfect number is of the form $2^{s-1} \times (2^s - 1)$ whenever $2^s - 1$ is prime. It is unknown whether any odd perfect numbers exist, though various results have been obtained:

Proposition 1.14. *Any odd perfect number N must satisfy the following conditions: $N > 10^{1500}$ and the largest prime factor of N is greater than 10^8 [13], [14].*

Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.

2. Numerical Calculations

Lemma 2.1.

$$\sum_{q} \left(\frac{1}{q \times (q + 0.5)} \right) < 0.38050392718998946944 1$$

Proof. Using the Proposition 1.10, we check by computer that,

$$\sum_{q} \left(\frac{1}{q \times (q + 0.5)} \right) < \sum_{q < 10^8} \left(\frac{1}{q \times (q + 0.5)} \right) + \sum_{q \geq 10^8} \left(\frac{1}{q^2} \right)$$

$$\leq 0.380503926673572 + \frac{1}{10^8 \times \log 10^8} - \frac{1}{10^8 \times \log^2 10^8} + \frac{2}{10^8 \times \log^3 10^8} - \frac{2.07}{10^8 \times \log^4 10^8}$$

$$< 0.380503927189989469441.$$

□

3. Central Lemma

Lemma 3.1. *For all prime numbers $q_n > 10^8$, we have that*

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^{0.0712132519795} \times \log q_n$$

is satisfied.

Proof. We apply the logarithm to the both sides of the inequality,

$$\sum_{q \leq q_n} \log(1 + \frac{1}{q}) > 0.0712132519795 + \log \log q_n.$$

We use the Proposition 1.13,

$$\sum_{q \leq q_n} \frac{1}{q + 0.5} > 0.0712132519795 + \log \log q_n.$$

This is the same as

$$\sum_{q \leq q_n} \left(\frac{1}{q}\right) - \sum_{q \leq q_n} \left(\frac{1}{q} - \frac{1}{q + 0.5}\right) > 0.0712132519795 + \log \log q_n.$$

We know that

$$\frac{1}{q} - \frac{1}{q + 0.5} = \frac{1}{2 \times q \times (q + 0.5)}.$$

Hence,

$$\sum_{q \leq q_n} \left(\frac{1}{q}\right) - \log \log q_n > 0.0712132519795 + \sum_{q \leq q_n} \left(\frac{1}{2 \times q \times (q + 0.5)}\right).$$

We use that Proposition 1.9,

$$B - \frac{1}{5 \times \log^3(q_n)} > 0.0712132519795 + \sum_{q \leq q_n} \left(\frac{1}{2 \times q \times (q + 0.5)}\right)$$

that is equivalent to

$$B > 0.0712132519795 + \sum_{q \leq q_n} \left(\frac{1}{2 \times q \times (q + 0.5)}\right) + \frac{1}{5 \times \log^3(q_n)}.$$

Using the numerical computation in the Lemma 2.1, we only need to prove that

$$B > 0.0712132519795 + \frac{0.380503927189989469441}{2} + \frac{1}{5 \times \log^3(10^8)}$$

since $\frac{1}{5 \times \log^3(q_n)}$ decreases as q_n increases. In this way, we obtain that

$$B > 0.261497212847634$$

and thus, the proof is done. □

4. Main Insight

Lemma 4.1. *Under the assumption of the Riemann Hypothesis, we prove that*

$$\frac{\pi^2}{6.4} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log \theta(q_n)$$

is satisfied for infinitely many prime numbers q_n.

Proof. We know there are infinitely many prime numbers q_n:

$$0 < \theta(q_n) - q_n < \frac{1}{8 \times \pi} \times \sqrt{q_n} \times \log^2 q_n$$

under the assumption of the Riemann Hypothesis because of the Proposition 1.6. That is the same as

$$\theta(q_n) \times \frac{\theta(q_n) - q_n}{\theta(q_n)} < \frac{1}{8 \times \pi} \times \sqrt{q_n} \times \log^2 q_n.$$

If we apply the logarithm to the both side of the inequality

$$\log \theta(q_n) + \log \frac{\theta(q_n) - q_n}{\theta(q_n)} < \log \left(\frac{1}{8 \times \pi} \times \sqrt{q_n} \times \log^2 q_n\right)$$

which is

$$\log \theta(q_n) < \log \left(\frac{1}{8 \times \pi} \times \sqrt{q_n} \times \log^2 q_n\right) + \log \frac{\theta(q_n)}{\theta(q_n) - q_n}.$$

In addition, we known that

$$\begin{aligned}
\log \frac{\theta(q_n)}{\theta(q_n) - q_n} &< \log \frac{(1 + \frac{0.15}{\log^3 q_n}) \times q_n}{(1 + \frac{0.15}{\log^3 q_n}) \times q_n - q_n} \\
&= \log \frac{(1 + \frac{0.15}{\log^3 q_n}) \times q_n}{q_n \times (1 + \frac{0.15}{\log^3 q_n} - 1)} \\
&= \log \frac{(1 + \frac{0.15}{\log^3 q_n})}{\frac{0.15}{\log^3 q_n}} \\
&= \log \left(1 + \frac{1}{\frac{0.15}{\log^3 q_n}}\right) \\
&= \log \left(1 + \frac{\log^3 q_n}{0.15}\right)
\end{aligned}$$

according to the Proposition 1.5. In this way, we obtain that

$$\frac{e^{\gamma}}{\frac{\pi^2}{6.4}} \times \log \theta(q_n) < \frac{e^{\gamma}}{\frac{\pi^2}{6.4}} \times \left(\log \left(\frac{1}{8 \times \pi} \times \sqrt{q_n} \times \log^2 q_n\right) + \log \left(1 + \frac{\log^3 q_n}{0.15}\right)\right)$$

after of multiplying the both sides by $\frac{e^{\gamma}}{\frac{\pi^2}{6.4}}$. Hence, it is enough to show there are infinitely many prime numbers q_n such that

$$\prod_{q\leq q_n}\left(1+\frac{1}{q}\right)\geq\frac{e^{\gamma}}{\frac{\pi^2}{6.4}}\times\left(\log\left(\frac{1}{8\times\pi}\times\sqrt{q_n}\times\log^2 q_n\right)+\log\left(1+\frac{\log^3 q_n}{0.15}\right)\right).$$

The previous inequality will be satisfied when

$$e^{0.0712132519795}\times\log q_n\geq\frac{e^{\gamma}}{\frac{\pi^2}{6.4}}\times\left(\log\left(\frac{1}{8\times\pi}\times\sqrt{q_n}\times\log^2 q_n\right)+\log\left(1+\frac{\log^3 q_n}{0.15}\right)\right)$$

due to the Lemma 3.1. That is equivalent to

$$\frac{\frac{\pi^2}{6.4}\times e^{0.0712132519795}}{e^{\gamma}}\times\log q_n\geq\log\left(\frac{1}{8\times\pi}\times\sqrt{q_n}\times\log^2 q_n\right)+\log\left(1+\frac{\log^3 q_n}{0.15}\right)$$

that is

$$q_n\geq\left(\frac{1}{8\times\pi}\times\sqrt{q_n}\times\log^2 q_n\times(1+\frac{\log^3 q_n}{0.15})\right)^{\frac{e^{\gamma}}{\frac{\pi^2}{6.4}\times e^{0.0712132519795}}}$$

which is true for large enough prime numbers q_n. □

5. Main Theorem

Theorem 5.1. *Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.*

Proof. Suppose that N is the smallest odd perfect number, then we will show its existence implies that the Riemann Hypothesis is false. There is always a large enough superabundant number n such that n is a multiple of N. We would have

$$f(n)\leq f(N)\times f(\frac{n}{N})$$

according to the Proposition 1.2. That is the same as

$$f(n)\leq 2\times f(\frac{n}{N})$$

since $f(N)=2$, because N is a perfect number. Hence,

$$\begin{aligned}\frac{f(n)}{2}&=\frac{(2-\frac{1}{2^{a_2}})\times f(\frac{n}{2^{a_2}})}{2}\\&=f(\frac{n}{2^{a_2}})\times\frac{(2-\frac{1}{2^{a_2}})}{2}\\&=f(\frac{n}{2^{a_2}})\times\frac{2^{a_2+1}-1}{2^{a_2+1}}\end{aligned}$$

when $2^{a_2}\parallel n$ due to the Proposition 1.2. In this way, we have

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})}\leq\frac{2^{a_2+1}}{2^{a_2+1}-1}.$$

However, we know that $p < 2^{a_2}$ because of $p > 10^8 > 11$ and the Propositions 1.8 and 1.14, where p is the largest prime factor of n. Consequently,

$$\frac{2^{a_2+1}}{2^{a_2+1} - 1} \leq \frac{2 \times p}{2 \times p - 1}$$

since $\frac{x}{x-1}$ decreases when $x \geq 2$ increases. In addition, we know that

$$\frac{2 \times p}{2 \times p - 1} \leq f(p)$$

where we know that $f(p) = \frac{p+1}{p}$ from the Proposition 1.2. Certainly,

$$\begin{aligned} 2 \times p^2 &\leq (p + 1) \times (2 \times p - 1) \\ &= 2 \times p^2 + 2 \times p - p - 1 \\ &= 2 \times p^2 + p - 1 \end{aligned}$$

where this inequality is satisfied for every prime number p. So,

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq f(p)$$

where we know that $p \parallel n$ from the Proposition 1.8. Under the assumption of the Riemann Hypothesis, we have that

$$\begin{aligned} e^{\gamma} &> G(n) \\ &= \frac{f(\frac{n}{p}) \times f(p)}{\log \log n} \\ &\geq \frac{f(\frac{n}{p}) \times f(\frac{n}{2^{a_2}})}{f(\frac{n}{N}) \times \log \log n} \end{aligned}$$

since $f(\ldots)$ is multiplicative and as a consequence of the Proposition 1.3. This is equivalent to

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} < \frac{e^{\gamma}}{f(\frac{n}{2^{a_2}})} \times \log \log n.$$

Using the Lemma 4.1, we deduce that:

$$\frac{\pi^2}{6.4} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log \theta(p)$$

which is the same as

$$\frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log((\theta(p))^{0.8}).$$

From the Propositions 1.1 and 1.8, we know that

$$f(\frac{n}{2^{a_2}}) = \left(\prod_{i=2}^{k} \frac{q_i}{q_i - 1}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

where $q_k = p$ and $q_1 = 2$. We know that

$$\frac{q_i}{q_i - 1} = \left(1 + \frac{1}{q_i}\right) \times \frac{q_i^2}{q_i^2 - 1}.$$

Using the previous inequality and the Lemma 4.1, we obtain that

$$\begin{aligned} e^{\gamma} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \log((\theta(p))^{0.8}) &< \frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= f(\frac{n}{2^{a_2}}) \times \frac{3}{2} \times \prod_{q > p} \frac{q^2}{q^2 - 1} \\ &\leq f(\frac{n}{2^{a_2}}) \times \frac{3}{2} \times e^{\frac{2}{p}} \end{aligned}$$

according to the Proposition 1.12. Taking into account that $p > 10^8 > 3$ and n is superabundant:

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} > \frac{e^{\gamma}}{f(\frac{n}{2^{a_2}})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

We use the previous inequality to show that

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < \frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log\log n.$$

For large enough superabundant number n and $p > 10^8$, then

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log\log n \leq \frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(((1 - \frac{0.068}{\log 10^8}) \times 10^8)^{0.8}\right)} \times \log\left((1 + \frac{0.5}{\log 10^8}) \times 10^8\right)$$

because of the Propositions 1.4 and 1.8. We obtain that

$$\frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(((1 - \frac{0.068}{\log 10^8}) \times 10^8)^{0.8}\right)} \times \log\left((1 + \frac{0.5}{\log 10^8}) \times 10^8\right) < 1.87811.$$

Thus,

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.87811.$$

For every prime p_j that divides N such that $p_j^{a_j} \parallel N$ and $p_j^{a_j+b_j} \parallel n$ for a_j, b_j two natural numbers, we have that

$$f(p_j^{a_j+b_j}) - f(p_j^{a_j}) \times f(p_j^{b_j}) = -\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

in the Proposition 1.2. This is equal to

$$\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})} = f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}.$$

Hence,

$$\begin{aligned}
\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right) &= \prod_{j}\left(\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})}\right) \times \prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\\
&= \prod_{j}\left(f(p_j^{a_j}) - \frac{(p_j^{a_j}-1)\times(p_j^{b_j}-1)}{f(p_j^{b_j})\times p_j^{a_j+b_j-1}\times(p_j-1)^2}\right) \times \prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\\
&> 1.999 \times \prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\\
&> 1.999 \times \left(1-\frac{1}{\log p}\times(1+\frac{1.5}{\log p})\right) \times \frac{1}{(1-\frac{1}{2^{a_2+1}})}\\
&> 1.999 \times \left(1-\frac{1}{\log p}\times(1+\frac{1.5}{\log p})\right)\\
&> 1.999 \times \left(1-\frac{1}{\log 10^8}\times(1+\frac{1.5}{\log 10^8})\right)\\
&> 1.88\\
&> 1.87811
\end{aligned}$$

using the Propositions 1.8 and 1.1 since we know that the expression

$$\frac{(p_j^{a_j}-1)\times(p_j^{b_j}-1)}{f(p_j^{b_j})\times p_j^{a_j+b_j-1}\times(p_j-1)^2}$$

tends to 0 as b_j tends to infinity for every odd prime p_j where

$$\begin{aligned}
\prod_{j}\left(f(p_j^{a_j}) - \frac{(p_j^{a_j}-1)\times(p_j^{b_j}-1)}{f(p_j^{b_j})\times p_j^{a_j+b_j-1}\times(p_j-1)^2}\right) &\approx \prod_{j}\left(f(p_j^{a_j})\right)\\
&= f(N)\\
&= 2.
\end{aligned}$$

Certainly, the fraction $\frac{f(\frac{n}{p})}{f(\frac{n}{N})}$ gets closer to 2 as long as we take n bigger and bigger. In addition, we note that

$$\begin{aligned}
\left(1-\frac{1}{\log p}\times(1+\frac{1.5}{\log p})\right) &< \prod_{i=1}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\\
&= \prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right)\times(1-\frac{1}{2^{a_2+1}})
\end{aligned}$$

after taking into account the Proposition 1.8. However,

$$1.87811 < \frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k}\left(1-\frac{1}{q_i^{a_i+1}}\right) < 1.87811$$

is a contradiction. By contraposition, the number N does not exist under the assumption of the Riemann Hypothesis. The smallest counterexample N must comply that $N > 10^{1500}$ and therefore, we will always be capable of obtaining a large enough superabundant number n that is multiple of N. Note that, this proof fails for even perfect numbers or for some other odd numbers N such that $f(N) > 2$, precisely when we consider a large enough superabundant number n. $\square$

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References

[1] A. Hertlein, Robin's Inequality for New Families of Integers, Integers 18, (2018).

[2] R. Vojak, On numbers satisfying Robin's inequality, properties of the next counterexample and improved specific bounds, arXiv preprint arXiv:2005.09307, (2020).

[3] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, J. Math. pures appl 63 (2) (1984) 187–213.

[4] S. Nazardonyavi, S. Yakubovich, Superabundant numbers, their subsequences and the Riemann hypothesis, arXiv preprint arXiv:1211.2147, (2012).

[5] C. Axler, New Estimates for Some Functions Defined Over Primes, Integers 18, (2018).

[6] P. Solé, M. Planat, Extreme values of the Dedekind ψ function, Journal of Combinatorics and Number Theory 3 (1) (2011) 33–38.

[7] Y. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin's criterion for the Riemann hypothesis, Journal de Théorie des Nombres de Bordeaux 19 (2) (2007) 357–372. doi:10.5802/jtnb.591.

[8] L. Alaoglu, P. Erdős, On highly composite and similar numbers, Transactions of the American Mathematical Society 56 (3) (1944) 448–469. doi:10.2307/1990319.

[9] F. Mertens, Ein Beitrag zur analytischen Zahlentheorie., J. reine angew. Math. 1874 (78) (1874) 46–62. doi:10.1515/crll.1874.78.46.

[10] J.-L. Nicolas, The sum of divisors function and the Riemann hypothesis, The Ramanujan Journal (2021) 1–45doi:10.1007/s11139-021-00491-y.

[11] H. M. Edwards, Riemann's Zeta Function, Dover Publications, 2001.

[12] A. Ghosh, An asymptotic formula for the Chebyshev theta function, Notes on Number Theory and Discrete Mathematics 25 (4) (2019) 1–7. doi:10.7546/nntdm.2019.25.4.1-7.

[13] P. Ochem, M. Rao, Odd perfect numbers are greater than 10^{1500}, Mathematics of Computation 81 (279) (2012) 1869–1877. doi:10.1090/S0025-5718-2012-02563-4.

[14] T. Goto, Y. Ohno, Odd perfect numbers have a prime factor exceeding 10^8, Mathematics of Computation 77 (263) (2008) 1859–1868.