A comparative study of dark matter flow & hydrodynamic turbulence and its applications

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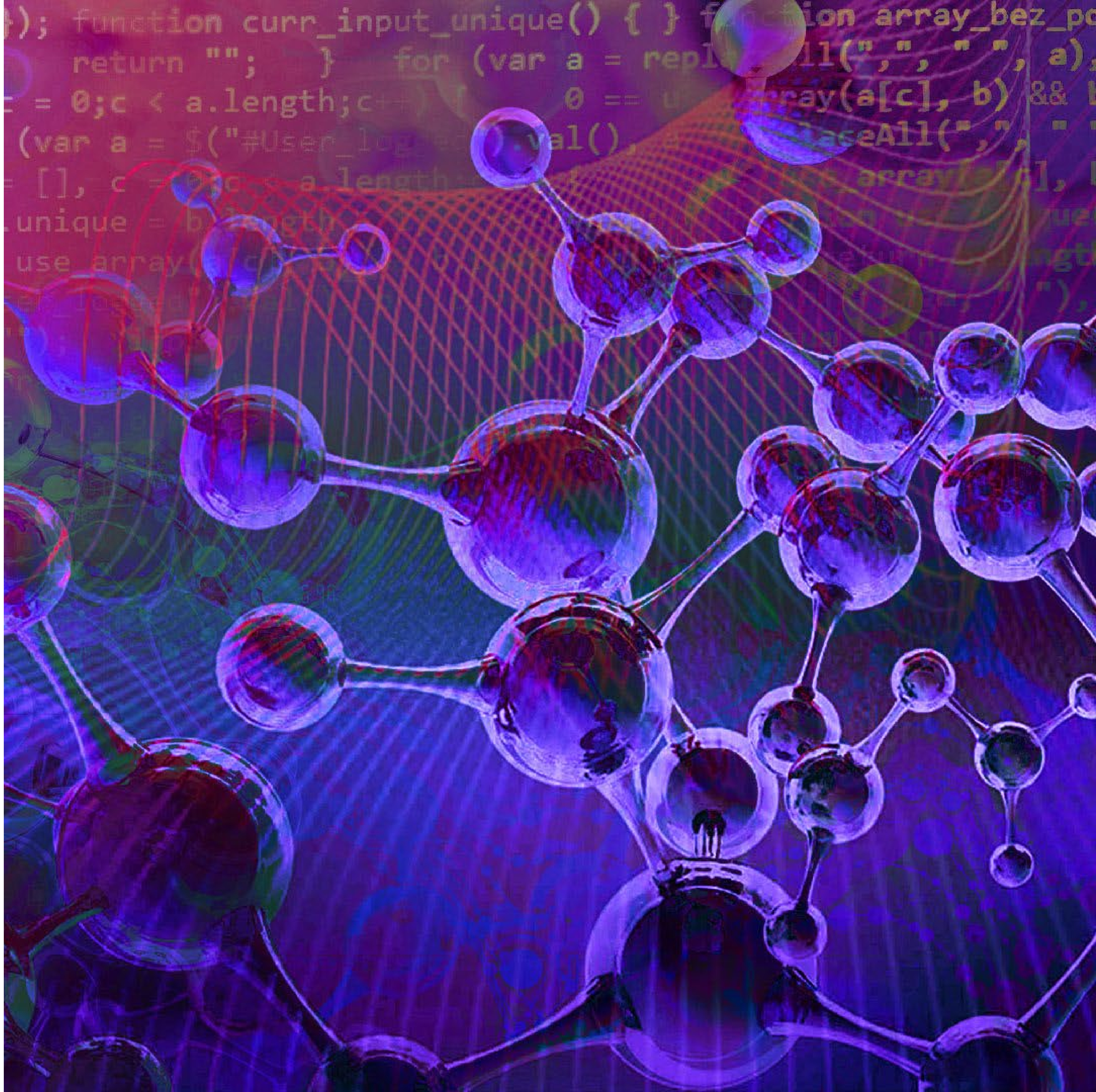


Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!

Data repository and relevant publications

Structural (halo-based) approach:

0.	**Data https://dx.doi.org/10.5281/zenodo.6541230**
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	**Data https://dx.doi.org/10.5281/zenodo.6569898**
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

Overview

- Some fundamentals of dark matter research
- Basic concepts in hydrodynamic turbulence
- Dark matter flow (SG-CFD) vs. hydrodynamic turbulence
- Theory of dark matter flow
  - Structural (halo-based) approach
  - Statistical (correlation-based) approach
- Applications of dark matter flow
  - Predicting dark matter particle properties
  - Understanding the origin of MOND
  - The baryonic-halo mass ratio and total baron faction

Some fundamentals of dark matter research

Overview of dark matter research

- Key questions: Does it exist? Where is it? How much is it? and What is it?

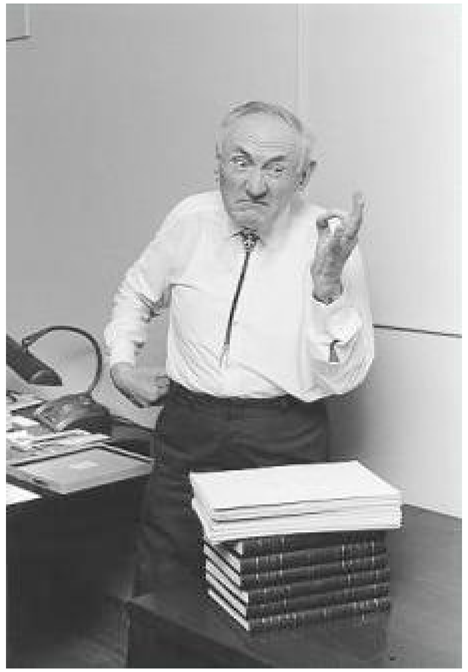
- Observational evidences (Does it exist? Where is it? How much is it?)
  - Motion of galaxies in galaxy clusters
  - Rotation curves of spiral galaxies
  - Gravitational lensing
  - Bullet clusters
  - Cosmic microwave background (CMB)
  - ……
- The nature of dark matter (What is it?)
  - Massive astrophysical compact halo object (MACHO)
  - Primordial black holes
  - Axions
  - sterile neutrino
  - WIMPs (Weakly Interacting Massive Particles) ⬅ Most popular
  - Superheavy dark matter
  - ……

Dark matter in galaxy clusters

Zwicky (1937):
Coma cluster is much larger than expected!

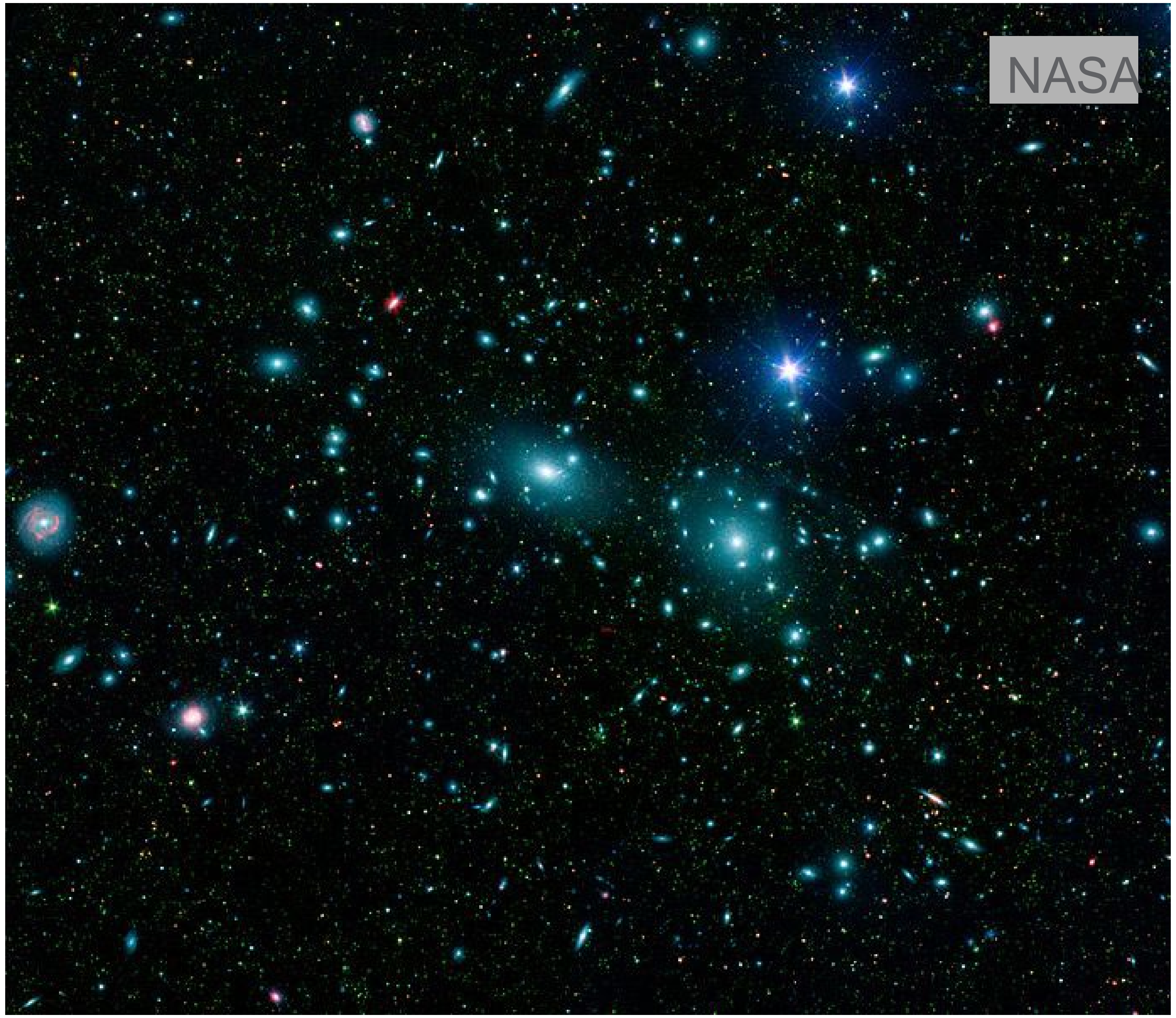
Coma cluster:
~1000 galaxies
~20Mpc in diameter
~100Mpc from Earth (320Mlys)

1Mpc = 3.086e+19 KM = 3Mlys



Fritz Zwicky

- Measuring speed of galaxies moving in Coma
- Enormous speed found ~1000km/s
- Fast enough to rip the cluster apart
- Unseen matter that holds all galaxies together

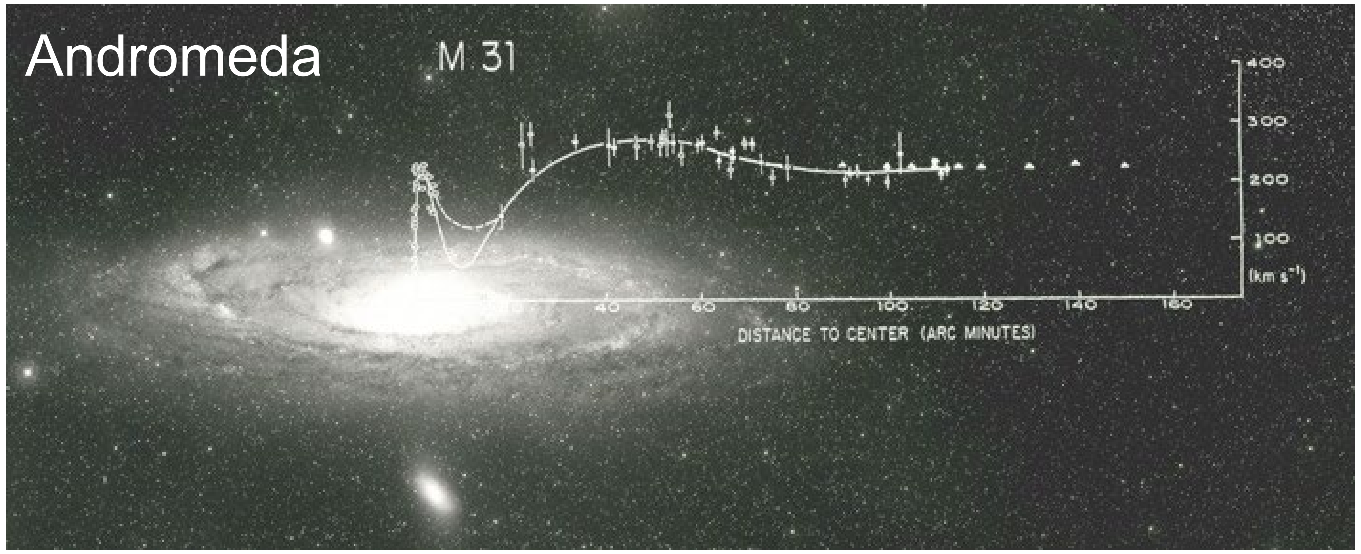


Coma cluster

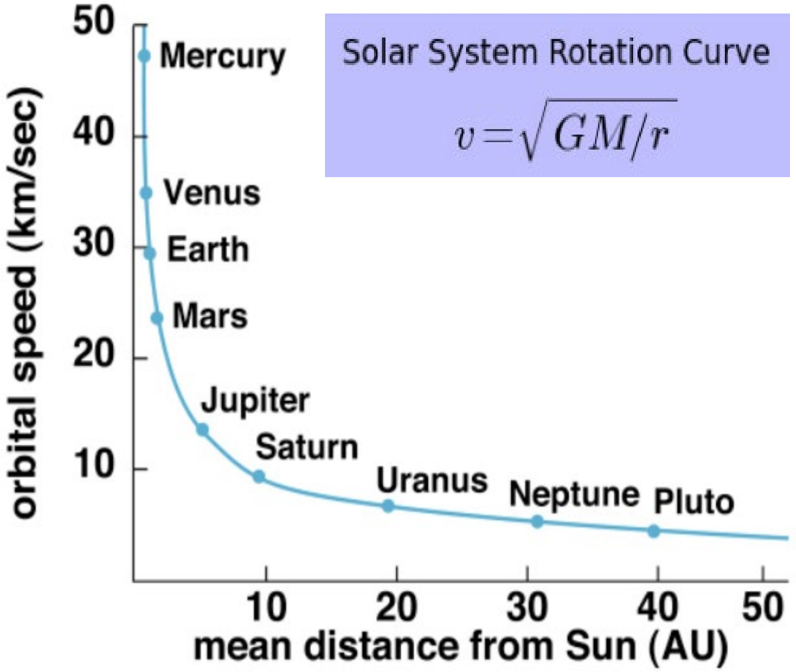
Dark matter in galaxy

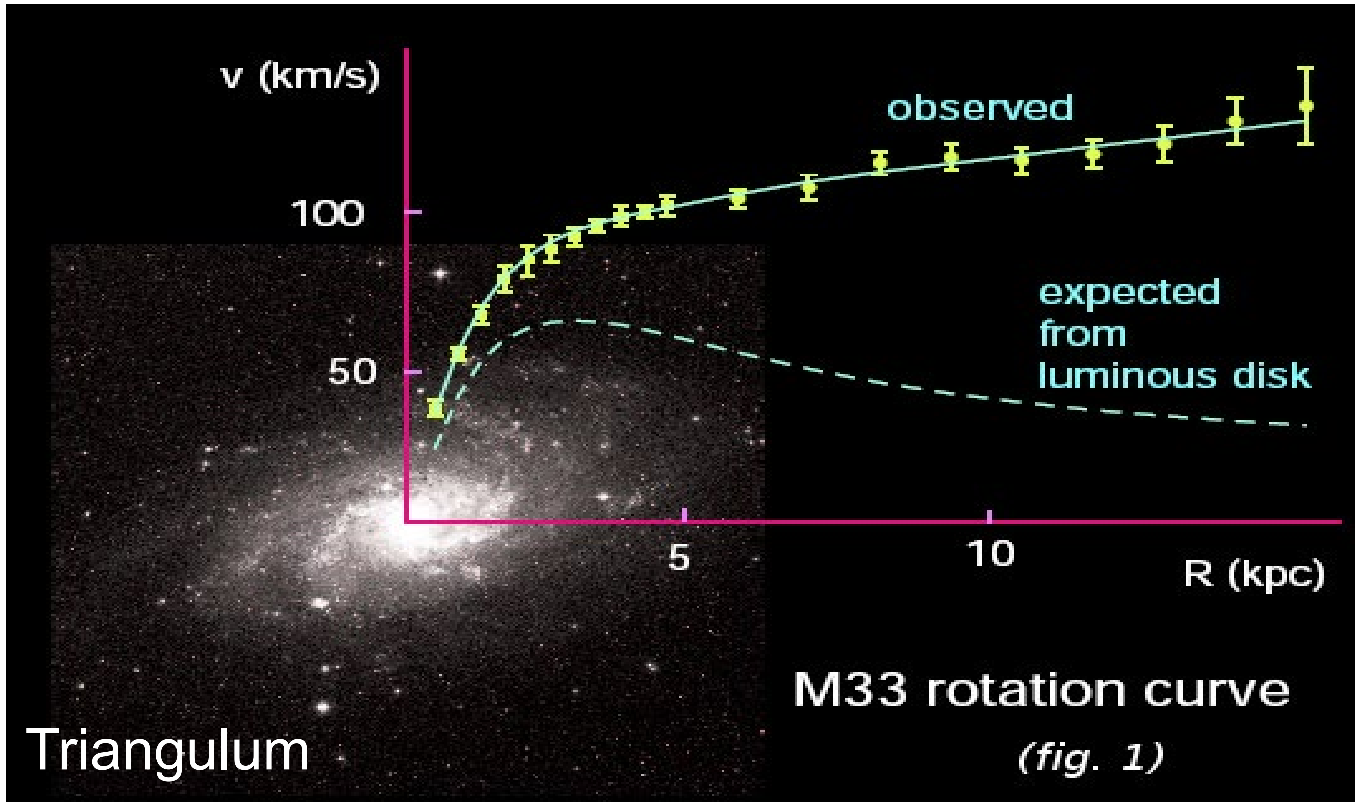
Rubin (1970s):
Rotation of M31
Andromeda Nebula

Vera Rubin

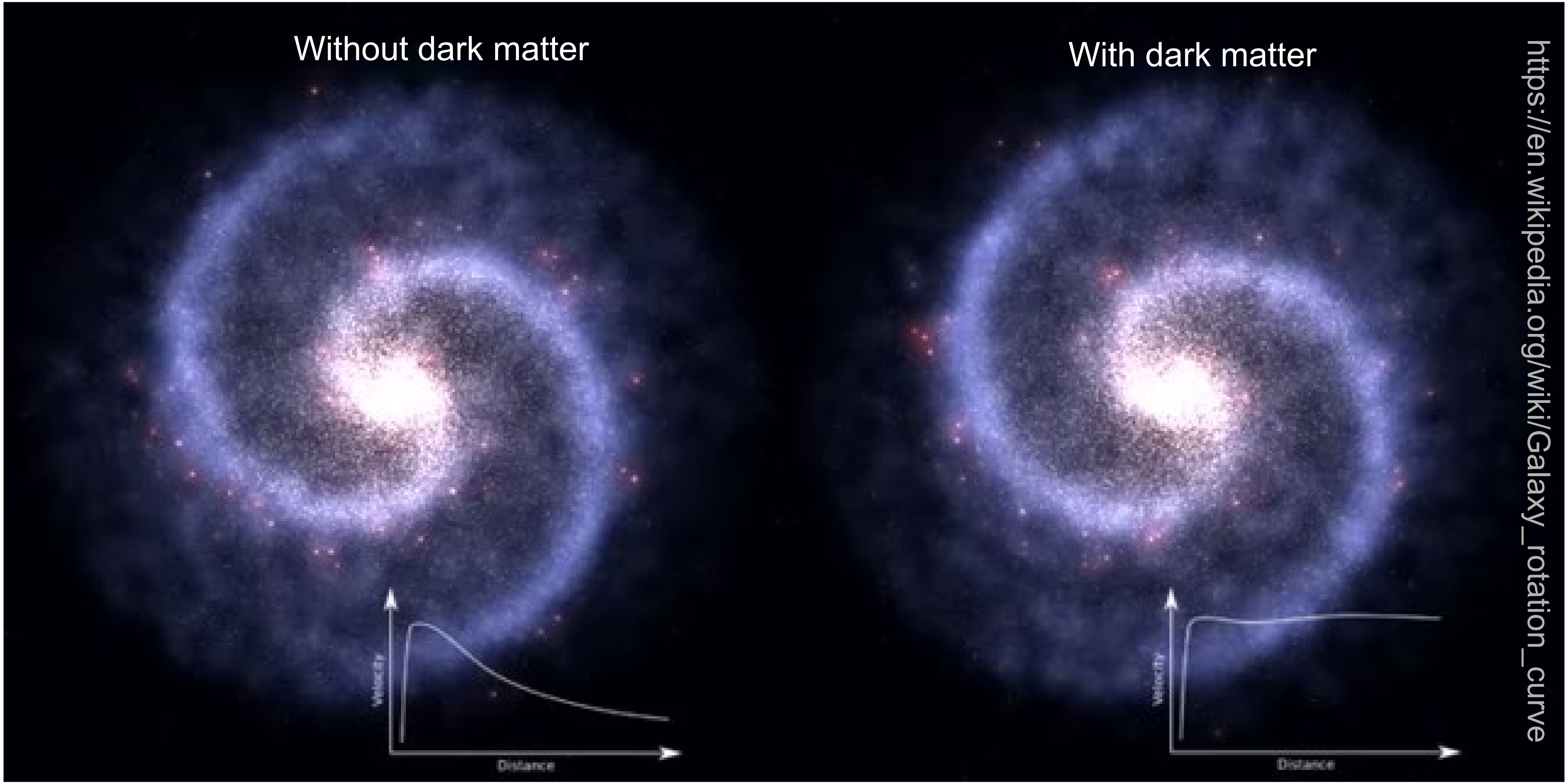


- Solar system rotation curve
- From Newtonian mechanics
- Galaxy flat rotation curve
- Unseen matter that holds galaxy

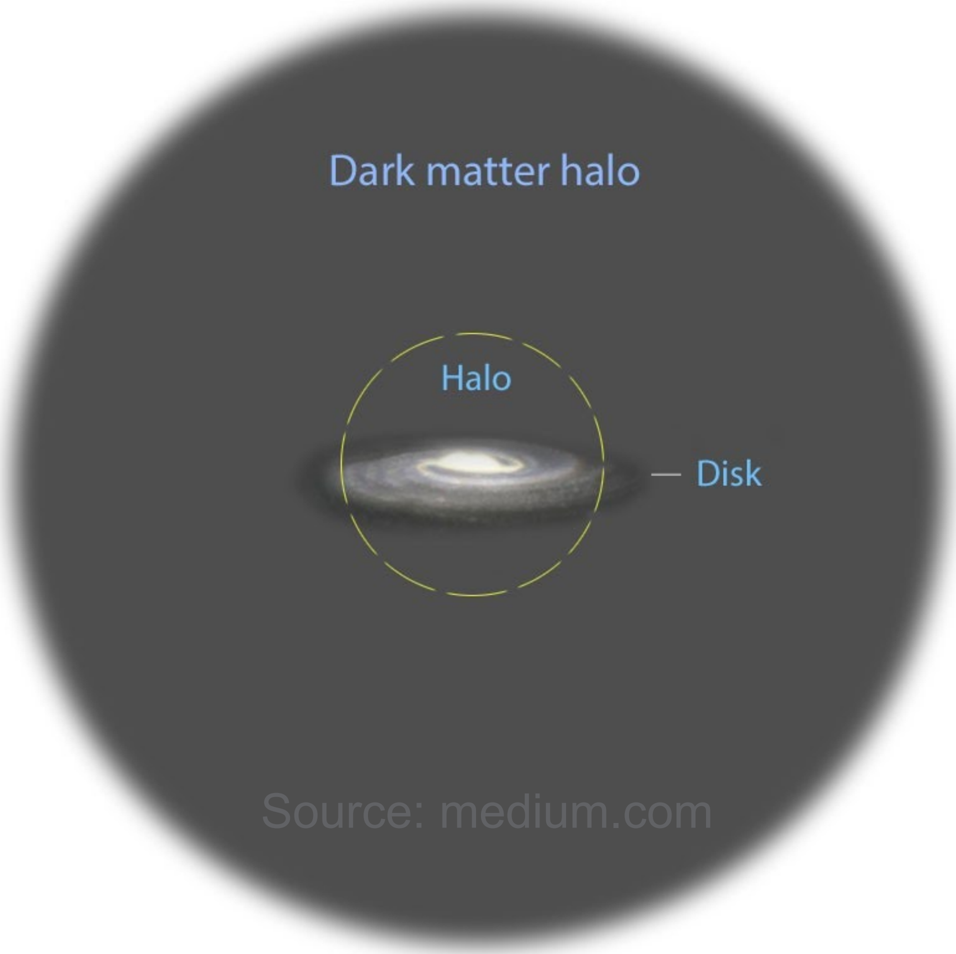




Effect of Dark matter on galaxy rotation curve

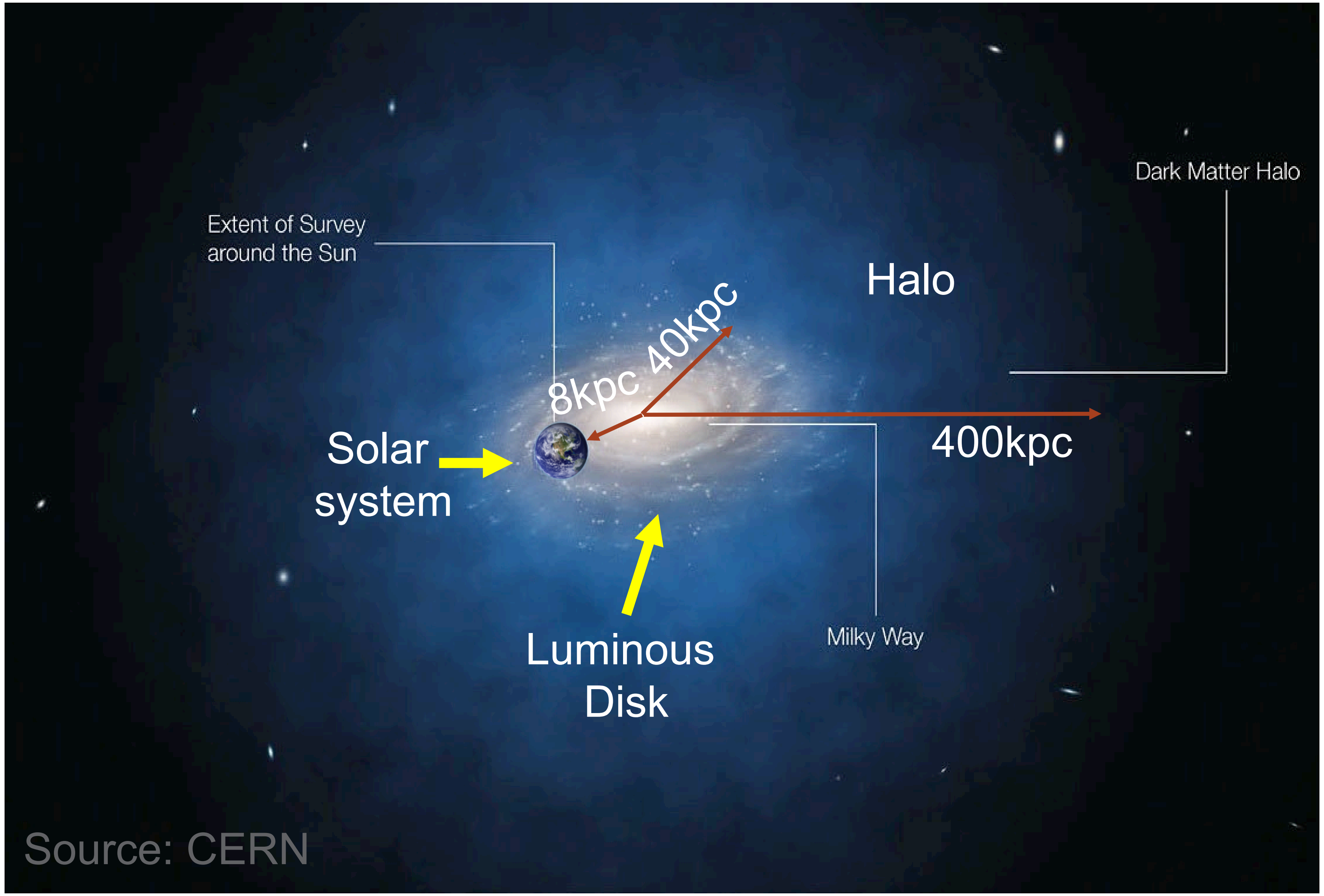


Dark matter in our galaxy

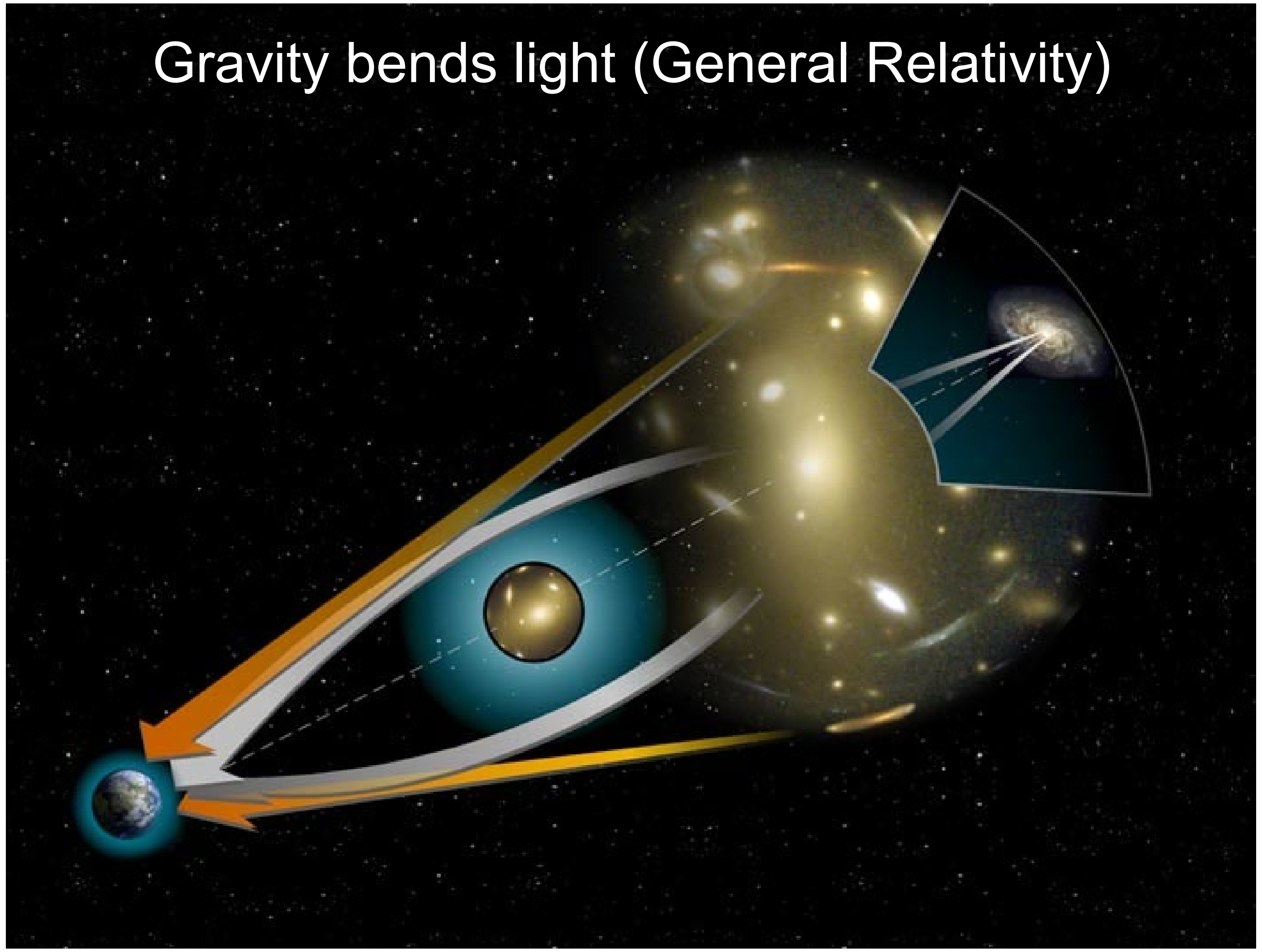


Milky Way model

Dark matter halo harbors our galaxy



Dark matter from gravitational lensing



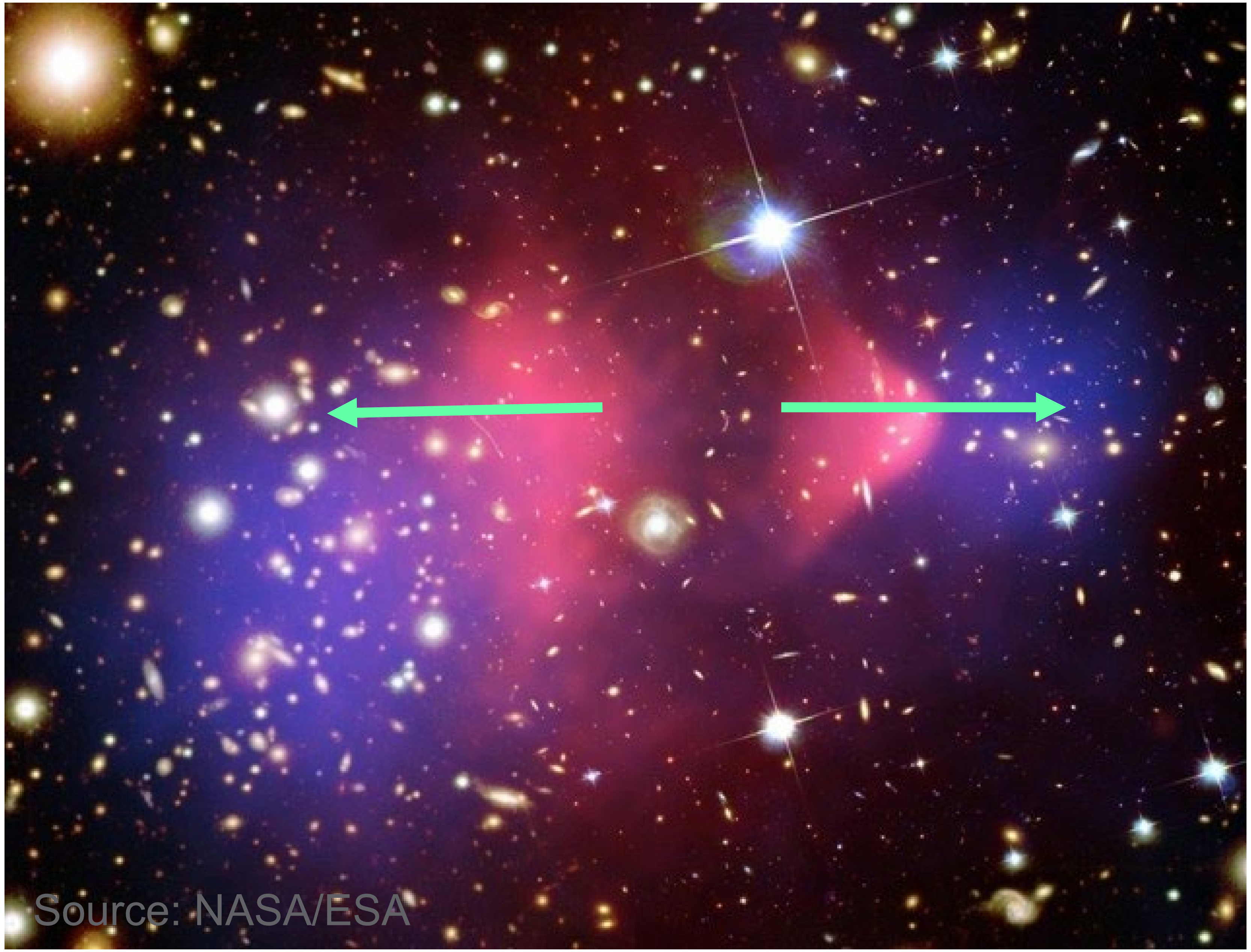
- Gravitational lensing by galaxy cluster
- Gravity from mass of matter bends light
- Effect of bending is stronger than expected from visible matter only

Dark matter from bullet cluster (2000s)

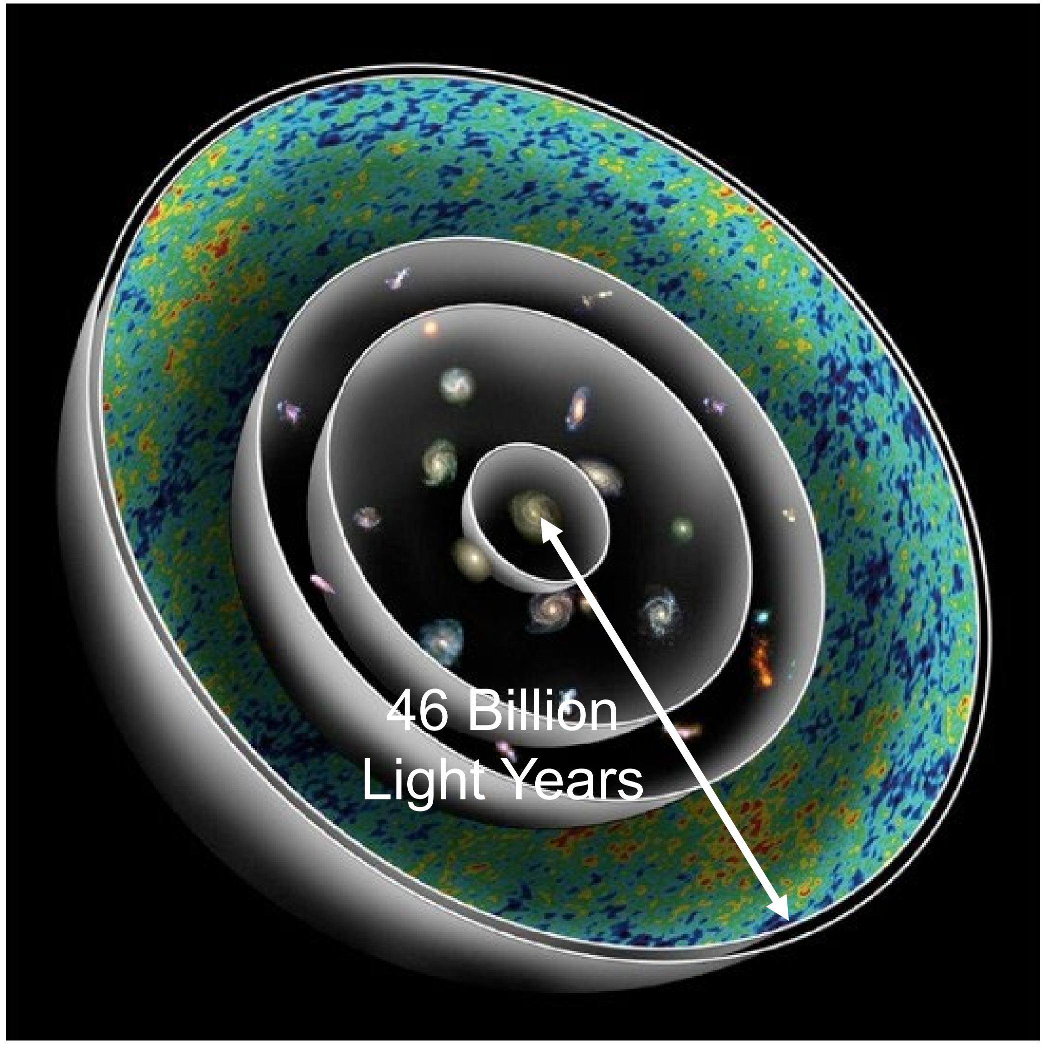
Composite image of X-ray (pink) and weak gravitational lensing (blue) of the famous Bullet Cluster of galaxies (colliding)

Red: gas and dust (baryonic matter)
Moving slower because of viscosity (collisional due to electromagnetic interactions)

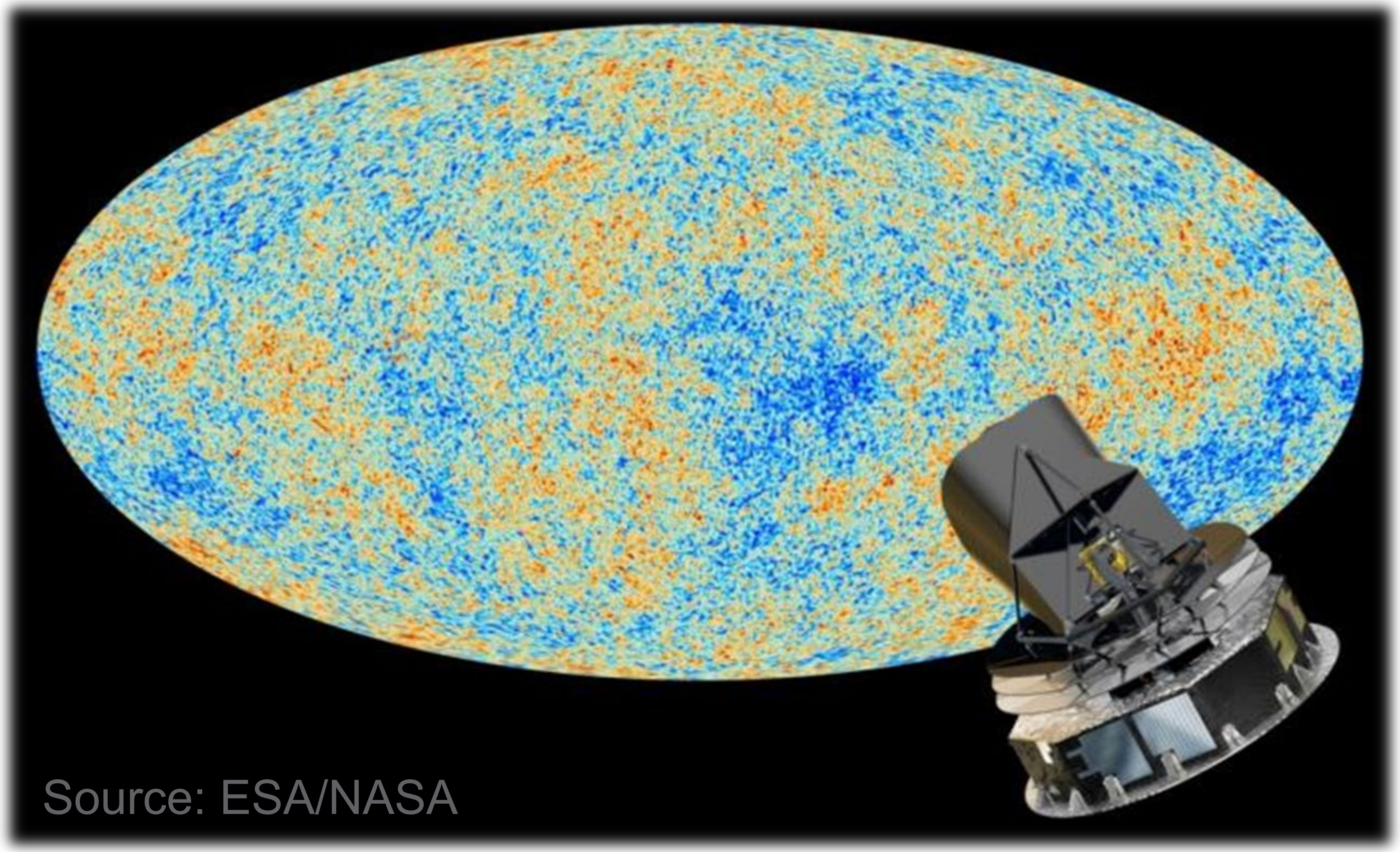
Blue: dark matter
Moving faster than baryonic matter because of collisionless nature



Dark matter from cosmic microwave background

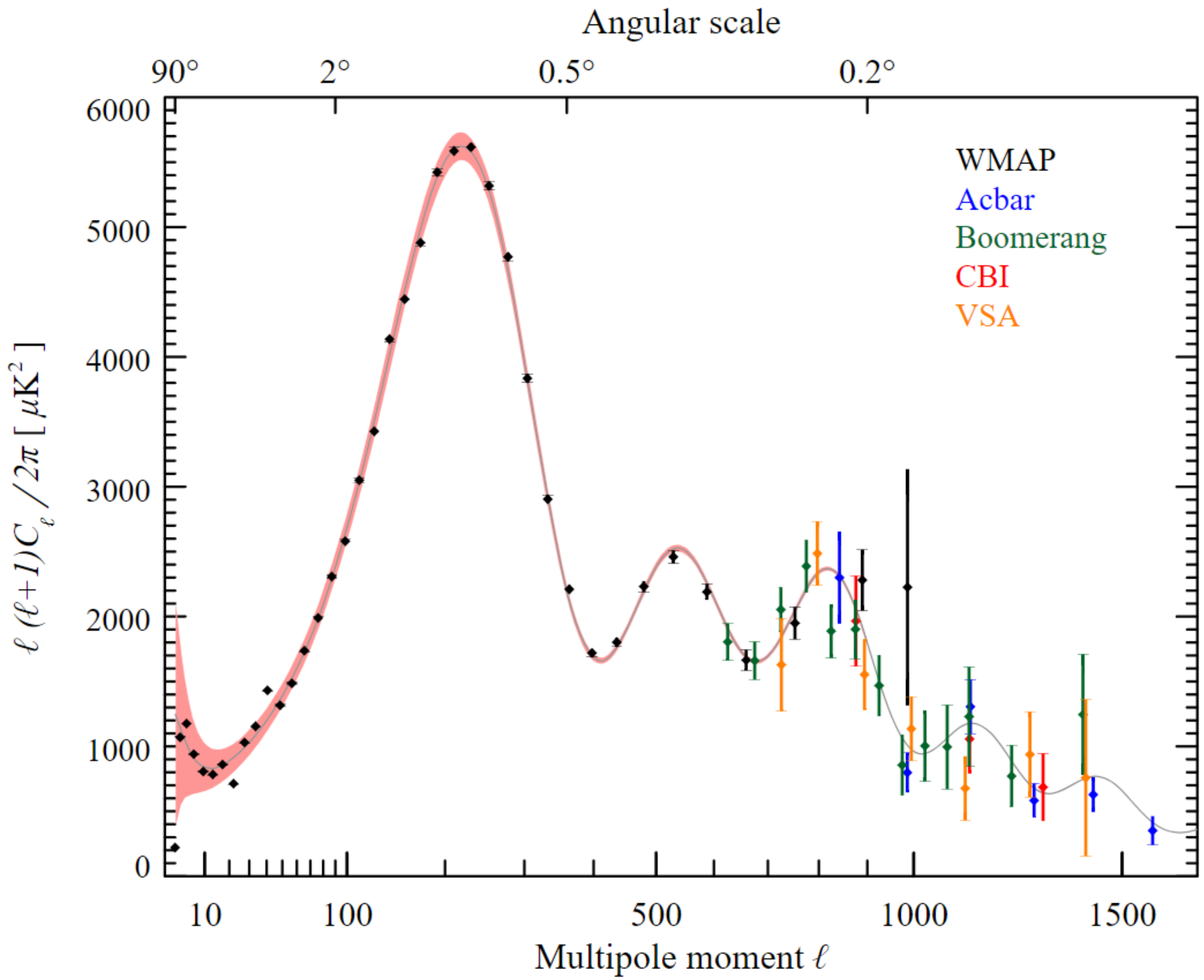


Cosmic Spheres of Time
http://new-universe.org/



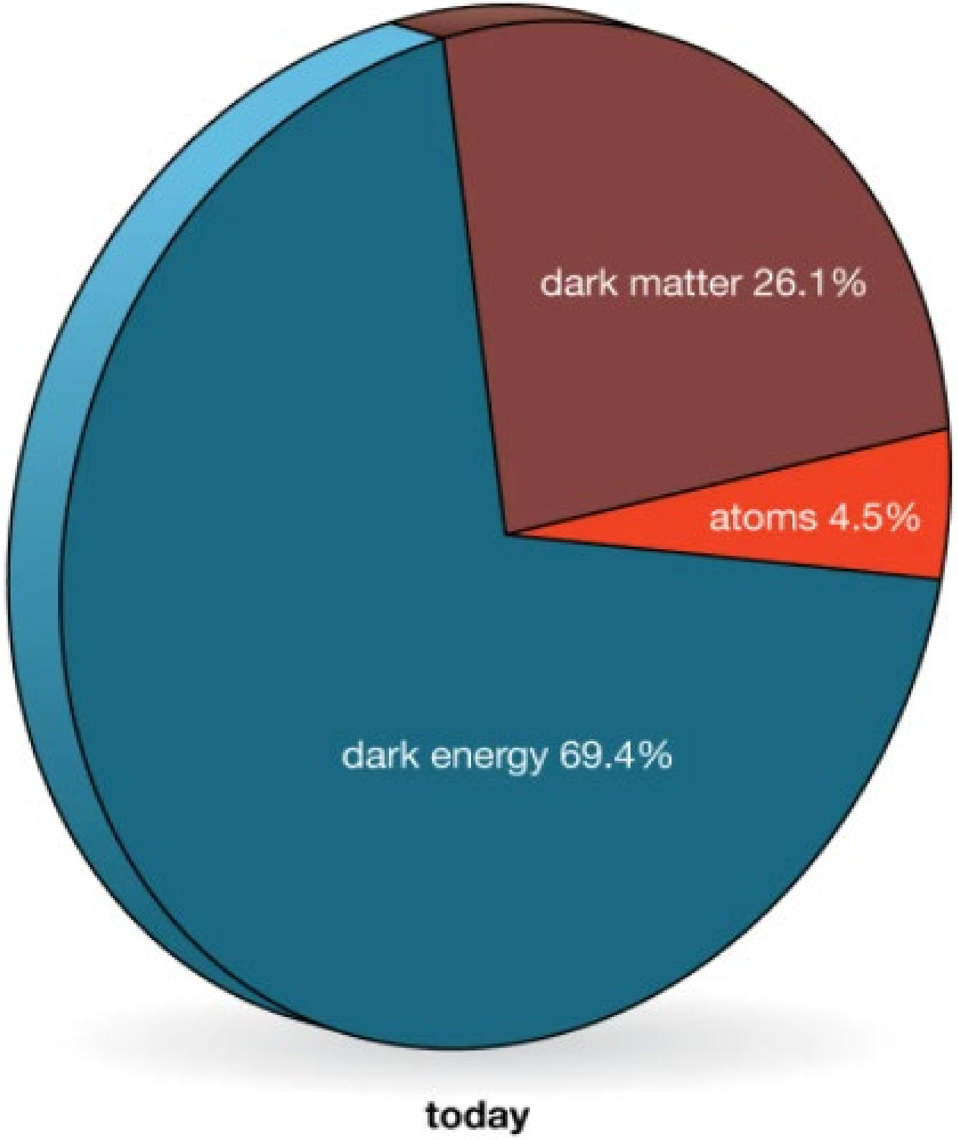
Planck results of CMB temperature anisotropy
(4-year survey from 2009-2013)
Baby universe: 400,000 years after Big Bang
cold (blue) and hot (red)

Quantifying amount of dark matter from CMB



Power spectrum of the CMB temperature anisotropy in terms of the angular scale. Also shown is a theoretical (double-dark ΛCDM) model (solid line)

Today's universe matter-energy content



Brief timeline for dark matter research (~100 years)

- **1930s**: Discovery of galaxy cluster velocity ~1000km/s
- **1960s**: Discovery of the CMB
- **1970s**: Discovery of flat galaxy rotation curves
- **1980s**: Cold Dark Matter (CDM) model proposed; MOND theory;
- **1992**: CMB fluctuations from COBE Confirms CDM prediction
- **1998**: Evidence for Dark Energy and accelerating expansion: Type Ia supernova
- **2000**: ΛCDM as the standard cosmological model
- **2003-2008**: WMAP and LSS data Confirm ΛCDM predictions
- **2013-2018**: Planck data of CMB anisotropies Confirm ΛCDM predictions
- **2022**: James Webb Space Telescope
- **2030s**: Discovery of dark matter particles??

COBE: COsmic Background Explorer (NASA)
WMAP: Wilkinson Microwave Anisotropy Probe (NASA)
LSS: Large Scale Structure (LSS) of the universe
CMB: Cosmic Microwave Background
ΛCDM: dark energy + cold dark matter (double dark)
Planck: European Space Agency (ESA)

What is dark matter?

No definite answer.

What it should not be?

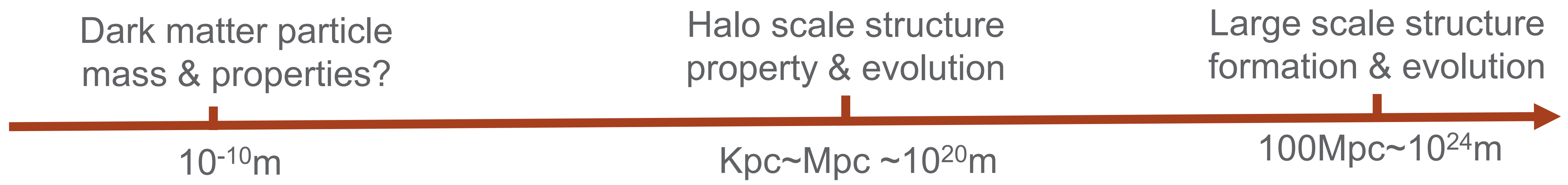
- No electric charge
- No color charge (strong interactions)
- No strong self-interaction
- No fast decay: stable and long-lived
- Not any particles in standard model of particle physics

What it should be?

- Non-baryonic
- Cold (non-relativistic)
- Collisionless
- Dissipationless (optically dark)
- Sufficiently smooth with a fluid-like behavior (justifies a fluid dynamics approach)

What is the nature of dark matter flow (DMF)?
Dark matter flow can be described by a non-relativistic, self-gravitating, collisionless fluid dynamics (SG-CFD).

Then why dark matter flow? understanding dark matter flow behavior on entire spectrum.



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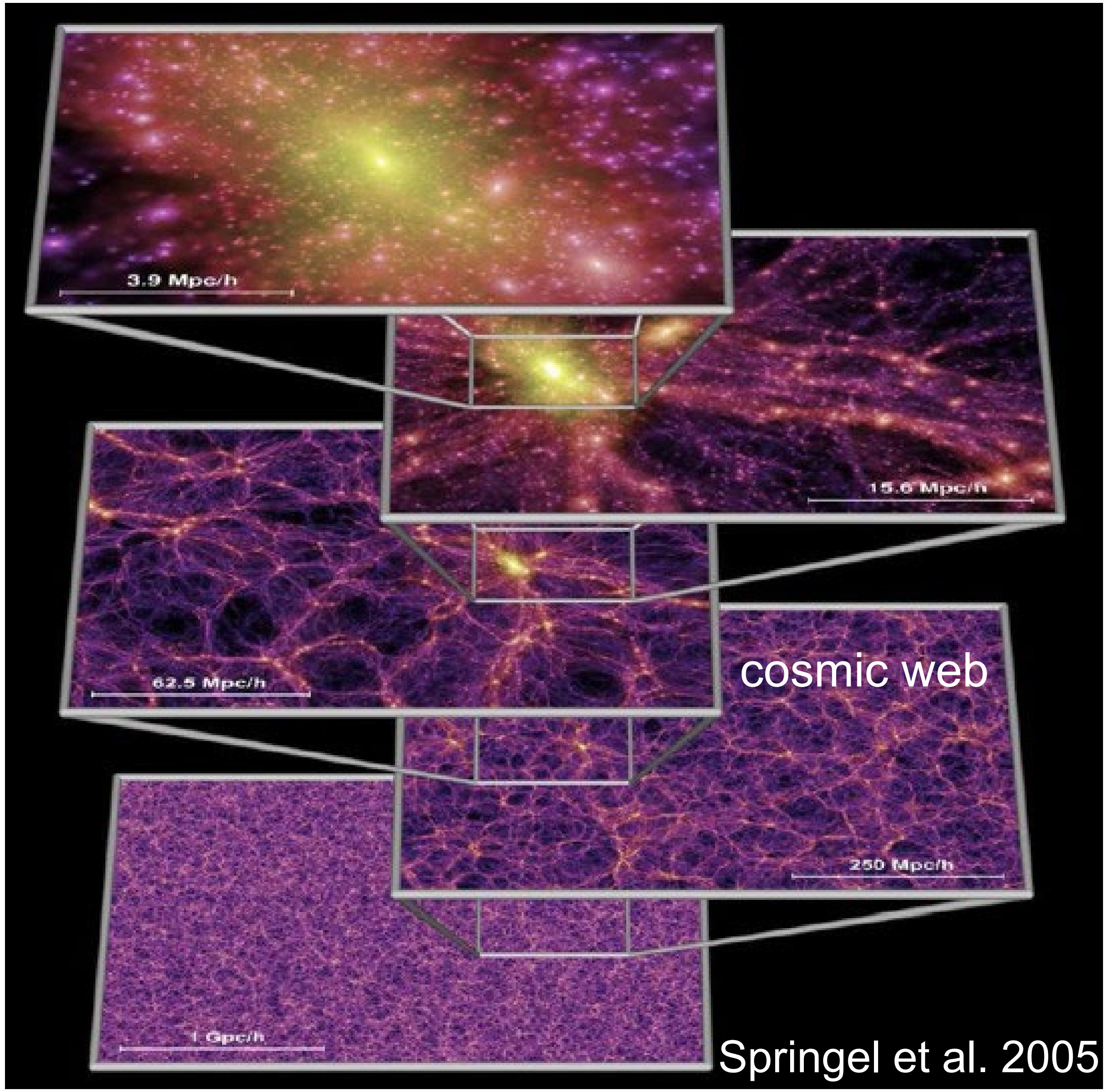
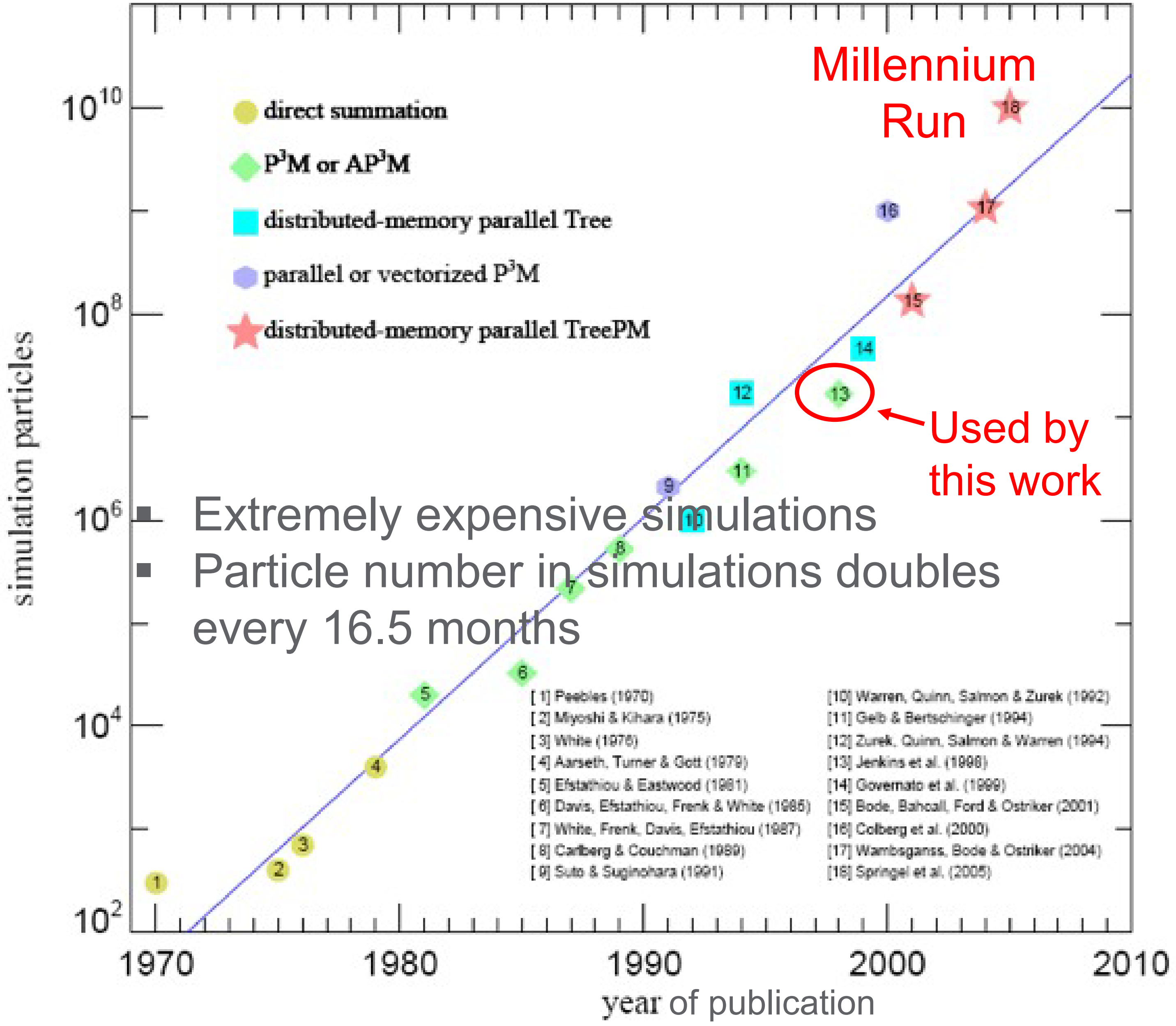
Cosmological N-body simulations



Springel et al. 2005



- Extremely expensive simulations
- Particle number in simulations doubles every 16.5 months

Dehnen and J. I. ReadThe European Physical Journal Plus, 126, 55

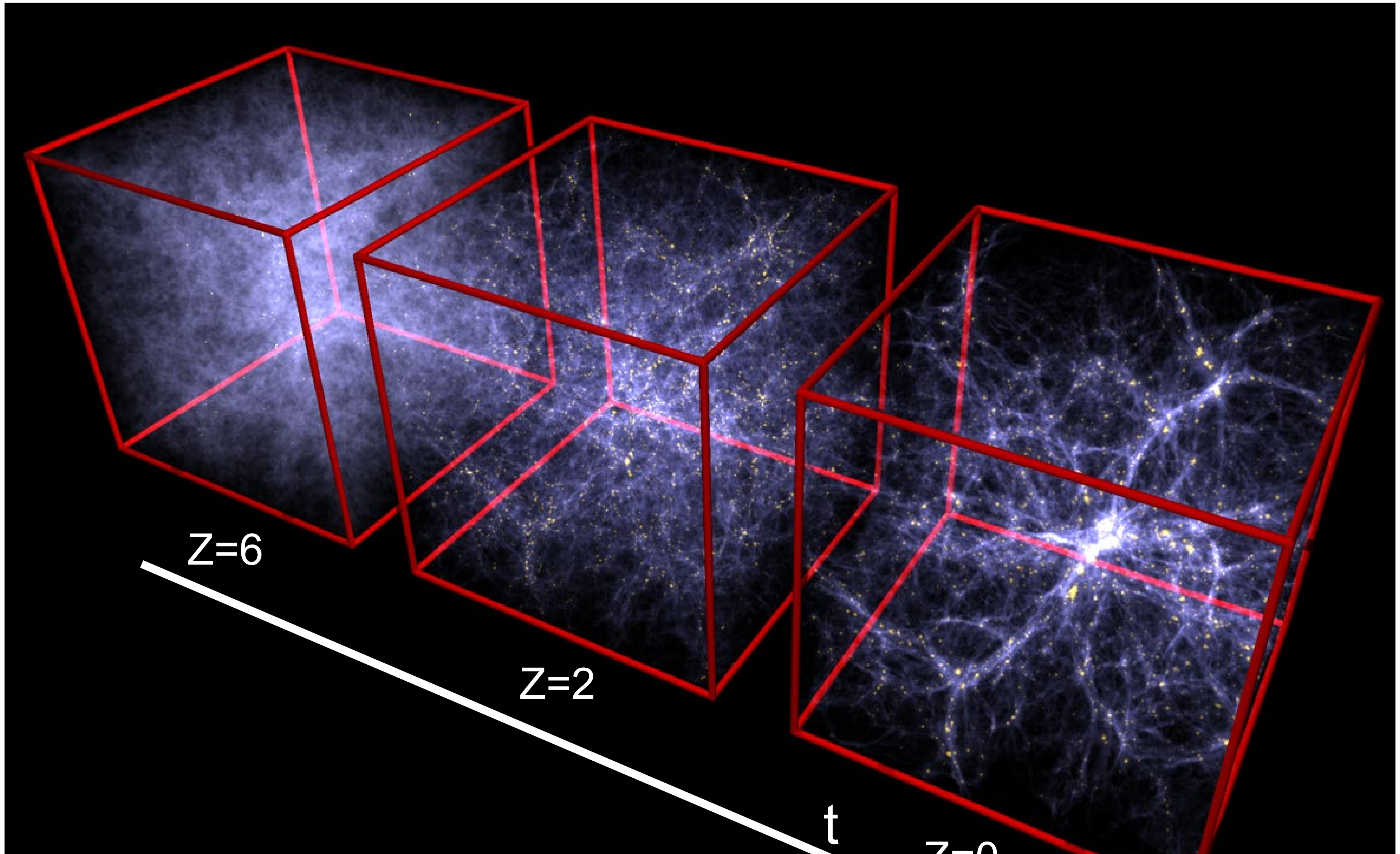
N-body simulations and dark matter flow

The Millennium Simulation:

- More than 10 billion "particles“
- Each with a billion Solar mass
- The large-scale structure of the universe ("the cosmic web")
- The largest simulation of dark matter structure at the time

Use N-body simulation:

- Self-gravitating collisionless fluid-like behavior
- Dark matter flow forms and evolves structures on both large and small scales



The Millennium Simulation

N-body simulations in this comparative study

Run	Ω_0	Λ	h	Γ	σ_8	$L(Mpc/h)$	N_p	$m_p(M_\odot/h)$	$l_{soft}(Kpc/h)$
SCDM1	1.0	0.0	0.5	0.5	0.51	239.5	256^3	2.27×10^{11}	36

- The numerical data are public available and generated from N-body simulations carried out by the Virgo consortium. https://wwwmpa.mpa-garching.mpg.de/Virgo/data_download.html
- As the first step, current study focus on the standard CDM power spectrum (SCDM) with matter-dominant gravitational flow.
- Similar analysis can be extended to other models with different assumptions and parameters.
- The same set of data has been widely used in many studies from clustering statistics to formation of halos in large scale environment, and test of models for halo abundances and mass functions.

Comparison of two non-equilibrium systems:

Dark matter flow (DMF or SG-CFD)
vs.
Hydrodynamic turbulence

SG-CFD: Self-Gravitating Collisionless Fluid Dynamics

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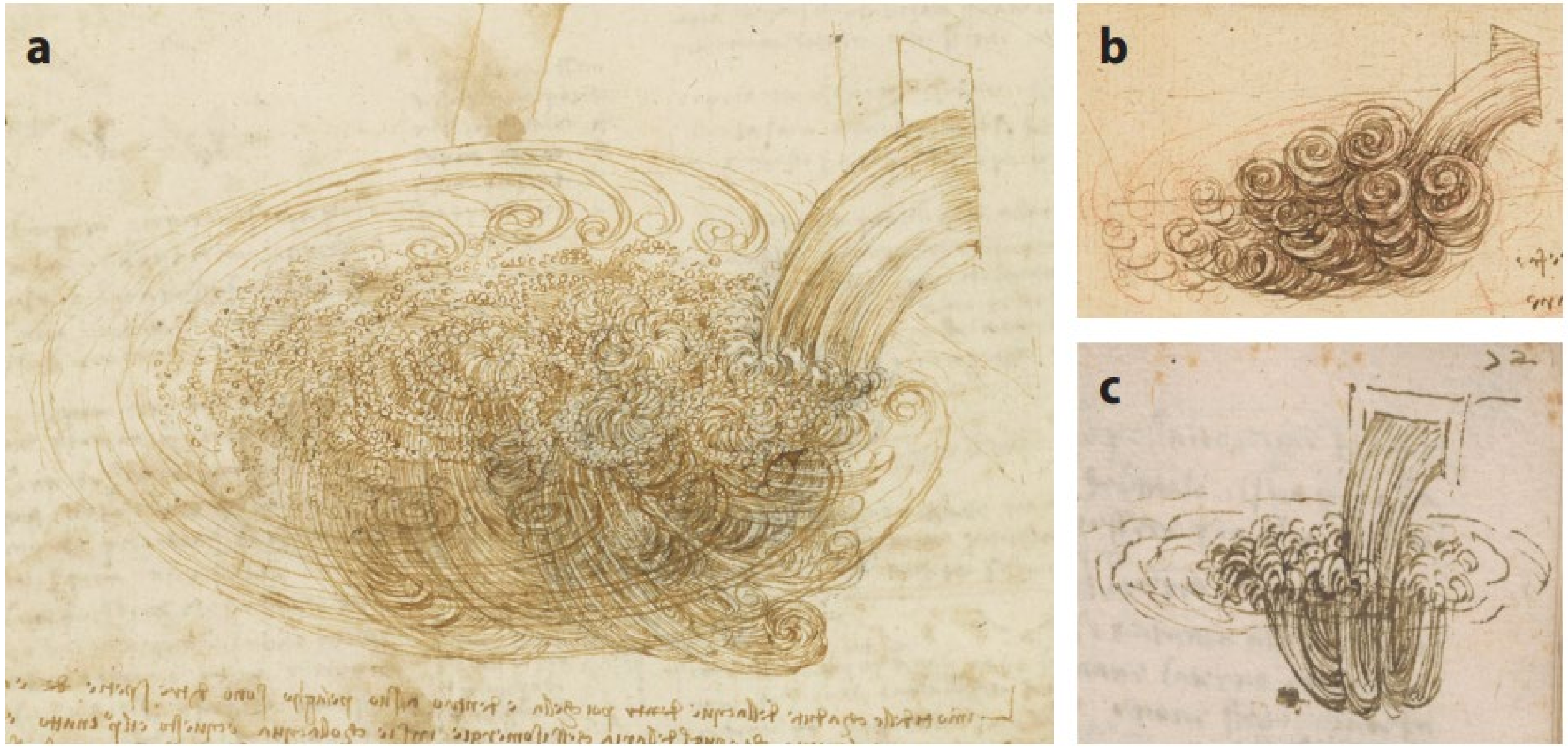
da Vinci’s sketch of turbulence (~1500 AD)



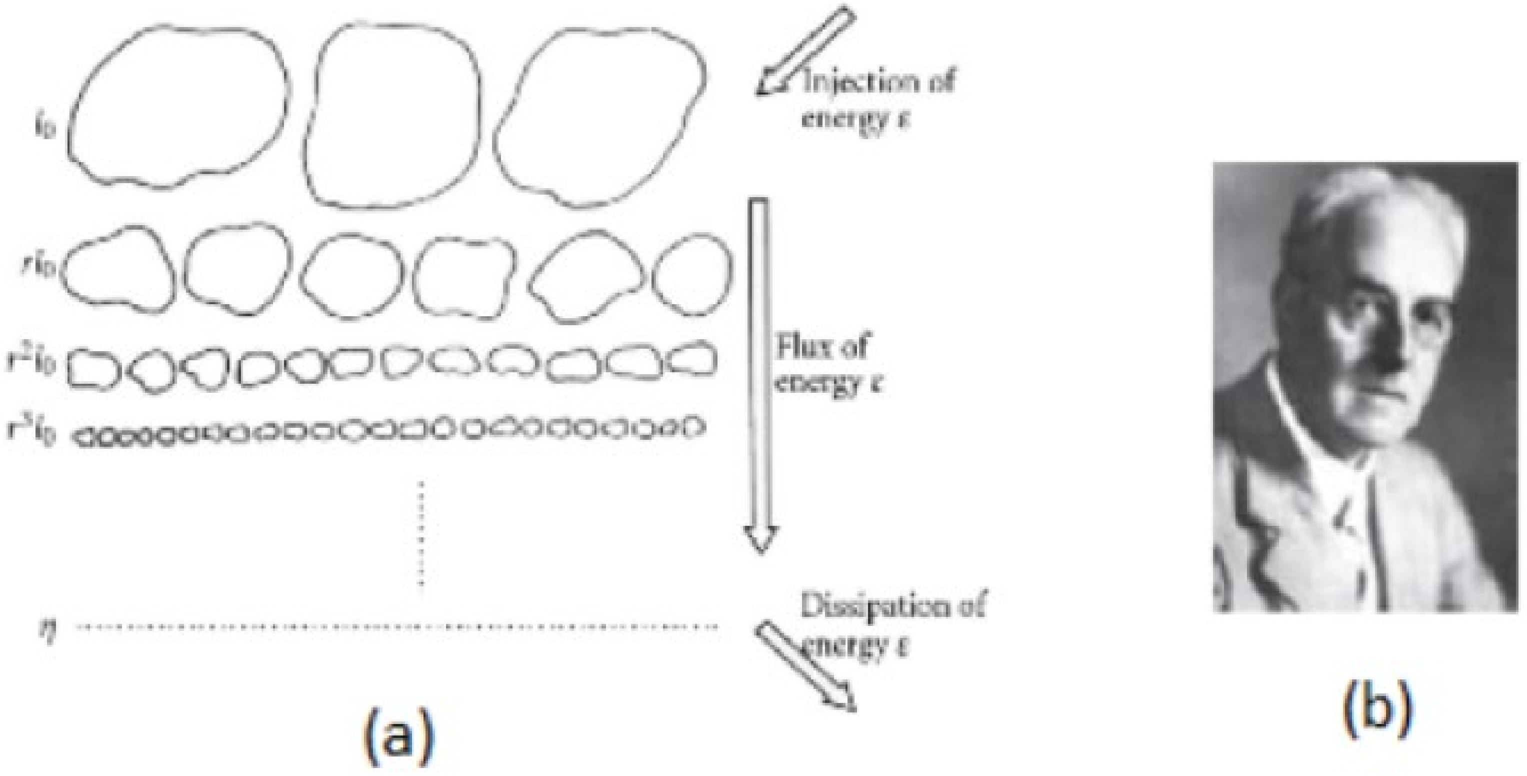
da Vinci sketch of turbulence: plunging water jet

- “turbolenza”: the origin of modern word “turbulence”
- The pattern of flow with vortexes in fluid
- The random chaotic nature

“. . . the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large.”

Richardson's direct cascade (1922)

"Big whorls have little whorls, That feed on their velocity;
And little whorls have lesser whorls, And so on to viscosity."



(a) : Cascade of energy, (b) : Lewis Richardson

[1] "Weather Prediction by Numerical Process", Richardson, L.F. 1922

Key attributes:

- Disorganized, chaotic, random;
- Nonrepeatability (sensitivity to initial conditions);

- Multiscale: large range of length and time scales;
- Dissipation mediated by viscosity;

- Three dimensionality;
- Time dependence;
- Rotationality (incompressible);
- Intermittency in space and time;

- Cascade: energy is injected at large scale, propagating, and dissipated at the smallest scale.

Existing approaches for turbulence

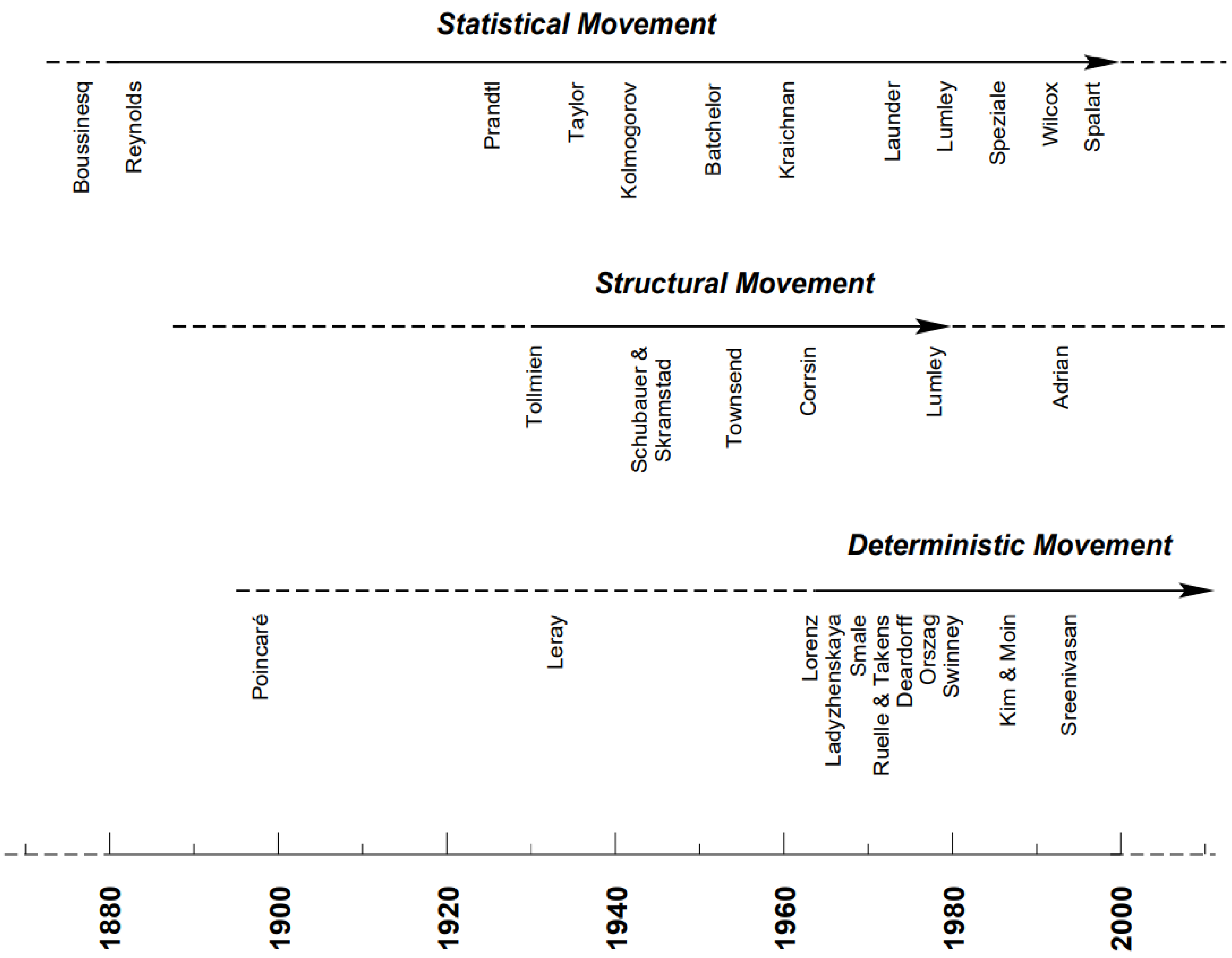


Figure 1.3: Movements in the study of turbulence, as described by Chapman and Tobak [1].

Statistic approach: (correlations etc.)

- Focusing on means and various averages
- Celebrated problem of closure
- Structureless without power of conceptualization

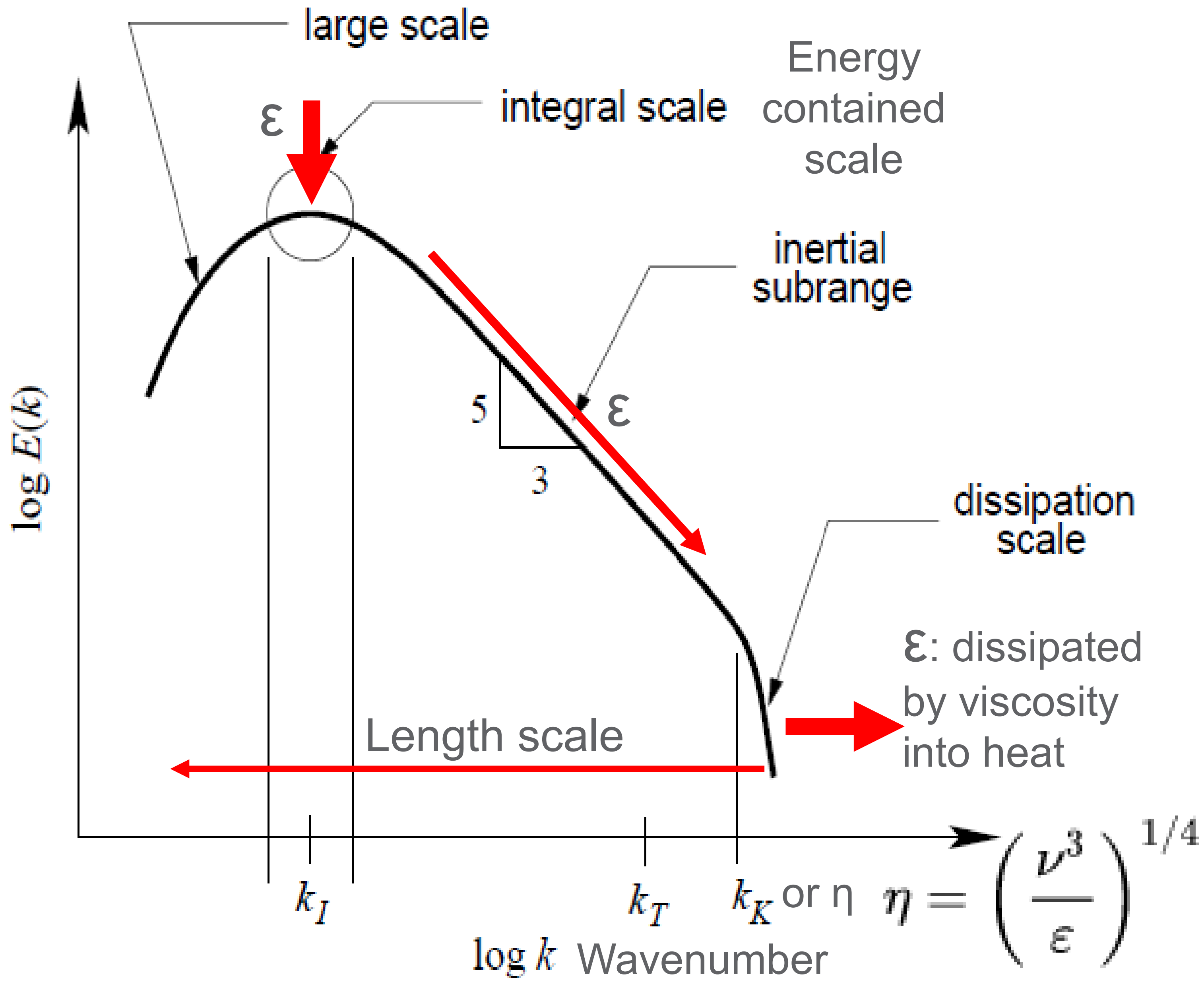
Structural approach: (vortex ect.)

- Existence of coherent structures
- Detecting and analyzing coherent structures in turbulent flows

Deterministic: (should be explored in DMF?)

- Chaotic behavior in simple deterministic systems
- Deterministic chaotic behavior can occur after just a few bifurcations
- Bifurcation theory, strange attractors, fractals, and renormalization group

Direct energy cascade in turbulence



- Freely decaying vs. forced stationary
- Integral scale: energy injection
- Inertial range: inertial >> viscous force
- Dissipation range: viscous dominant
- Dissipation scale: determined by viscosity (m^2/s) and rate of cascade (m^2/s^3)
- Is there energy cascade in dark matter flow? If yes, how it initiates, propagates, and dies ??

Inertial range, scaling laws, and intermittence

- There exist an inertial range with a scale-independent rate of energy cascade (ε does not depend on eddy size *l*) for eddy size η< l <L. *L is the integral length scale where energy is injected.*

Dissipation (Kolmogorov) scale:

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

- In this range, inertial force is dominant over viscous force. For eddies with a characteristic velocity *u* and size l , the lifetime (turnaround time) of eddy is *l*/*u*. The rate ε can be computed as the kinetic energy passed per lifetime.

$$\varepsilon \approx \frac{u^2}{(l/u)} \approx \frac{u^3}{l} \Rightarrow u^3 \propto l$$

turnaround time

- In this range, a general scaling for velocity structure functions for pairwise velocity can be identified (the most important results in turbulence)

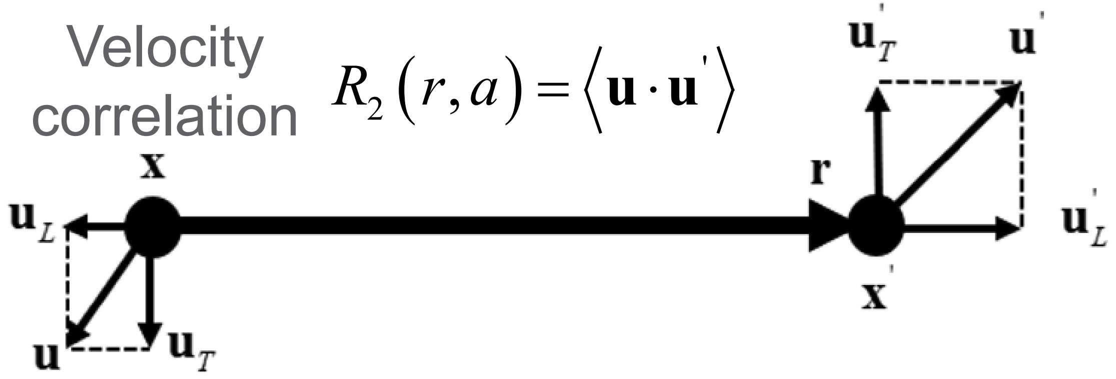
$$S_m(r,a) = \left\langle (\Delta u_L)^m \right\rangle = \left\langle (u_L' - u_L)^m \right\rangle \Rightarrow S_m(r) \propto (\varepsilon_u)^{m/3} r^{m/3}$$

m=2

$$S_2 \propto (-\varepsilon_u)^{2/3} r^{2/3}$$

two-thirds law in hydrodynamic turbulence

Velocity correlation $R_2(r,a) = \left\langle \mathbf{u} \cdot \mathbf{u}' \right\rangle$



- Intermittence of cascade in space and time can be identified from the deviation from ideal scaling law

- What is the dissipation scale η in DMF?
- Is there any simple expression for ε?
- What are the scaling laws in DMF?
- What about the intermittence in DMF?
  - Touched here but need to be further studied.

Large scale dynamics of freely decaying turbulence

- Freely decaying turbulence is free from any external force to maintain the turbulence (Coffee example).
- There is no energy injection on large scale and total energy is continuously decaying with time.
- Both integral scale l (energy-contained scale) and energy dissipation rate ε vary with time.
- What is the large-scale dynamics of freely decaying turbulence? How does energy evolve with time?

$$\varepsilon \equiv A\frac{u^2}{(l/u)} = A\frac{u^3}{l}$$

Loitsyansky integral invariant (integral of velocity correlation):

$$\int \langle \mathbf{u}\cdot\mathbf{u}' \rangle \mathbf{r}^2 d\mathbf{r} \approx u^2 l^5 = const$$

$$u^2 \sim t^{-10/7}$$

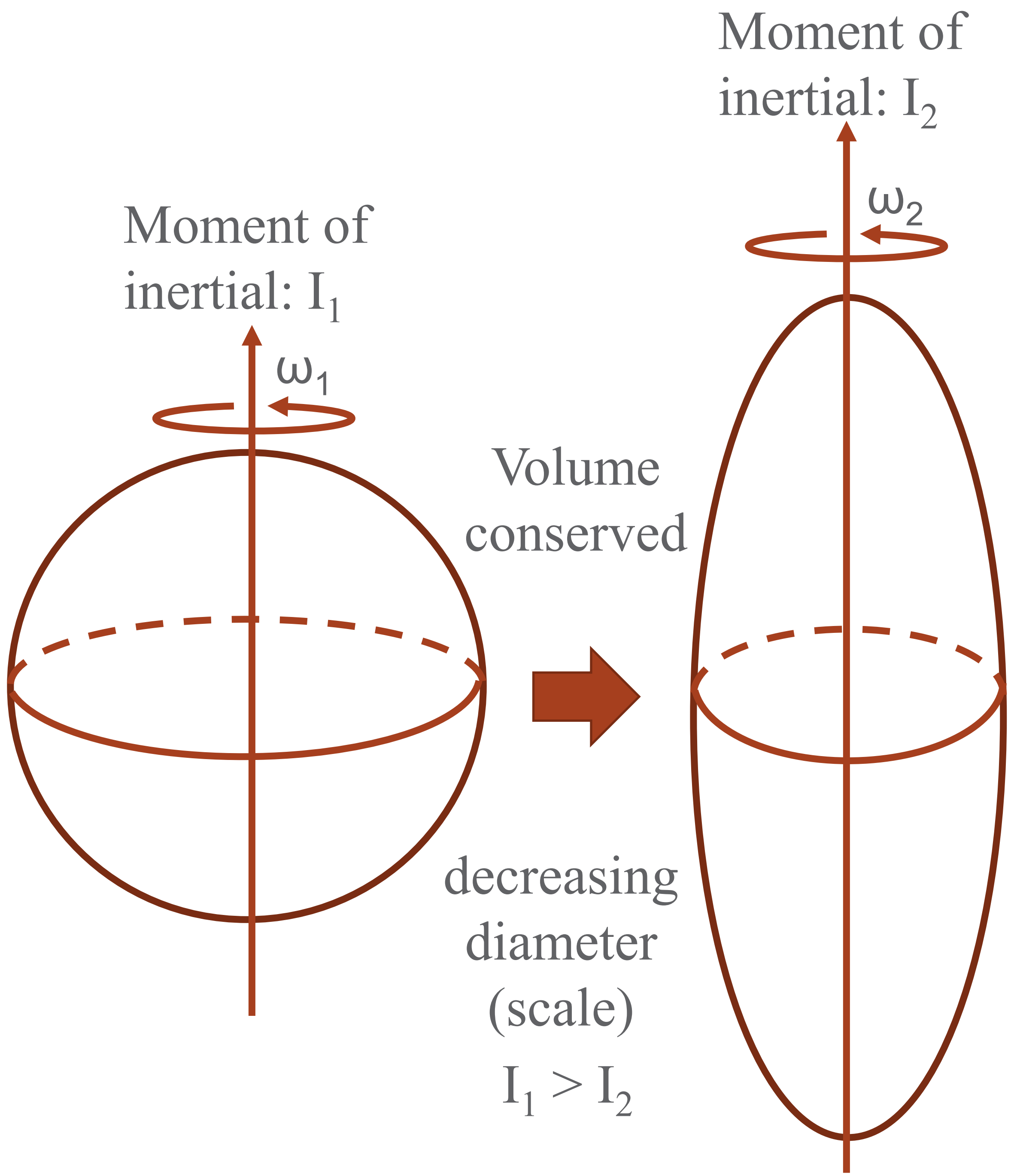
$$l \sim t^{2/7}$$

$$\varepsilon \sim t^{-17/7}$$

Due to the formation and virilization of halos, the kinetic energy in dark matter flow continuously increases with time. In this regard, dark matter flow is a freely growing turbulence.

- What is the large-scale dynamics in DMF?
- How energy and momentum (both radial and angular) evolve on large scale?
- Loitsyansky integral invariant is related to the conservation of angular momentum
- Do we have similar integral “constants” of motion in dark matter flow? Are they still constant or varying with time?

Vortex Stretching mechanism for energy cascade



Conservation of angular momentum:

$$I_1\omega_1 = I_2\omega_2 \Rightarrow \omega_2 > \omega_1$$

Ratio of rotational kinetic energy:

$$\frac{I_2\omega_2^2}{I_1\omega_1^2} = \frac{I_1}{I_2} \Rightarrow I_2\omega_2^2 > I_1\omega_1^2$$

Rotational kinetic energy is passing down the scales (direct energy cascade) !

- Does similar mechanism hold for halos in dark matter flow?
- What is the major mechanism for energy cascade in dark matter flow? (facilitated by mass cascade)

Reynolds stress for energy transfer between mean flow and random fluctuation

Reynolds decomposition: $u_i = \overline{u_i} + u_i'$,

Navier–Stokes equation (self-closed):

$$\nabla \cdot \mathbf{u} = 0$$

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right)$$

Reynolds Averaged Navier–Stokes (RANS, not closed):

$$\rho\left[\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}\right] = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \overline{u_i}}{\partial x_j} - \underbrace{\rho \overline{u_i' u_j'}}_{\text{Reynolds stress}}\right)$$

- Reynolds stress facilitates the **one-way** energy exchange from coherent (mean) flow to random fluctuation and enhances system entropy.
- Eddy viscosity models the Reynolds stress using the rate of strain of mean flow $\tau_{ij}^{ev} = -2\nu_{sgs}\overline{S}_{ij}$.

Jeans' equation (not self-closed):

$$\rho\left[\underbrace{\frac{\partial \langle u_i \rangle}{\partial t}}_{\text{Mean flow}} + \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j}\right] = -\underbrace{\frac{\partial \langle \rho \sigma_{ij}^2 \rangle}{\partial x_j}}_{\text{Pressure from Fluctuation}} - \underbrace{\rho \frac{\partial \Phi}{\partial x_i}}_{\text{Potential}}$$

$$\sigma_{ij}^2 = \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle = \langle u_i' u_j' \rangle$$

- Is it possible to obtain a self-closed equation for dark matter flow? (closure problem)
  - Any similar concept as eddy viscosity in dark matter flow?
- How energy/momentum exchanges between mean flow and random fluctuation in dark matter flow?

Brief timeline for turbulence research (~500 years)

- **1500s**: Earliest recognition of turbulence as a physical phenomenon
- **1877**: Boussinesq's concept of "eddy viscosity" replacing molecular viscosity
- **1894**: Reynolds number for transition from laminar to turbulence; Reynolds's decomposition (RANS) $v = \langle v \rangle + v'$ Reynolds stress;
- **1925**: Prandtl "mixing-length theory" predicts the eddy viscosity
- **1930s**: Taylor's statistical methods involving correlations power spectra; Karman and Howarth
- **1940s**: Kolmogorov: the "K41 theory" Two-thirds law -5/3 law
- **1963**: Lorenz propose possible links between "deterministic chaos" and turbulence
- **1967**: Kraichnan's inverse cascade in 2-dimensional turbulence

RANS: Reynolds-averaged Navier-Stokes Equation;

For all concepts listed here, we can identify their counterparts in dark matter flow!

Hydrodynamic turbulence vs. dark matter flow

Key attributes of hydrodynamic turbulence:

- Disorganized, chaotic, random;
- Nonrepeatability (sensitivity to initial conditions);
- Multiscale in length and time scales;
- Intermittency in space and time;
- Dissipative and collisional
- Short-range interaction
- Velocity fluctuation
- Vortex as fundamental building block
- Maximum entropy distribution (Gaussian)
- Incompressible on all scales $\nabla \cdot \mathbf{v} = 0$
  - Divergence-free
  - Constant density
- Energy cascade from large to small length scales
- Vortex stretching responsible for energy cascade
  - Volume conserving
  - Shape changing
  - Uniform density
- Reynolds decomposition
- Reynolds stress for energy transfer between mean flow and random motion (turbulence)
- Closure problem, eddy viscosity, etc…
- Statistical theory: correlation/structure functions scaling laws in inertial range

Key attributes of dark matter flow:

- Disorganized, chaotic, random;
- Nonrepeatability;
- Multiscale in mass/length/time scales;
- Intermittency in space and time;
- Dissipationless and collisionless
- Long-range gravity
- Velocity & acceleration fluctuation
- Halos as fundamental building block
- Maximum entropy distribution?? (X dist.)
- Flow behavior is scale-dependent (peculiar velocity)
  - Small scale: constant divergence $\nabla \cdot \mathbf{v} = \theta$
  - Large scale: irrotational (curl-free) $\nabla \times \mathbf{v} = 0$
- Mass/energy cascade from small to large mass scales
- Role of halos for energy cascade??
  - Halos are growing, rotating, with nonuniform density
  - Is halo shape changing important?
  - Mass cascade facilitates energy cascade?
- Velocity/acceleration decomposition?
- What facilitates the energy transfer between mean flow and random motion??
- Self-closed model (analogue of NS) ?? Closure problem?
- Statistical theory: Kinematic and dynamic relations? Scaling laws?

Common features

Critical MOND acceleration a_0?

Deep MOND?

Theory and applications of dark matter flow

Theory of dark matter flow

- Structural (halo-based) approach
  - Inverse mass cascade in dark matter flow
    - Impact on halo mass functions
    - Impact on halo energy and density profiles
  - Energy cascade in dark matter flow
  - Properties of spherical, axisymmetric, rotating, and growing halos (from mass accretion)
  - Maximum entropy distributions in dark matter flow
  - Halo mass function from maximum entropy distribution
  - Two-body collapse model (TBCM): an elementary step of mass cascade
  - Energy and momentum evolution and integral constants

- Statistical (correlation-based) approach
  - One-point statistics: velocity, density, acceleration distributions in dark matter flow
  - Two-point statistics:
    - Kinematic relations for second order statistics (correlation, structure, spectrum functions)
    - Kinematic and dynamic relations for high order statistics

Applications

- Predicting dark matter mass and properties
- Origin of MOND acceleration
- Baryonic-to-halo mass relation and total baryons in halos

Structural (halo-based) approach for dark matter flow

Inverse mass cascade in dark matter flow and effects on halo mass functions

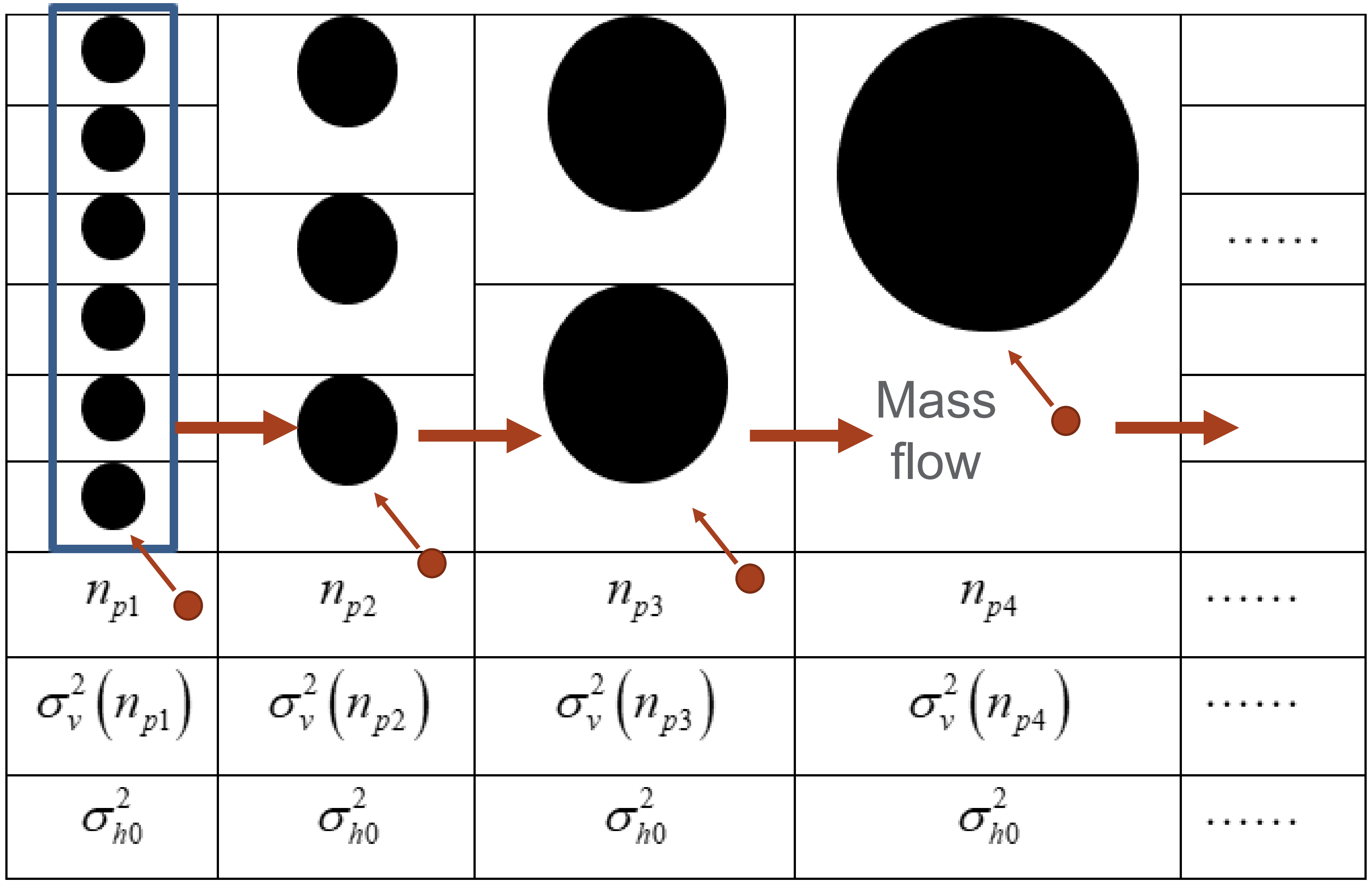
Xu Z., 2021, arXiv:2109.12244v1 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2109.12244

Introduction

Review: In hydrodynamic turbulence, "energy cascade" involves the energy transfer from large eddies to small eddies with a scale-independent rate of energy cascade.

The dark matter flow, a self-gravitating collisionless flow, involves a continuous mass transfer from small to large mass scales with a scale-independent rate of mass cascade ε_m.

- Goal 1: Identify and formulate mass cascade
- Goal 2: Explore the random walk of halos in mass space
- Goal 3: Derive the halo mass function based on the theory of mass cascade



- Identify all halos of different sizes
- Group halos according to the halo size n_p
- Mass flow across halo groups from small to large mass scale (inverse) through the merging with "single merger"
- Cascade leads to random-walk of halos in mass space

Mass redistribution among halo groups

Backward function: fraction of mass inherited from all other halo groups at an earlier time

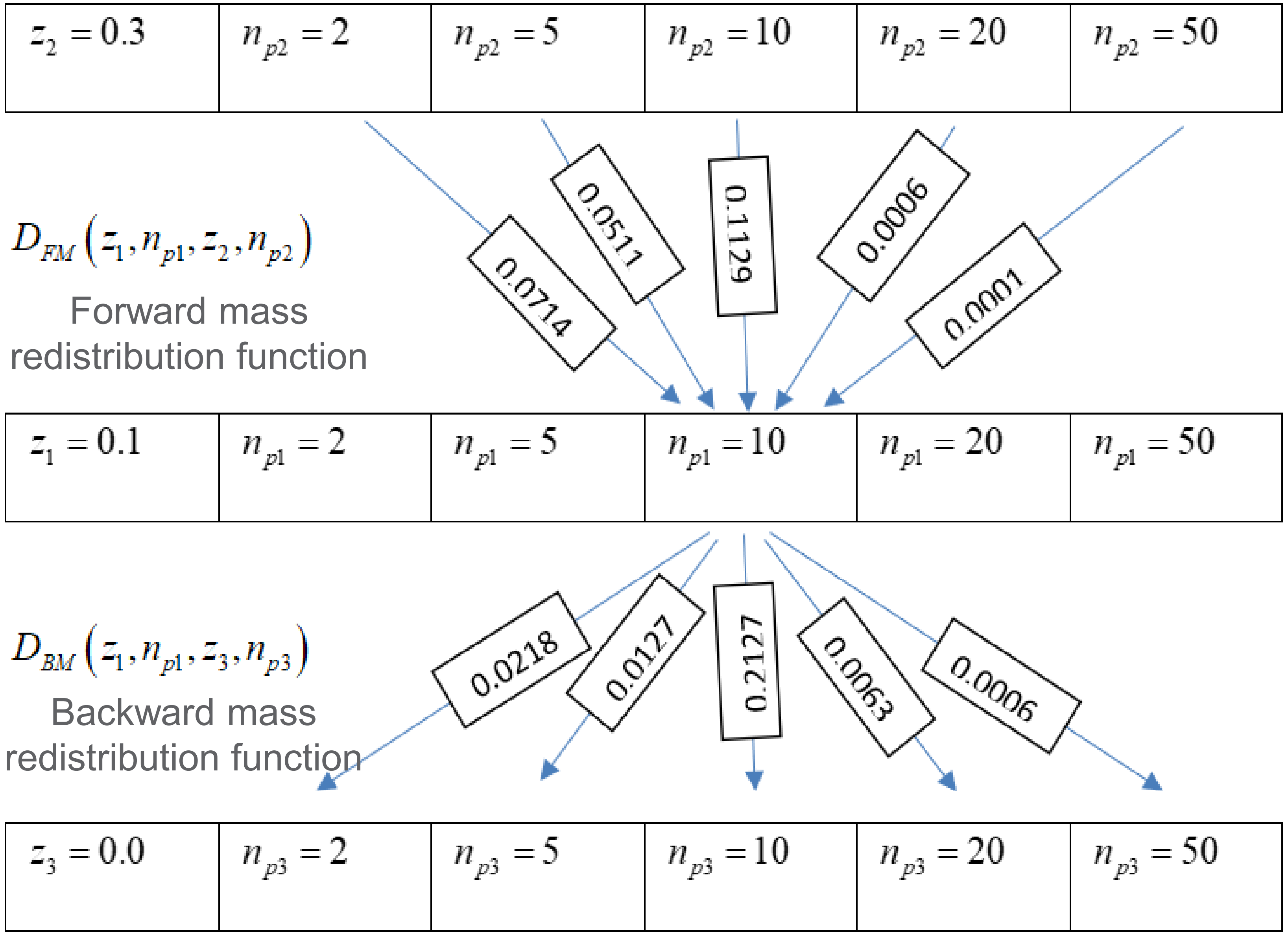
Forward function: fraction of mass passed to all other halo groups at a later time

Minus sign

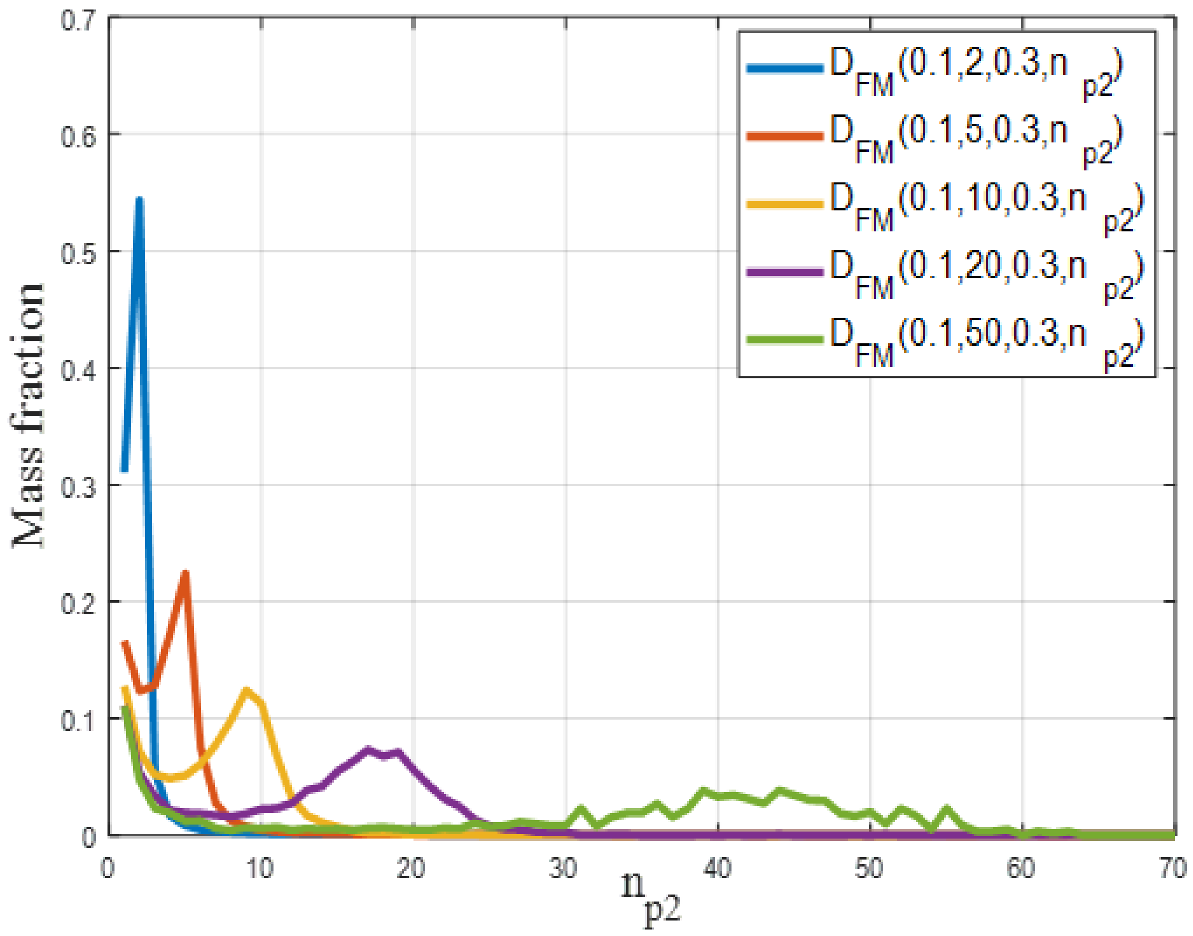
Backward mass redistribution function

-- Forward mass redistribution function

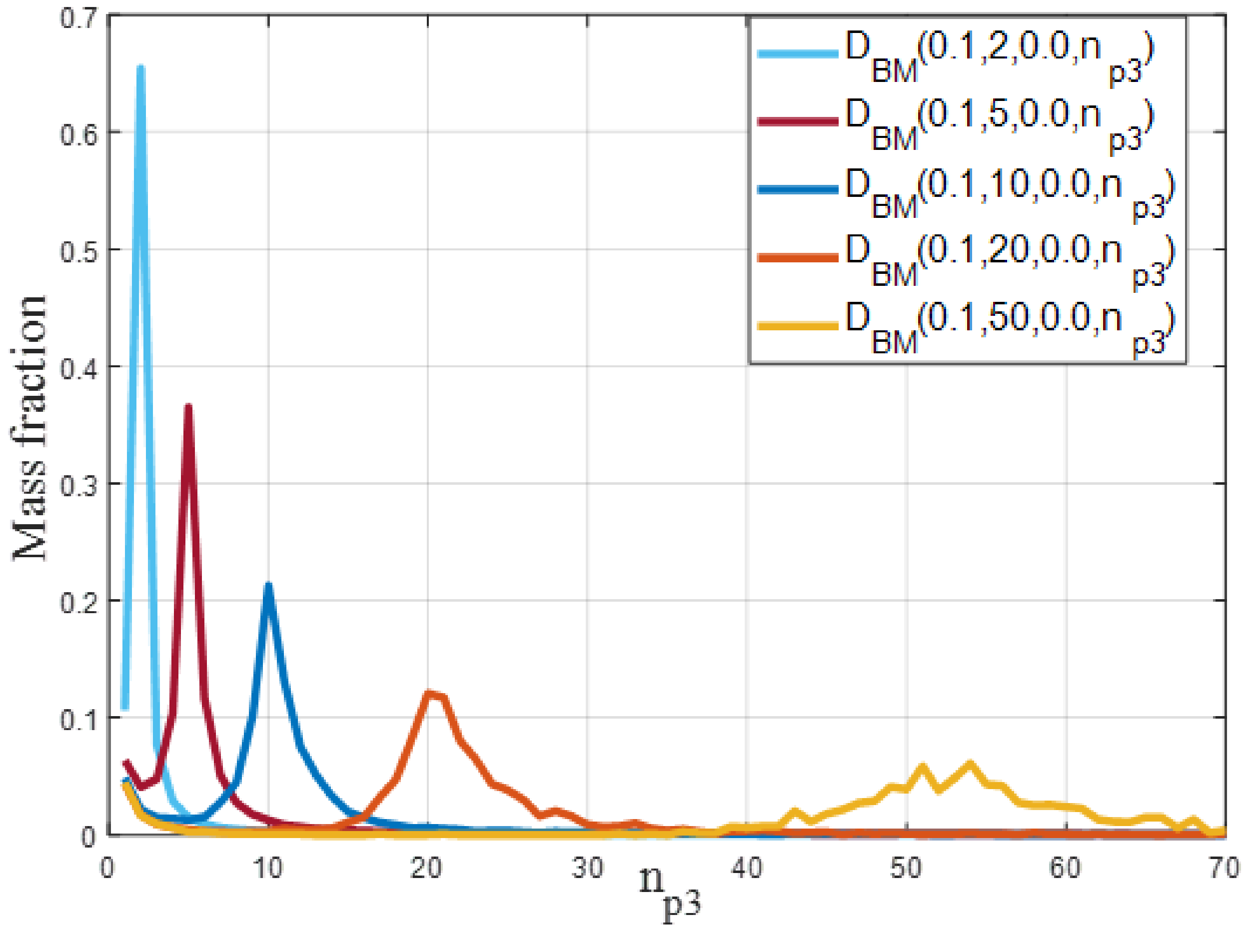
= Net mass redistribution function



Properties/features of mass cascade



Forward mass redistribution function



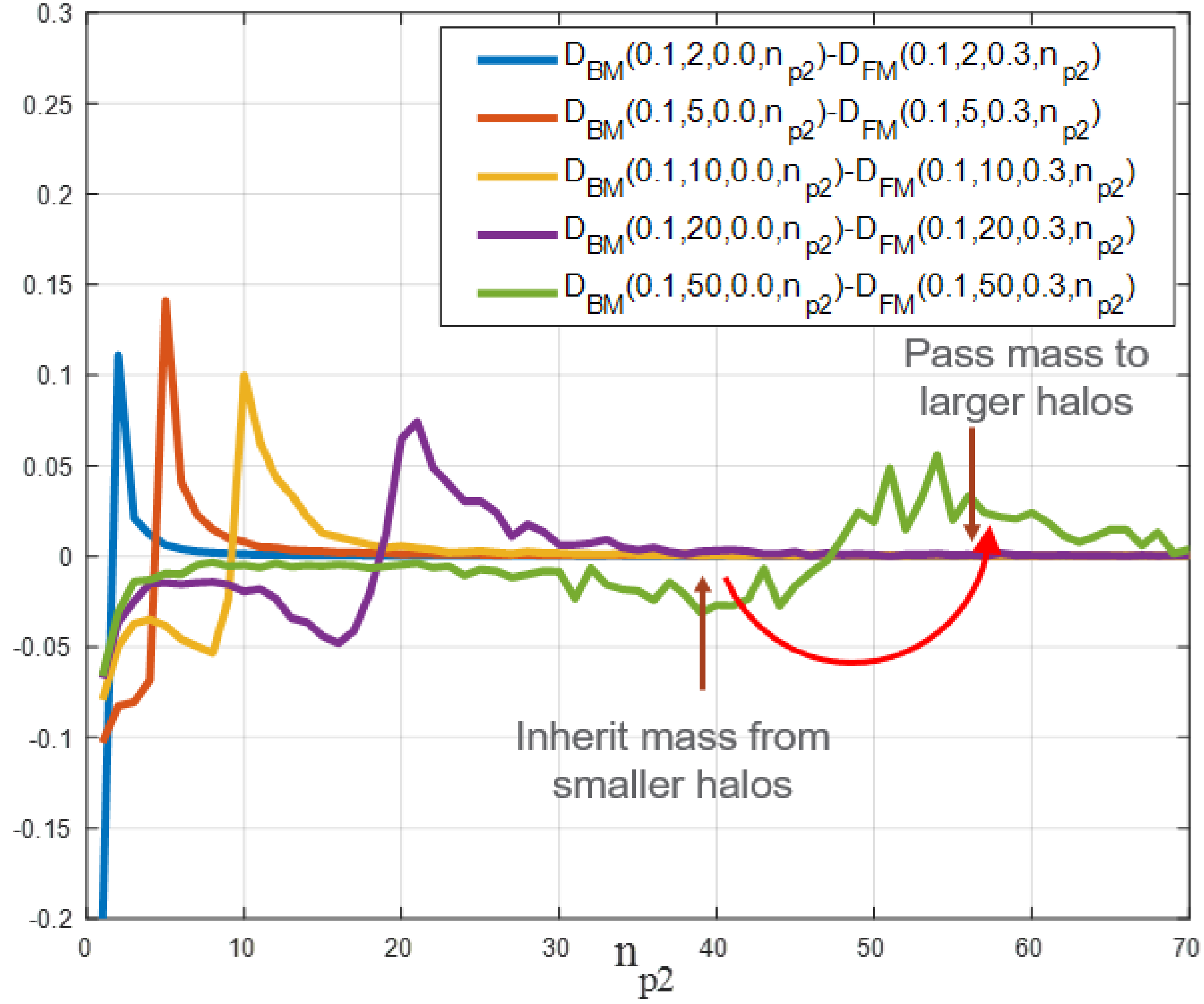
Backward mass redistribution function

- *Local: cascade is local in mass space*

Halos inherit/pass their mass mostly from/to halos of the same or similar size.

(energy cascade in turbulence is also local in wavenumber space)

Properties of mass cascade



Net mass redistribution function

Net mass redistribution function D_{NM}:
<0: inherit more mass than pass mass
>0: pass more mass than inherit mass
Sum of D_{NM} =0

Net effect: halos transfers mass from below to above.

- *Asymmetric: cascade is two-way in mass space but not symmetric*
- *Inverse: from small to large mass scales*

(energy cascade in turbulence is a direct cascade from large to small scales)

Time and mass scales in inverse mass cascade

Average waiting time of a merging event with a single merger in a given halo group of halo mass m_h

$$\tau_h\left(m_h,a\right)$$

The rate at which mass is passed up from this group:

$$\varepsilon_m \sim -m_h/\tau_h$$

Average waiting time (halo lifespan) of a merging event for a given halo in halo group with n_h halos of mass m_h

$$\tau_g\left(m_h,a\right)=n_h\tau_h=-\frac{m_h n_h}{\varepsilon_m}=-\frac{m_g}{\varepsilon_m}$$

Average time required to form halo of mass m_h via a sequence of merging events (n_p times):

$$\tau_f\left(m_h,a\right)=\tau_g n_p=\tau_g m_h/m_p$$

Time required to cascade entire mass M_h in all halos:

$$\tau_M\left(a\right)=-M_h\left(a\right)/\varepsilon_m\left(a\right)\sim t$$

Time required to form halo of a characteristic mass m_h^* should be on the order of the current physical time t:

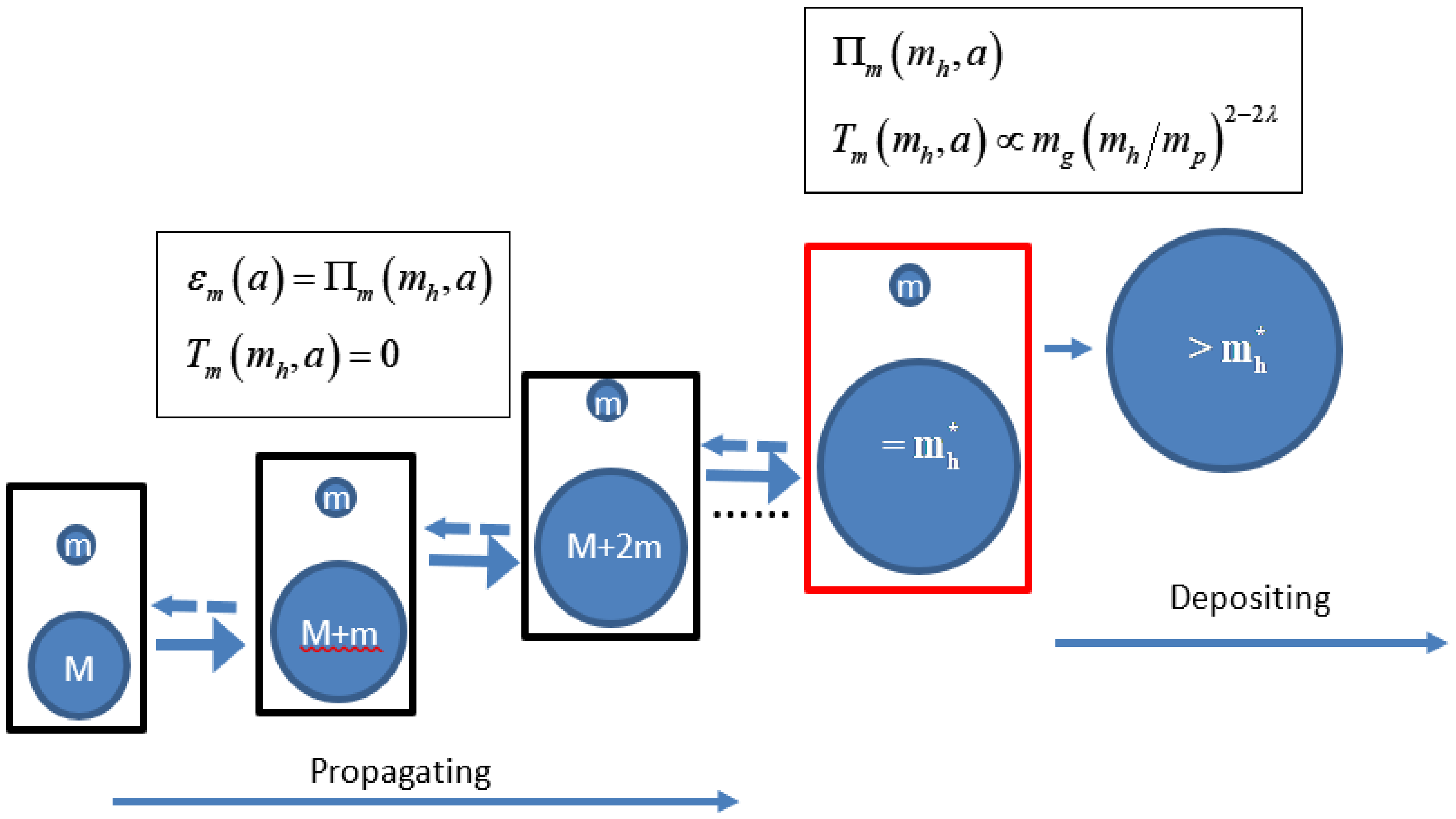
$$\tau_f\left(m_h^*,a\right)\sim t$$

$$\boxed{\tau_M\left(a\right)\geq\tau_f\left(m_h,a\right)\geq\tau_g\left(m_h,a\right)\geq\tau_h\left(m_h,a\right)}$$

$$m_h^*\sim\frac{M_h\left(a\right)m_p}{n_h^* m_h^*}\sim-\frac{\varepsilon_m\left(a\right)}{H n_h^* n_p^*}$$

Chain reaction description of mass cascade

"Little halos have big halos, That feed on their mass; And big halos have greater halos, And so on to growth"

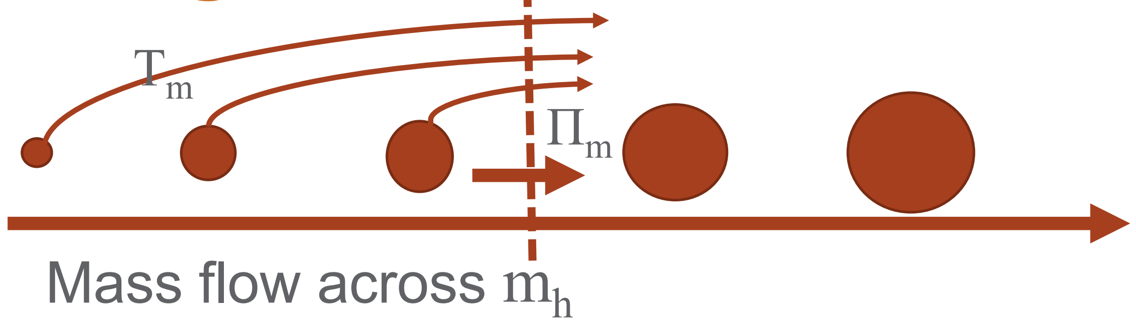


Chain reactions provide non-equilibrium systems a potential mechanism to continuously release energy and increase the system entropy.

- Mass cascade is Local, Asymmetric, Inverse;
- Justifies a chain reaction description of mass cascade;
- The initial stage: initiation/generation of the chain carriers (free radicals)
- The propagation stage: a sequence of accretion of single mergers to propagate the mass along the reaction chain
- The termination stage: the deposition of the mass cascaded from the scales below to grow halos

Formulating mass cascade

Mass flux function (kg/s): total mass flux from all halos below m_h



$$\Pi_m(m_h,a) = -\frac{\partial}{\partial t}\left[M_h(a)\int_{m_h}^{\infty} f_M(m,m_h^*)dm\right]$$

Mass transfer function (1/s): rate of mass transfer for halos of mass m_h

$$T_m(m_h,a) = \frac{\partial \Pi_m(m_h,a)}{\partial m_h} = \frac{\partial\left[M_h(a) f_M(m_h,m_h^*)\right]}{\partial t} = \frac{\partial m_g(m_h,a)}{m_p \partial t}$$

In mass propagation range: $m_h \ll m_h^*$

$$\varepsilon_m(a) = \Pi_m(m_h,a)$$

$$T_m(m_h,a) = \frac{\partial m_g(m_h,a)}{m_p \partial t} = 0 \qquad m_g(m_h) \equiv m_g(m_h,a)$$

Mass flux function: $\Pi_m(m_h,a)$

Total mass of all halos: $M_h(a)$

Halo mass function: $f_M(m_h,a)$

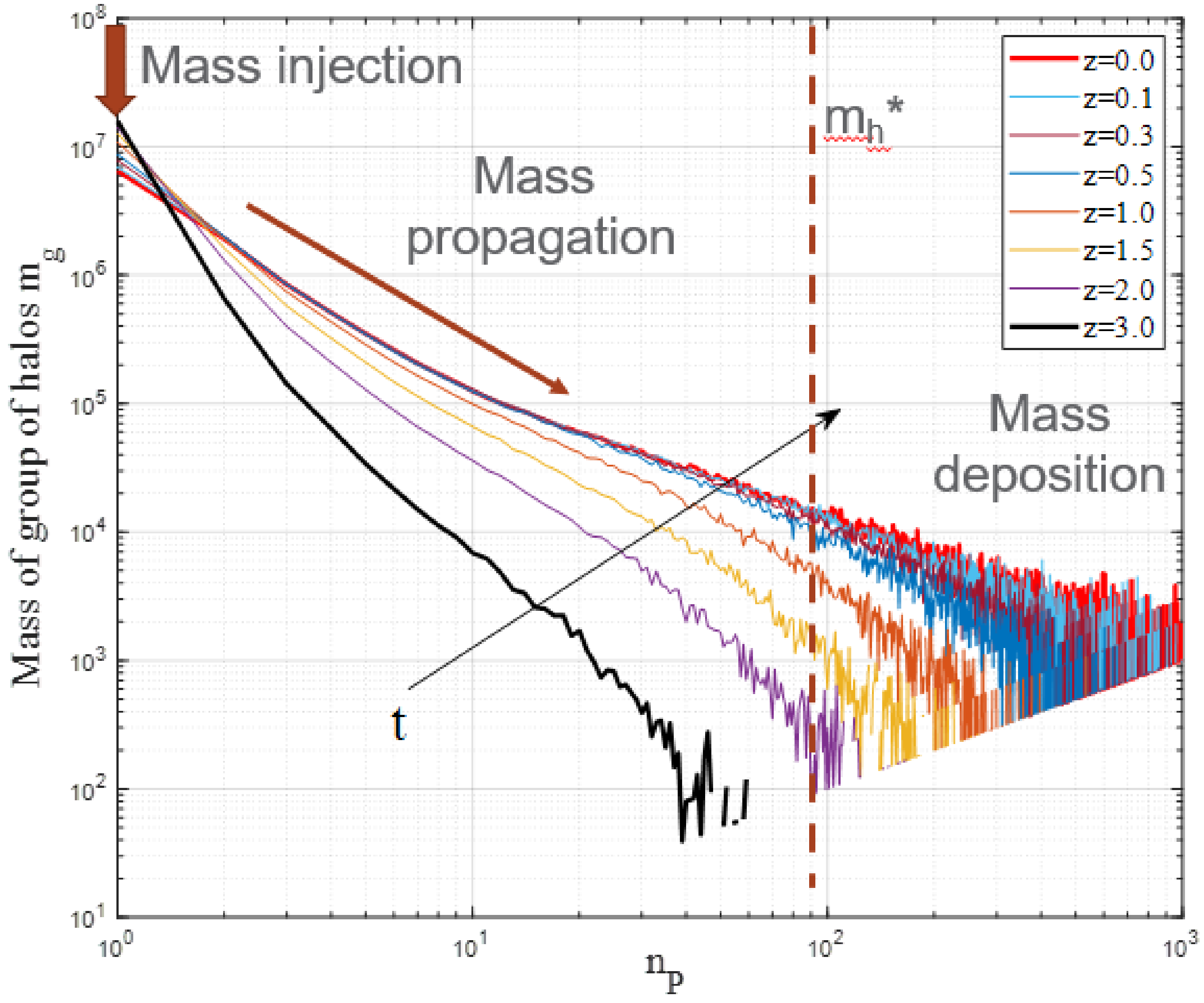
Halo group mass: $m_g(m_h,a)$

Halo mass: m_h

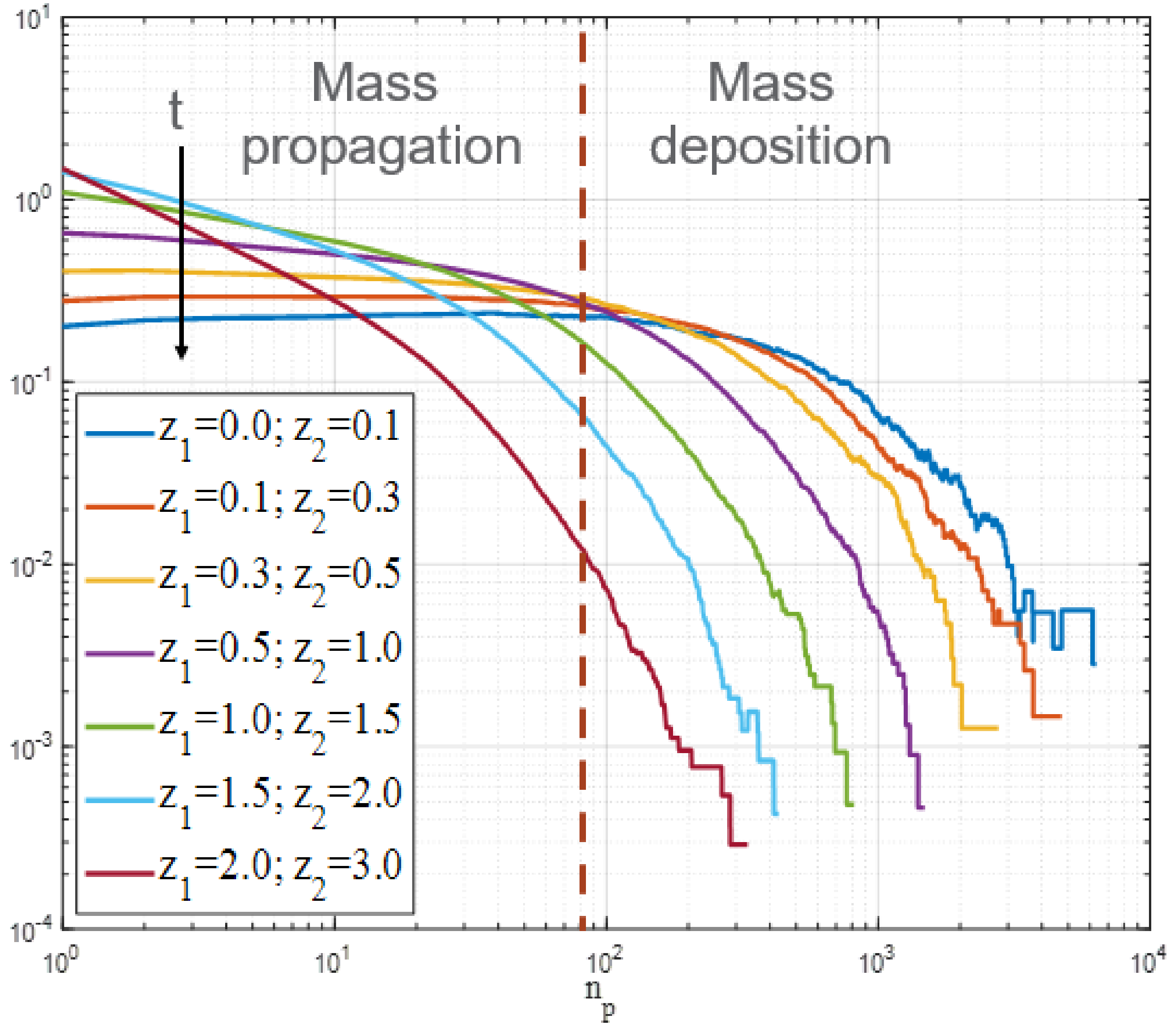
Particle mass: m_p

- Rate of mass cascade is Mass-scale independent;
- Halo group mass is time-independent (steady-state);

Halo group mass and mass flux function



Halo group mass $m_g(m_h, a)$
(time-independent in mass propagation range)



Mass flux function $\Pi_m(m_h, a)$ (normalized by Nm_p/t_0) varying with halo size
(scale-independent in mass propagation range)

Formulating mass cascade

In mass propagation range: $m_h \ll m_h^*$

$$\varepsilon_m(a) = \Pi_m(m_h = 0, a) = -\frac{\partial M_h(a)}{\partial t}$$

$$\varepsilon_m(a) = -m_h f_h(m_h, a)$$

$$f_h(m_h, a) = \underbrace{f_0(a) M_h(a) f_M(m_h, m_h^*(a)) \frac{m_p}{m_h}}_{1} \underbrace{\left(\frac{m_h}{m_p}\right)^{\lambda}}_{2}$$

Term 1: proportional to the number of halos in group;

Term 2: proportional surface area of halo in group;

$$\lambda \approx 2/3$$

Merging frequency for halo group: $f_h(m_h, a)$

Halo geometry parameter: λ

Fundamental frequency for merging of two single mergers: $f_0(a) \propto a^{-\tau_0}$

Independent variables: m_h a

Free parameters: m_p λ τ_0

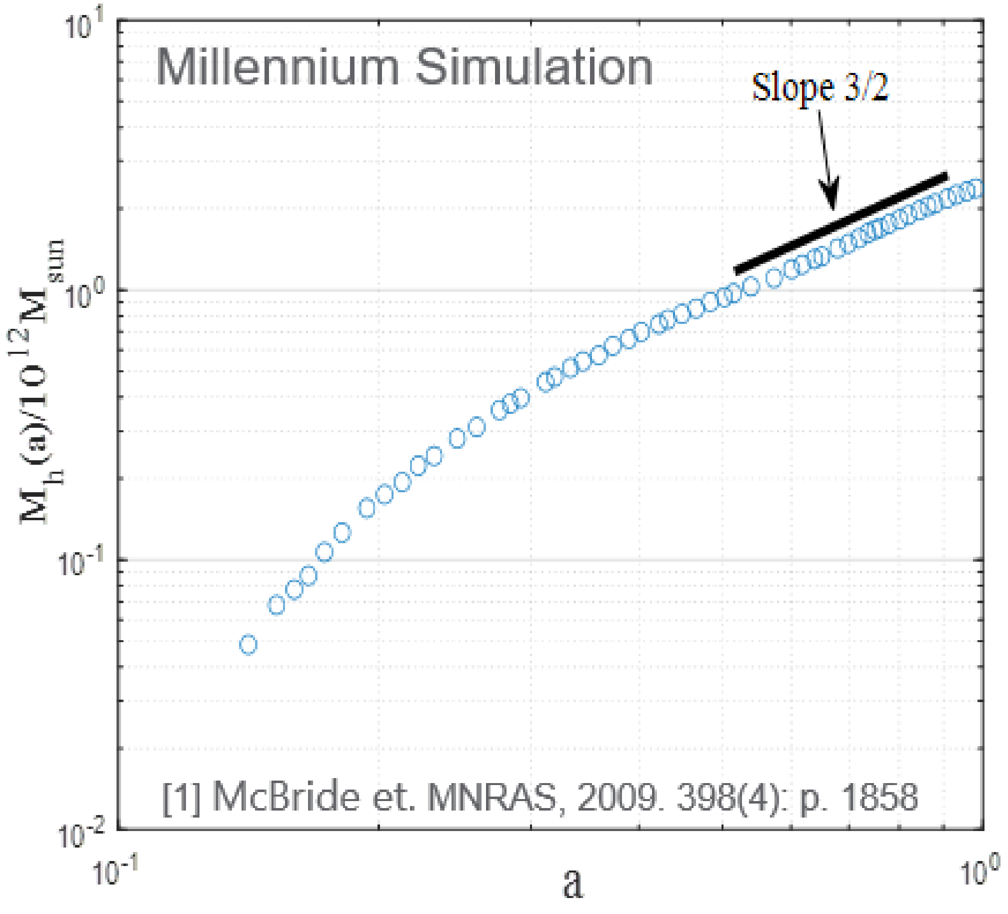
Formulating mass cascade

In mass propagation range: $m_h \ll m_h^*$

Dimensional analysis requires mass function: $$f_M\left(m_h, m_h^*\right) = \beta_0 m_h^{-\lambda}\left(m_h^*\right)^{\lambda-1}$$

Table 2. List of dependence on the scale factor a for different values of τ_0 and λ

λ	τ_0	f_0	ε_m	M_h	f_M	m_h^*	τ_h^*	n_h^*	m_g^*	τ_g^*
λ	τ_0	$a^{-\tau_0}$	$a^{-\tau_0}$	$a^{3/2-\tau_0}$	$a^{\tau_0-3/2}$	$a^{\frac{(3/2-\tau_0)}{(1-\lambda)}}$	$a^{\frac{(3/2-\lambda\tau_0)}{(1-\lambda)}}$	$a^{-\left(\frac{3}{2}-\tau_0\right)\frac{(1+\lambda)}{(1-\lambda)}}$	$a^{-\left(\frac{3}{2}-\tau_0\right)\frac{\lambda}{(1-\lambda)}}$	$a^{\frac{(\tau_0-3\lambda/2)}{(1-\lambda)}}$
$2/3$	1	a^{-1}	a^{-1}	$a^{1/2}$	$a^{-1/2}$	$a^{3/2}$	$a^{5/2}$	$a^{-5/2}$	a^{-1}	a^{0}
$2/3$	$1/2$	$a^{-1/2}$	$a^{-1/2}$	a^{1}	a^{-1}	a^{3}	$a^{7/2}$	a^{-5}	a^{-2}	$a^{-3/2}$
$3/4$	1	a^{-1}	a^{-1}	$a^{1/2}$	$a^{-1/2}$	a^{2}	a^{3}	$a^{-7/2}$	$a^{-3/2}$	$a^{-1/2}$



The halo mass for type II halos (the dominant type for large halos, Fig. 2 in ref. [1]) exhibits a power law scaling

Dependence on halo mass m_h and mass resolution m_p

Table 3. List of dependence on the halo mass m_h

f_M	ε_m	f_h	τ_h	τ_g	τ_f	m_g	n_h
$m_h^{-\lambda}$	m_h^0	m_h^{-1}	m_h^1	$m_h^{-\lambda}$	$m_h^{1-\lambda}$	$m_h^{-\lambda}$	$m_h^{-1-\lambda}$

Table 4. List of dependence on the mass resolution m_p

f_M	f_h	τ_f	ε_m	τ_h^*	τ_h	m_h^*	M_h	c_0	β_0
m_p^0	m_p^0	m_p^0	m_p^0	m_p^0	m_p^0	m_p^0	m_p^0	m_p^0	m_p^0
n_h	τ_g	m_g	n_p	f_0	b_0	λ_0	n_h^*	m_g^*	n_p^*
m_p	m_p	m_p	m_p^{-1}	$m_p^{\lambda-1}$	$m_p^{\lambda-1}$	$m_p^{1-\lambda}$	m_p	m_p	m_p^{-1}

Fundamental frequency f_0 for merging between two single mergers depends on particle mass (same as cosmological redshift for photon frequency $f \sim a^{-1}$):

$$f_0 \propto a^{-1} m_p^{-1/3}$$

Can we detect f_0 from any experiment or observation?

Random walk of halos and halo mass function

Merging frequency for halo group: $$f_h\left(m_h, a\right)$$

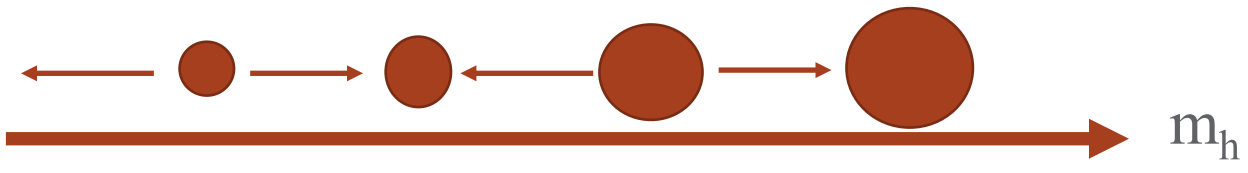
Characteristic merging time for halo group: $$\tau_h\left(m_h, a\right) = 1/f_h$$

Characteristic merging time (lifetime) for a given halo: waiting time to merge

\# of halos in group

$$\tau_g\left(m_h, a\right) = n_h \tau_h$$

The exponential distribution of waiting time to merge: $$P\left(\tau_{gr}\right) = \frac{1}{\tau_g}\exp\left(-\frac{\tau_{gr}}{\tau_g}\right)$$



1D Random walk equation in mass space:

$$\frac{\partial m_h(t)}{\partial t} = \frac{m_p \xi(t)}{\tau_g(m_h)} = \sqrt{2D_p(m_h)}\varsigma(t)$$

Fokker-Planck equation for distribution function:

$$\frac{\partial P_h}{\partial t} = \frac{\partial}{\partial m_h}\left[\sqrt{D_p}\frac{\partial}{\partial m_h}\left(\sqrt{D_p}P_h\right)\right] = D_{p0}\frac{\partial}{\partial m_h}\left[m_h^{\lambda}\frac{\partial}{\partial m_h}\left(m_h^{\lambda}P_h\right)\right]$$

Halo mass function:

$$f_M\left(m_h, a\right) = \frac{(1-\lambda)}{\sqrt{\pi\eta_0}}\left(\frac{m_h^*}{m_h}\right)^{\lambda}\frac{1}{m_h^*}\exp\left[-\frac{1}{4\eta_0}\left(\frac{m_h}{m_h^*}\right)^{2-2\lambda}\right]$$

Reduce to Press-Schechter (PS) mass function if $\lambda=2/3$!

Double-λ mass function from mass cascade

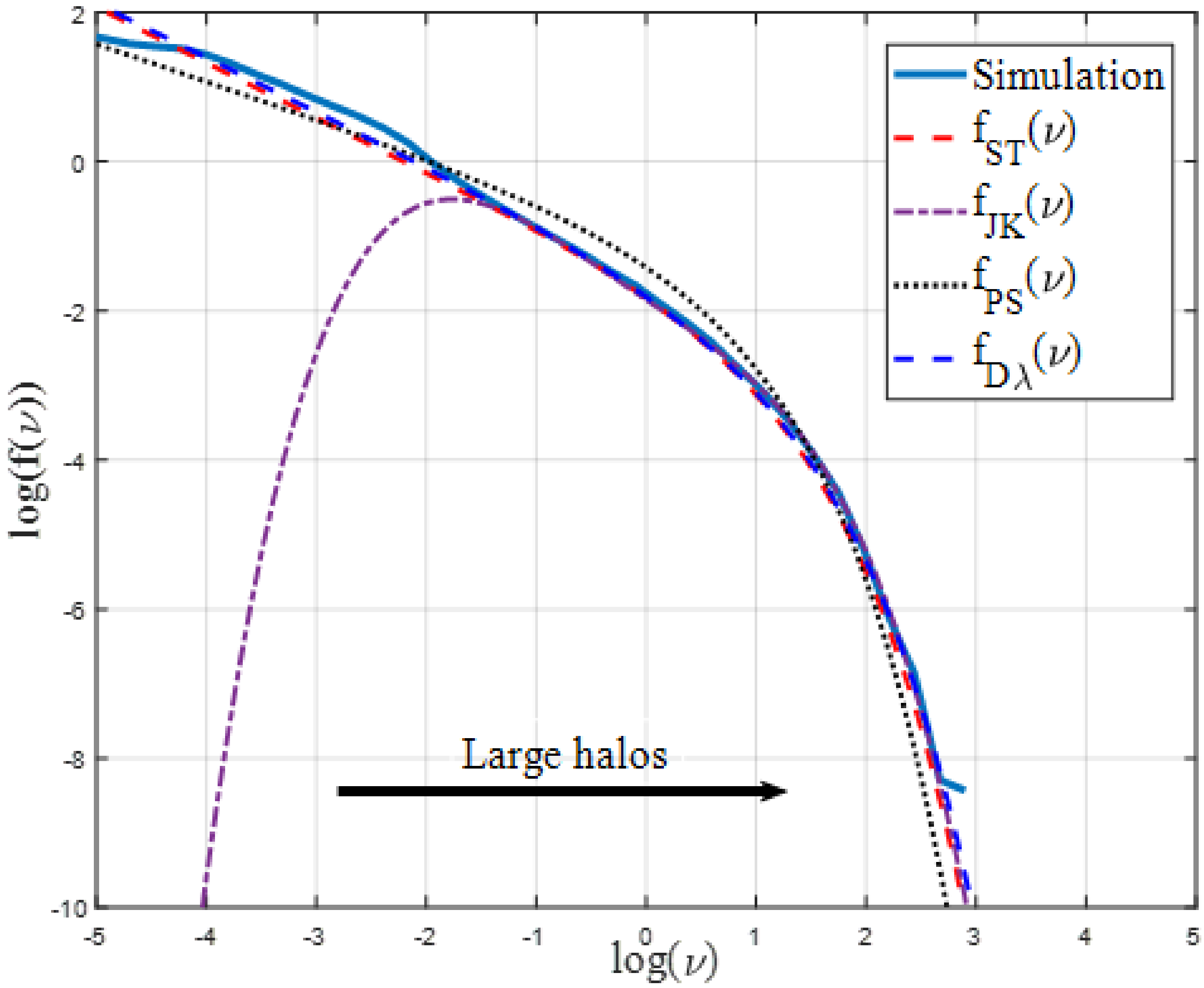
λ: halo geometry parameter; naturally, we can have different λ for different range.
λ_1 for mass propagation range (small halos);
λ_2 for mass deposition range (large halos);

$$f_M(m_h,a)=\frac{(1-\lambda)}{\sqrt{\pi\eta_0}}\left(\frac{m_h^*}{m_h}\right)^{\lambda_1}\frac{1}{m_h^*}\exp\left[-\frac{1}{4\eta_0}\left(\frac{m_h}{m_h^*}\right)^{2-2\lambda_2}\right]$$

Double-λ mass function:

$$f_{D\lambda}(\nu)=\frac{\left(2\sqrt{\eta_0}\right)^{-q}}{\Gamma(q/2)}\nu^{q/2-1}\exp\left(-\frac{\nu}{4\eta_0}\right)$$

- PS mass function
- ST model (modified PS) from ellipsoid collapse
- JK mass function by data fitting
- More generally, λ_1 can be a function of halo mass m_h



Comparison between different mass functions and simulation

Summary and key words

Hydrodynamic turbulence	Dark matter flow
Direct energy cascade from large to small length scales	Inverse mass cascade from small to large mass scales
“inertial range” & “dissipation range”	propagation range & deposition range

Mass redistribution
Random walk
Heterogeneous diffusion
Waiting time
Chain-reaction
Halo mass function

- Strong connections between dark matter flow and hydrodynamic turbulence
- The mass cascade is local, two-way, and asymmetric in mass space
- Scale-independent rate of mass cascade and time-independent halo group mass
- Chain reaction description for mass cascade to release energy and maximize entropy
- Random-walk of halos in mass space with an exponential distribution of waiting time
- Press-Schechter mass function is a special solution from halo random-walk
- New Double-λ halo mass function (based on the mass cascade)
- Extend double-λ halo mass function to consider λ as some function of halo size.

Effect of mass cascade on halo energy, size, and density profile

Xu Z., 2021, arXiv:2109.12244v1 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2109.12244

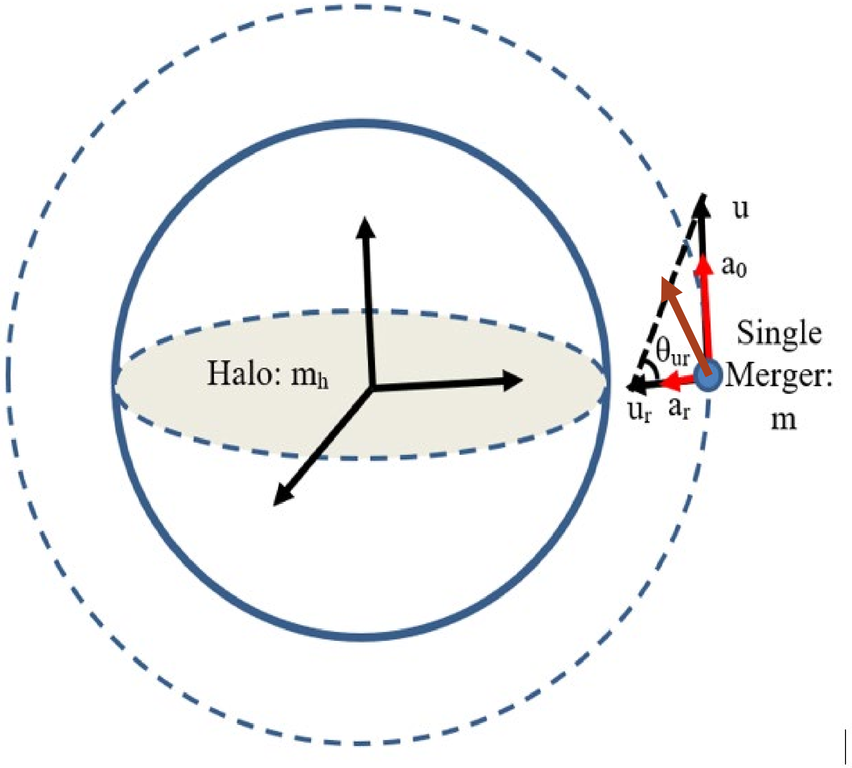
Introduction

Review: In hydrodynamic turbulence, “Energy cascade” involves the energy transfer from large eddies to small eddies with a scale-independent rate of energy cascade. No mass cascade!

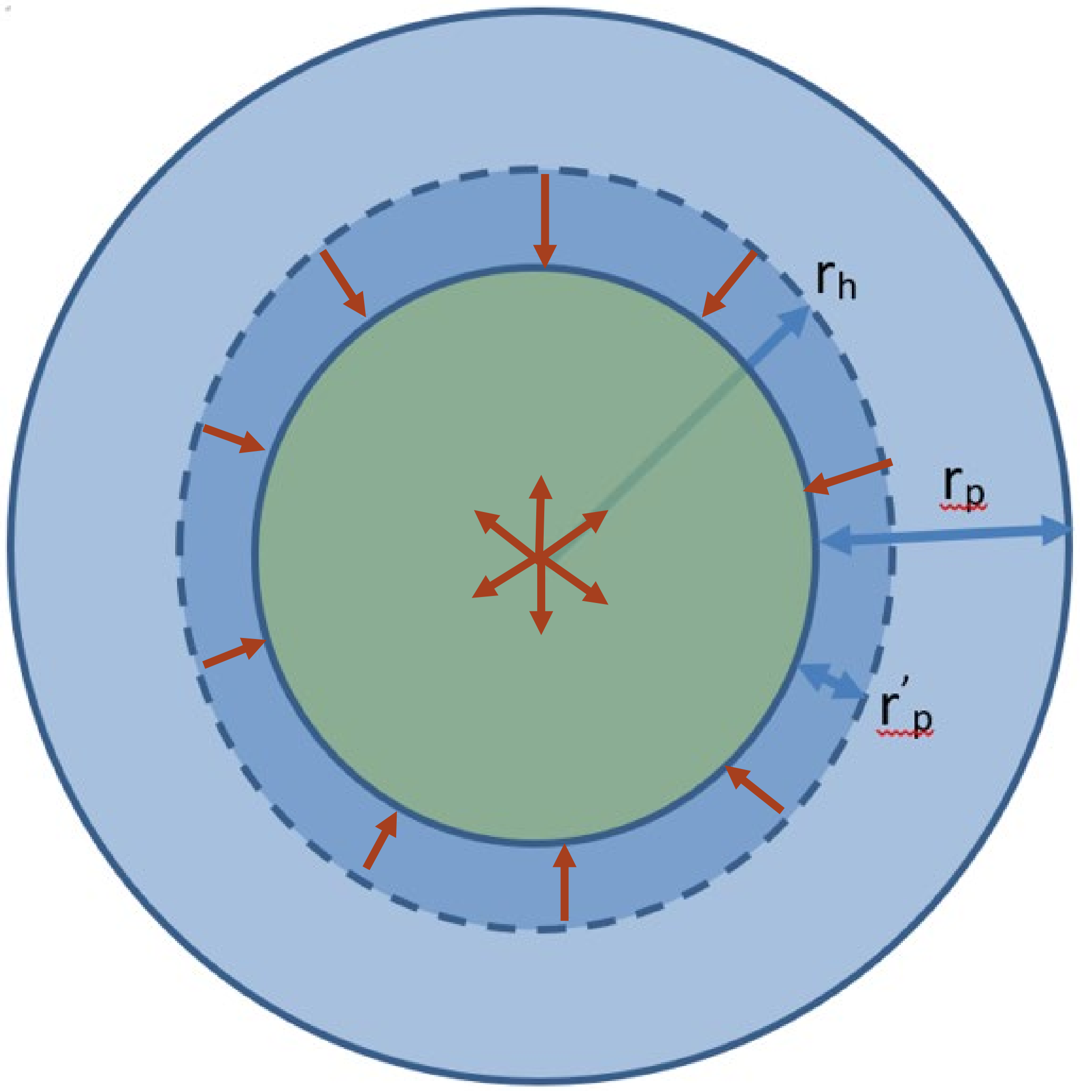
“Little halos have big halos, That feed on their mass; And big halos have greater halos, And so on to growth”

“Eddy” is not a well-defined object in turbulence literature. However, “halo” are well-defined dynamical objects, whose abundance and internal structure have been extensively studied over several decades.

- Goal 1: Explore effects of inverse mass cascade on halo energy, momentum, halo size and internal structure (density) evolution.
- Goal 2: Explore the dynamic evolution of halo size (geometric Brownian motion)
- Goal 3: Explore the random walk of particle in halos with a randomly evolution size. This leads to a universal halo density profile.



Halo mass accretion, deformation, and radial flow



Schematic plot of halo mass accretion and deformation

- Halo grows with a new layer of particles of thickness r_p formed due to halo mass accretion (mass cascade)
- Original halo (dash line) deforms in size (shrinks to green) by r_p' due to gravity of new layer
- The net change in halo size is r_p - r_p'
- Halo deformation at halo surface induces a non-zero inward radial flow u_r
- What about the radial flow at halo center??
  - Must be outwards if no blackhole considered

Halo deformation parameter $$\alpha_h = 1 - r_p^{'} / r_p$$

Isothermal profile (vanishing radial flow, no time to relax or deform due to extremely fast mass accretion): $\alpha_h = 1$

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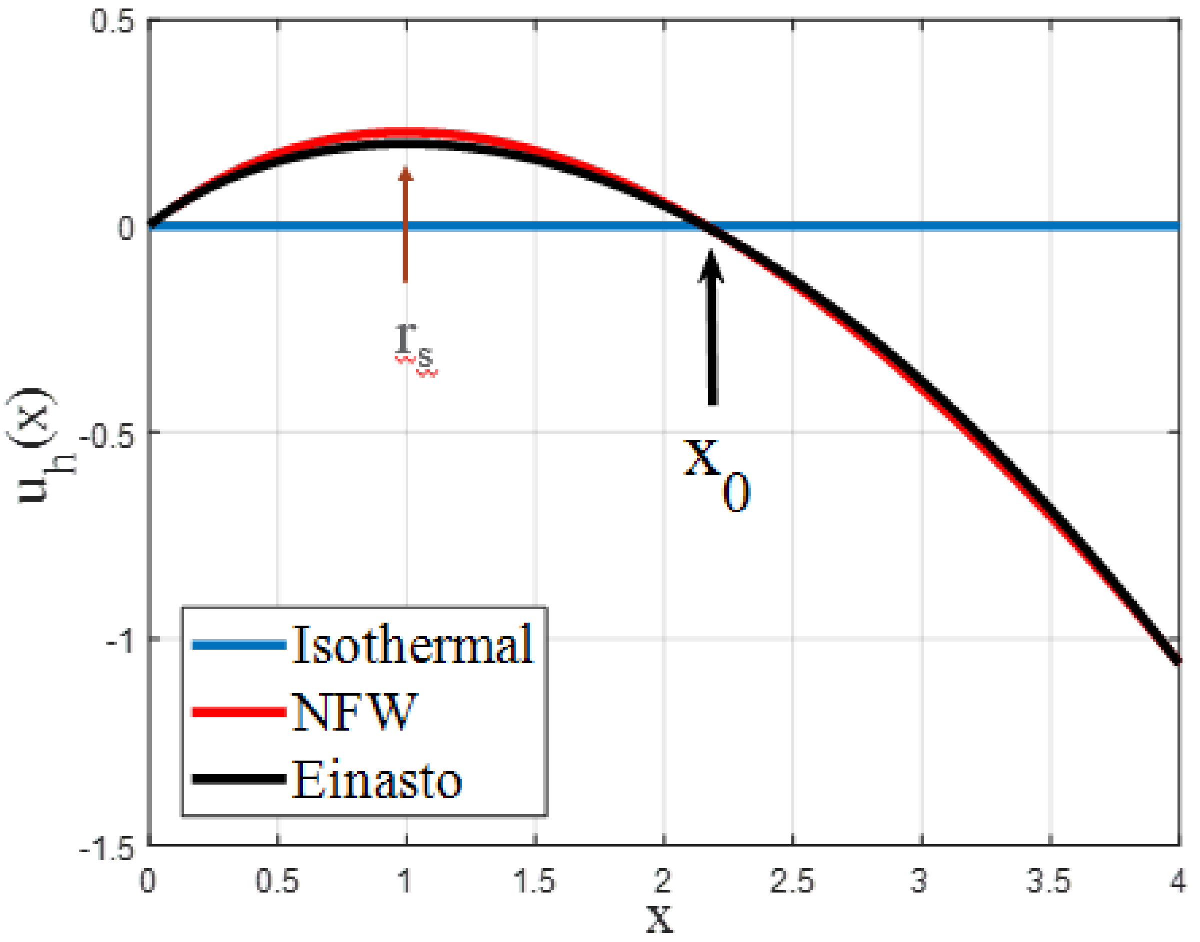
Effect of radial flow on halo density profile



- Outward flow in core and inward flow in outer region
- Radial flow creates a new length scale for any halo density: the scale radius r_s
- Vanishing radial flow for isothermal: extremely fast mass accretion and no time for halo to deform

Reduced spatial/ temporal coordinate: $x(r,a)=\frac{r}{r_s(a)}=\frac{cr}{r_h(a)}$

Function *F(x)* for enclosed mass at given *r*: $m_r(r,a)=m_h(a)\frac{F(x)}{F(c)}$

Halo density: $\rho_h(r,a)=\frac{1}{4\pi r^2}\frac{\partial m_r(r,a)}{\partial r}=\frac{m_h(a)}{4\pi r_h^3}\frac{c^3F'(x)}{x^2F(c)}$

Radial continuity equation: $\frac{\partial \rho_h(r,a)}{\partial t}+\frac{1}{r^2}\frac{\partial\left[r^2\rho_h(r,a)u_r(r,a)\right]}{\partial r}=0$

Radial flow equation: $u_h(x)=\left[x-\frac{F(x)}{F'(x)}\right]\frac{\partial \ln r_h}{\partial \ln t}$ ← Mass cascade

Density ρ_h ⟷ F(x) ⟷ Radial flow $u_h(x)$

NFW: $F(x)=\ln(1+x)-x/(1+x)$

Einasto: $F(x)=\Gamma(3/\alpha)-\Gamma(3/\alpha,2x^{\alpha}/\alpha)$

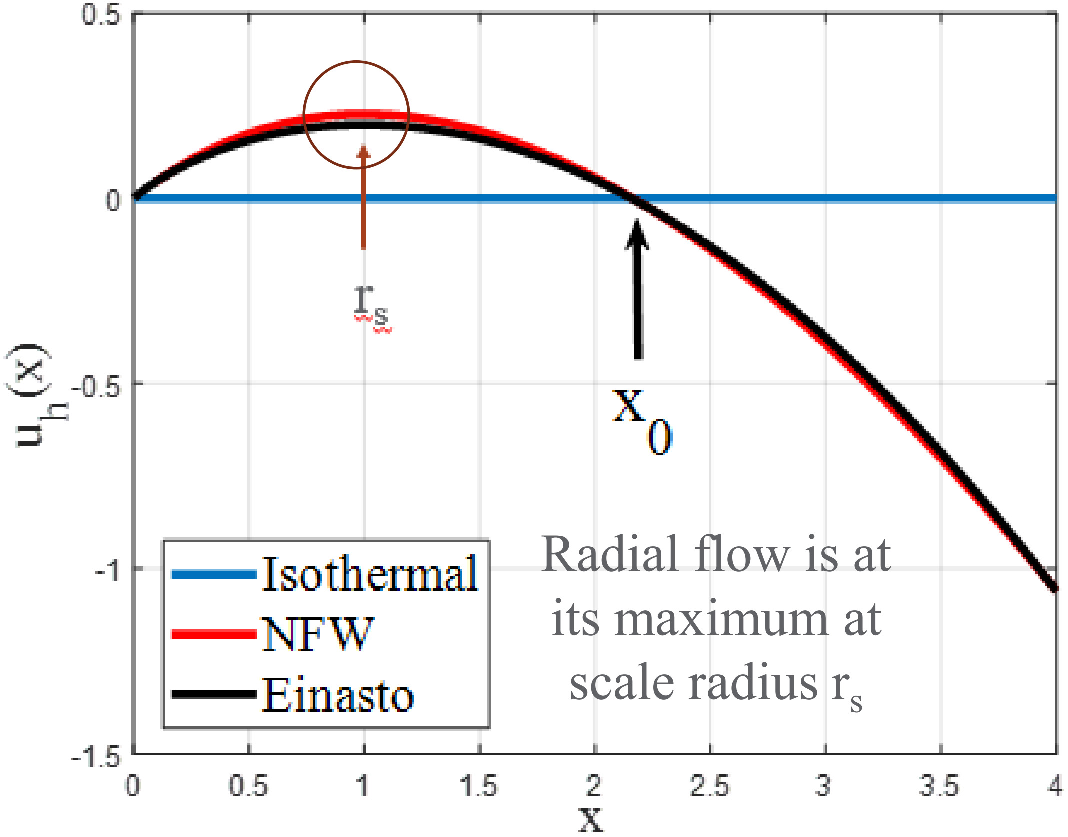
Isothermal: $F(x)=x/c$

Radial flow and angle of incidence

Logarithmic slope of density:

$$\frac{\partial \ln \rho_h}{\partial \ln x} = \frac{\partial \ln F^{'}(x)}{\partial \ln x} - 2 = \frac{\partial u_h / \partial x}{1 - u_h / x} - 2$$

At $r = r_s$ $\quad \frac{\partial \ln \rho_h}{\partial \ln x} = -2$ and $\frac{\partial u_h}{\partial x} = 0$



- Single mergers merging with halo at an angle: angle of incidence
- Neither perpendicular nor tangential
- Angle of incidence determined by peculiar radial flow u_p and circular velocity v_{cir}

$$\cot(\theta_{vr}) = \frac{u_p}{v_{cir}} = \frac{1}{2\pi}\left(\frac{1}{\alpha_h} - \frac{1}{3}\right)$$

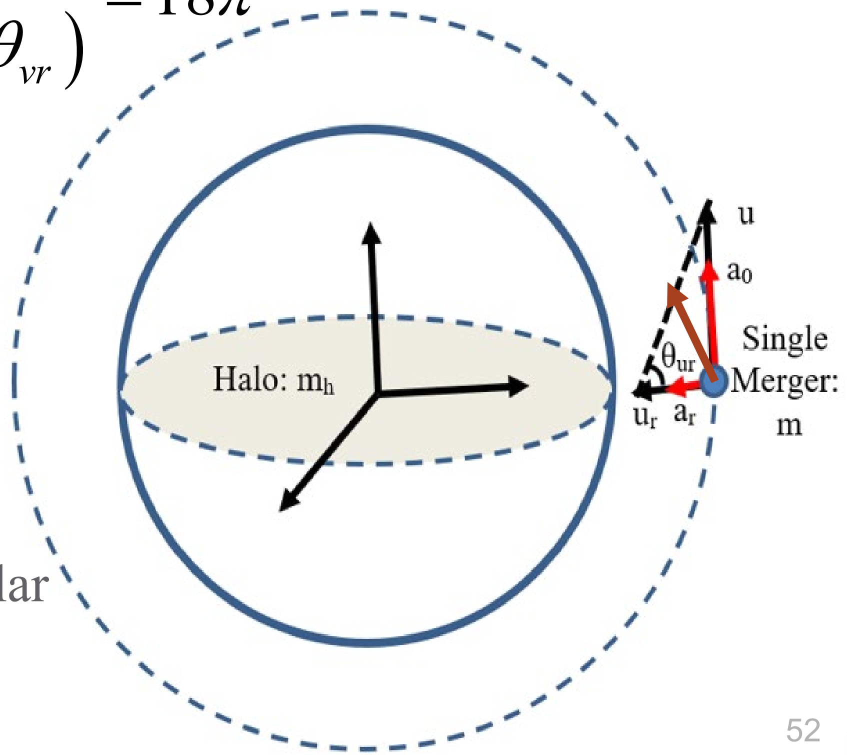
Deformation parameter for Isothermal profile: $\alpha_h = 1$

$$\cot(\theta_{vr}) = \frac{1}{3\pi} = \sqrt{\frac{2}{\Delta_c}}$$

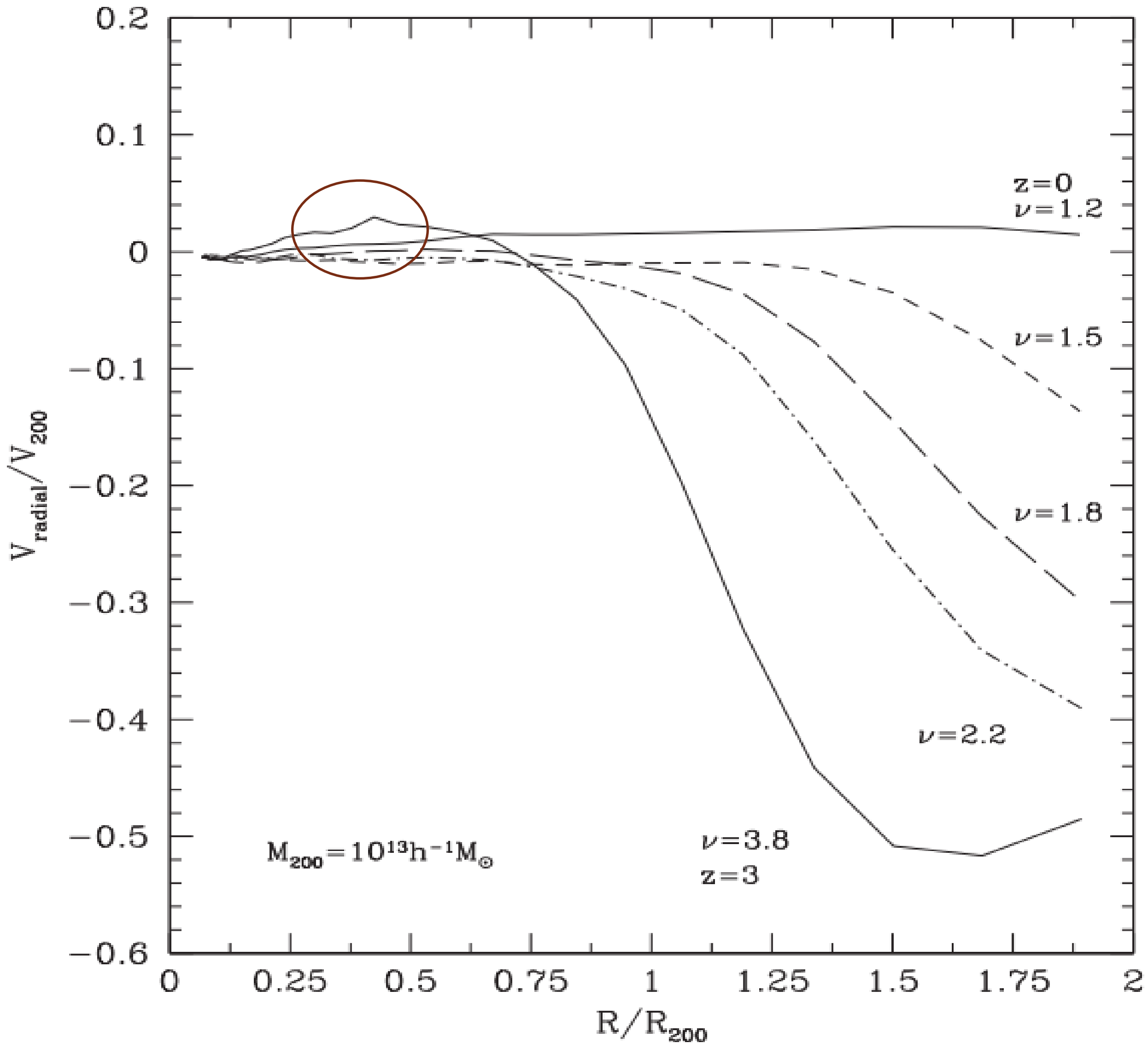
and

$$\Delta_c = \frac{2}{\cot^2(\theta_{vr})} = 18\pi^2$$

- Determine critical halo density Δ_c, (two-body collapse model)
- Determine the rate of energy cascade
  - No energy cascade if tangential
  - Maximum cascade if perpendicular
- Understand the critical MOND acceleration a_0



Radial flow from simulation



Radial flow from simulation

Klypin A. etc.., 2016, Mon. Not. R. Astron. Soc., 457, 4340

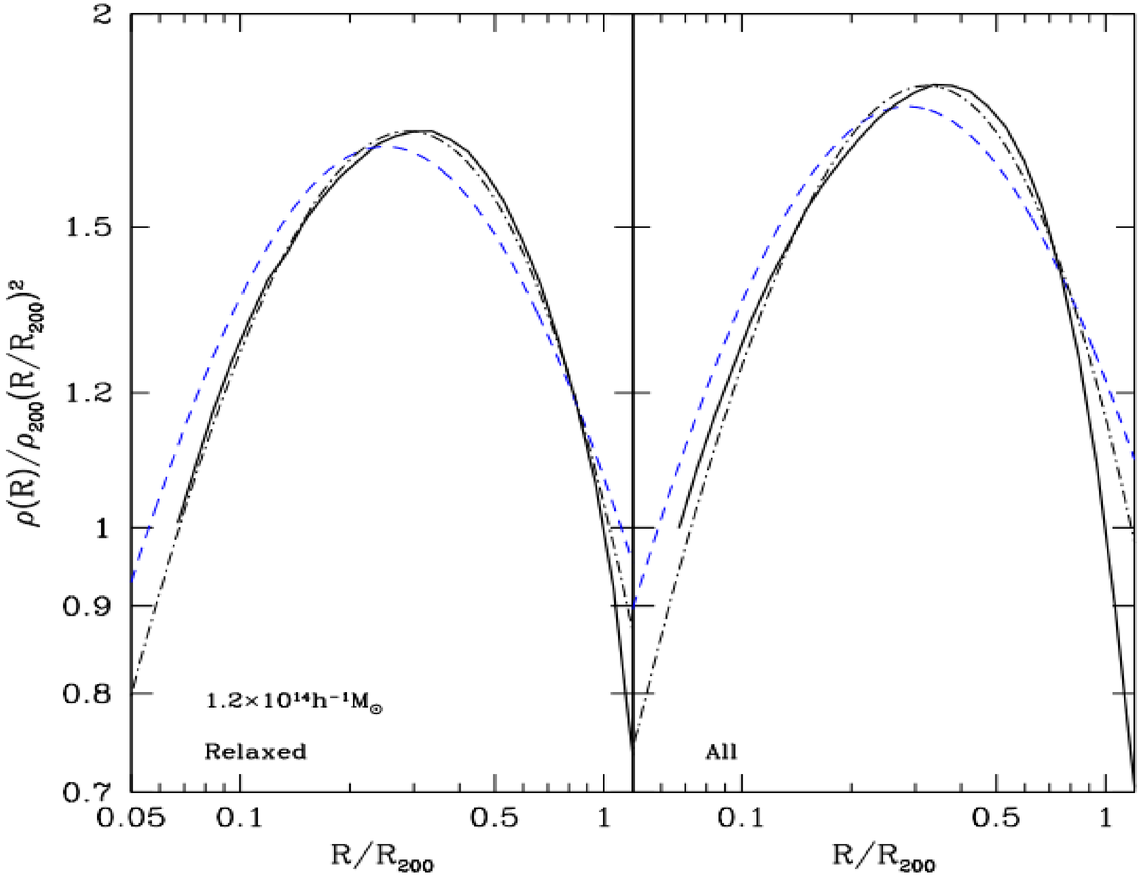


Figure 8. Density profiles of haloes with mass $M_{200} \approx 1.2 \times 10^{14}\, h^{-1}\, M_{\odot}$ at $z = 1.5$ (full curves). Left (right) panels show relaxed (all) haloes. Dot–dashed curves show Einasto fits, which have the same virial mass as haloes in the simulation. The NFW profiles (dashed curves) do not provide good fits to the profiles and significantly depend on what part of the density profile is chosen for fits.

Einasto profile is better than NFW for massive halos (high peak height ν), why?

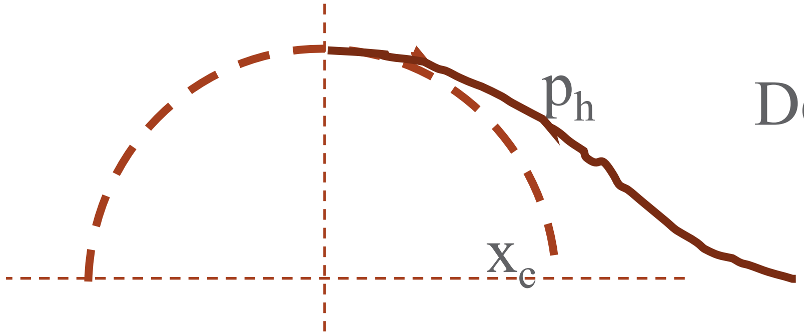
Radial flow u_r and pressure around halo center

Radial flow at halo center:

- Term 1 from mass cascade usually neglected
- The radial flow should vanish for virialized small halos with extremely slow mass accretion (late stage); gravity exactly balances pressure; stable clustering hypothesis (SCH)
- The radial flow should be the Hubble flow for large halos with extremely fast mass accretion (early stage).
- In spherical collapse model, the initial velocity of mass shells is simply the Hubble flow

Define a halo deformation rate:

$$\gamma_h=\partial u_h/\partial x\big|_{x=0}$$



Jeans' equation:

$$\underbrace{\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}}_{1}+\frac{1}{\rho_h}\frac{\partial\left(\rho_h\sigma_r^2\right)}{\partial r}=-\frac{\partial\phi_h(r,a)}{\partial r}=-\frac{Gm_r(r,a)}{r^2}$$

$$\underbrace{\sigma_r^2\frac{\partial\ln\left(\rho_h\sigma_r^2\right)}{\partial\ln x}}_{1}=x\underbrace{\frac{r_s^2}{t^2}\left[\frac{\partial u_h}{\partial x}\left(x\frac{\partial\ln r_s}{\partial\ln t}-u_h\right)+u_h\left(1-\frac{\partial\ln r_s}{\partial\ln t}\right)\right]}_{2}-\underbrace{v_c^2}_{3}$$

1: from pressure; 2: from radial flow; 3: from gravity

Parabolic pressure around halo center:

$$p_h(x)\equiv\rho_h(x)\sigma_r^2(x)=p_h(x=0)-\frac{\rho_h^2(x=0)v_{cir}^2}{2\bar{\rho}_h(a)c^2}x^2$$

Define a halo core size x_c :

$$p_h(x_c)=0 \quad\Rightarrow\quad x_c=\sqrt{\frac{2\bar{\rho}_h(a)}{\rho_h(0)}}\,\frac{c\sigma_r(0)}{v_{cir}}$$

Double power-law for halo density

Density profiles	Concentration c	Deformation parameter α_h	Deformation rate parameter γ_h	$\rho_h(r < r_s)$
Isothermal	3.5	1	0	r^{-2}
NFW	3.5	0.8329	1/2	r^{-1}
Einasto (α=0.2)	3.5	0.8371	2/3	r^{0}
			3/4	r^{1}

Density ρ_h ⟷ F(x) ⟷ Radial flow $u_h(x)$

$$\frac{\partial \ln \rho_h}{\partial \ln x} = \frac{\partial \ln F'(x)}{\partial \ln x} - 2 = \frac{\partial u_h / \partial x}{1 - u_h / x} - 2 \qquad \frac{\partial u_h(x)}{\partial x} = \frac{F(x) F''(x)}{F'^2(x)}$$

Double power-law:

$$\rho_h(r < r_s) \propto r^{(3\gamma_h - 2)/(1-\gamma_h)}$$

$$\rho_h(r > r_s) \propto r^{\frac{c(\alpha_h - 1)}{c - x_0} - 2}$$

$$\gamma_h = \partial u_h / \partial x \big|_{x=0}$$

$$\alpha_h = c \frac{F'(c)}{F(c)}$$

- Double power-law is a natural result due to radial flow in outer and inner regions
- Halo deformation parameter from mass cascade controls density in outer region
- Halo deformation parameter controls density in inner region
- The larger deformation rate at center, the larger logarithmic slope (baryonic feedback for core-cusp?)

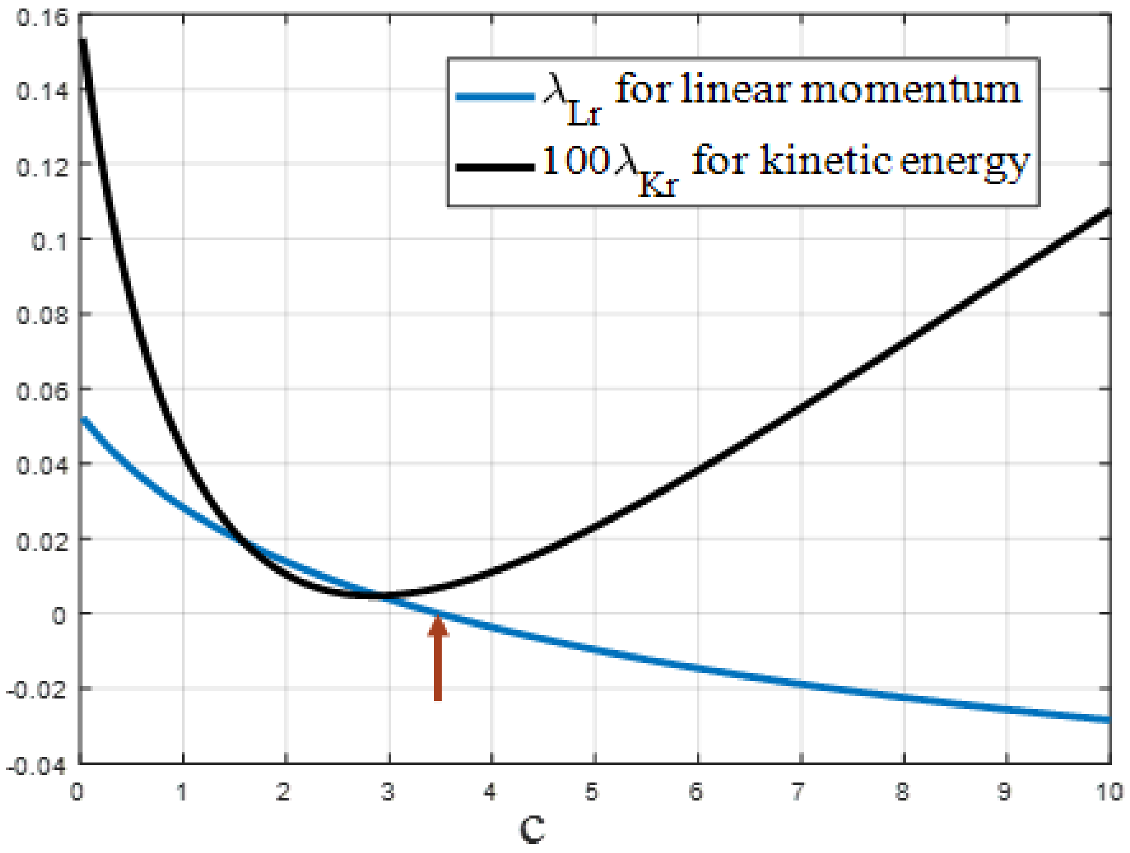
The limiting concentration c for large halos and radial momentum and kinetic energy

Vanishing radial Linear momentum (halos at turn-around):

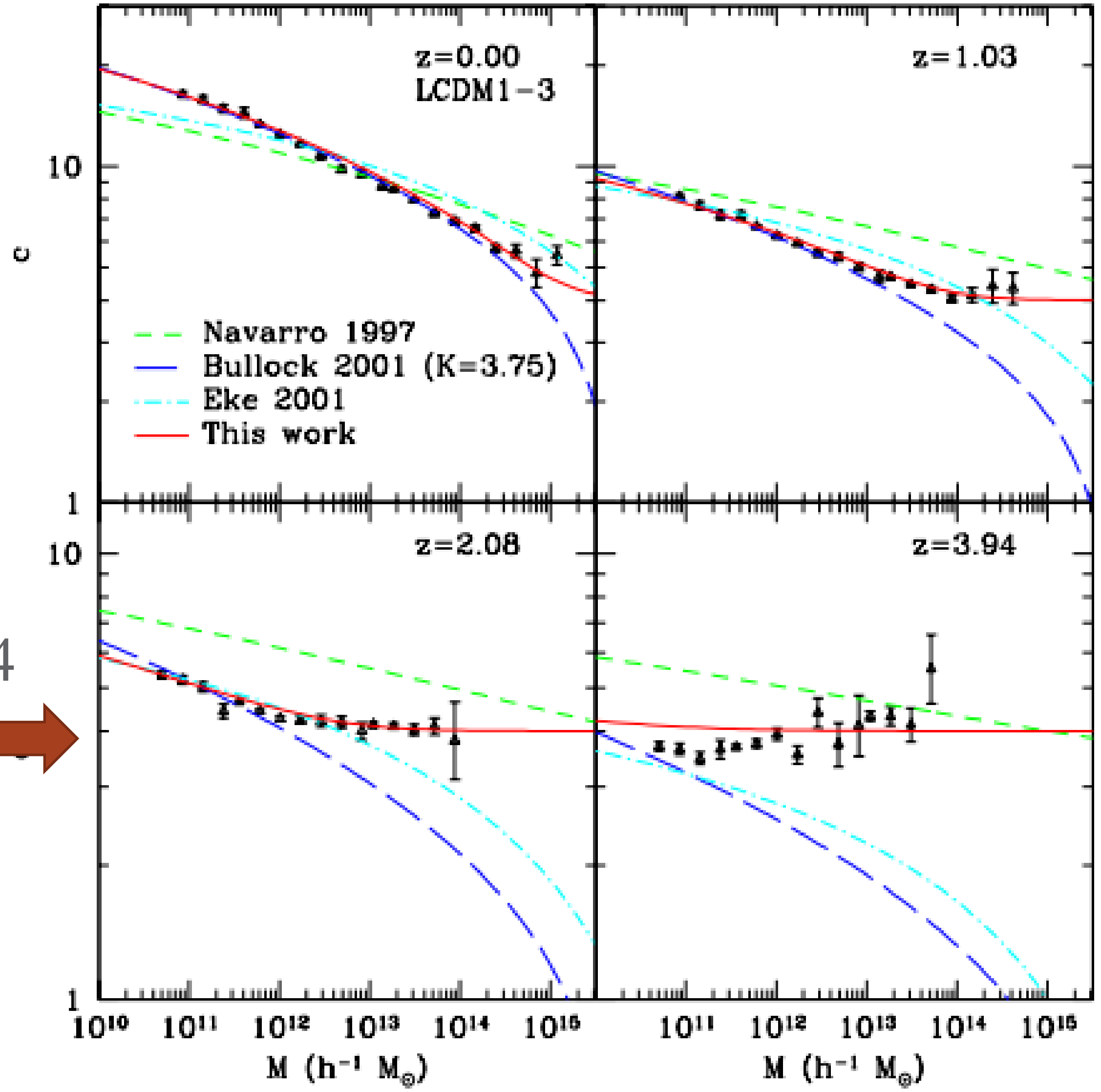
$$L_{hr}(a)=\int_0^{r_h} u_r(r,a)4\pi r^2\rho_h(r,a)dr=\frac{m_h v_{cir}}{2\pi cF(c)}\left(cF(c)-2\int_0^c F(x)dx\right)$$

$$cF(c)=2\int_0^c F(x)dx$$

Limiting concentration for large halos $c=3.5$ for NFW profile

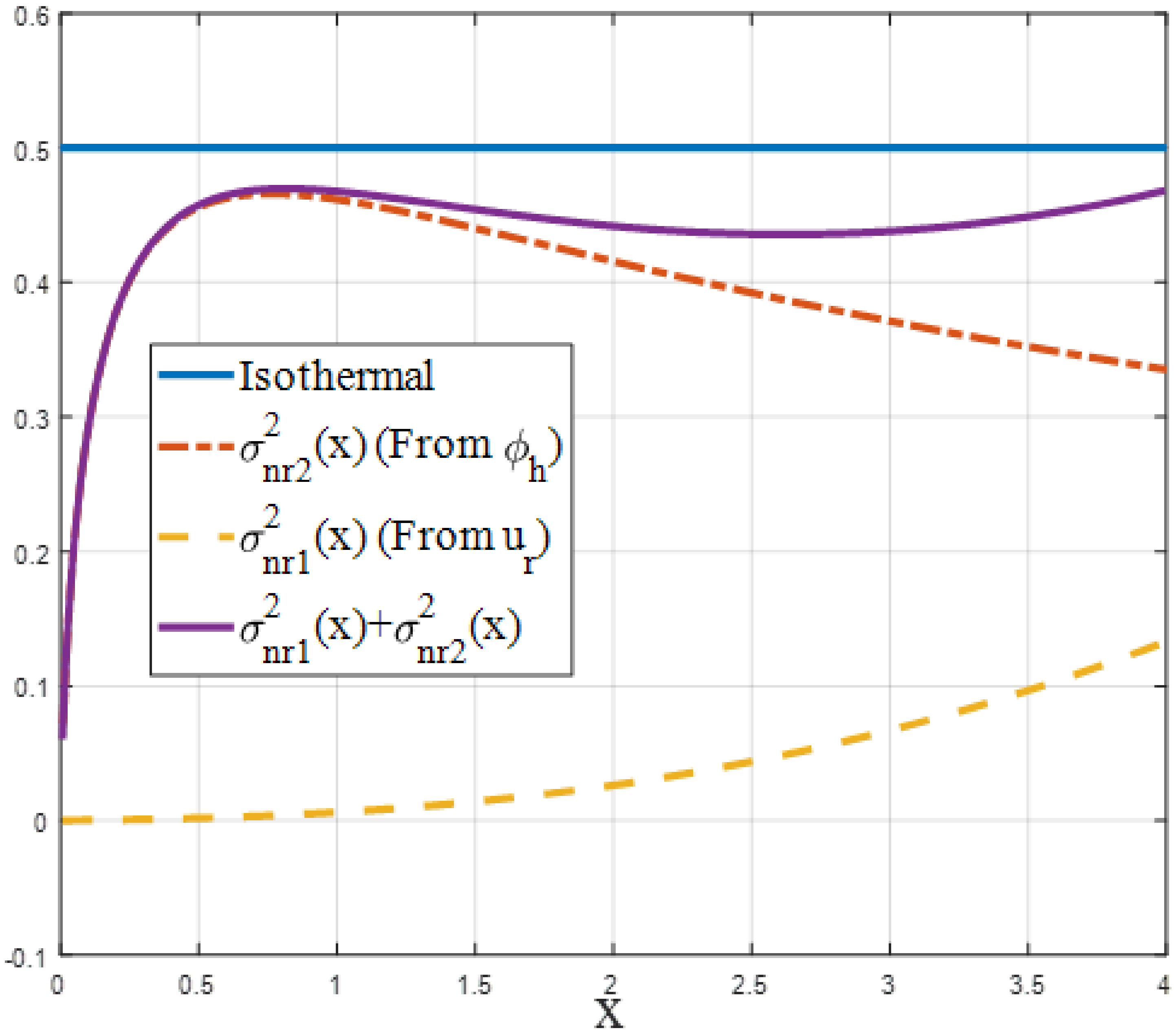


Limiting c~4 from simulation



Zhao etc.., 2009, Astrophys. J., 707, 354

Effect of radial flow on velocity dispersion



- Radial flow usually neglected for virialized halos;
- Effect of radial flow can be significant for halos in their early life before fully virialized (high peak height v);
- The radial flow tends to enhance the radial random motion and is only significant in the halo outer region.

Mass cascade induced halo surface energy

Standard virial theorem for static halos with a vanishing radial flow (K_σ is 1D kinetic energy):

$$6K_\sigma - n\Phi_h = 0$$ Potential exponent $n = -1$

Jeans' equation for isotropic growing halos with non-zero radial flow:

$$\frac{\partial(\rho_h u_r)}{\partial t} + \frac{1}{r^2}\frac{\partial(\rho_h r^2 u_r^2)}{\partial r} + \frac{\partial(\rho_h \sigma_r^2)}{\partial r} + \rho_h \frac{Gm_r(r,a)}{r^2} = 0$$

Integrating Jeans' Equation leads to a generalized virial theorem for growing halos with fast mass accretion:

mean flow

$$6K_\sigma + \Phi_h = I_h - 2K_u + S_u + S_\sigma$$

Rewrite to introduce effective exponent n_e:

$$6K_\sigma - n_e\Phi_h = 0 \text{ and } n_e = -1 + \frac{\cancel{I_h - 2K_u} + S_u + S_\sigma}{\Phi}$$

Halo surface energy:

$$S_{eh} = (S_u + S_\sigma), \quad n_e \approx -1 + \frac{S_{eh}}{\Phi} \approx -1.3 \neq -1$$

Halo surface tension:

$$S_{th} = S_{eh}/(2A_h)$$ Surface area: $A_h = 4\pi r_h^2$

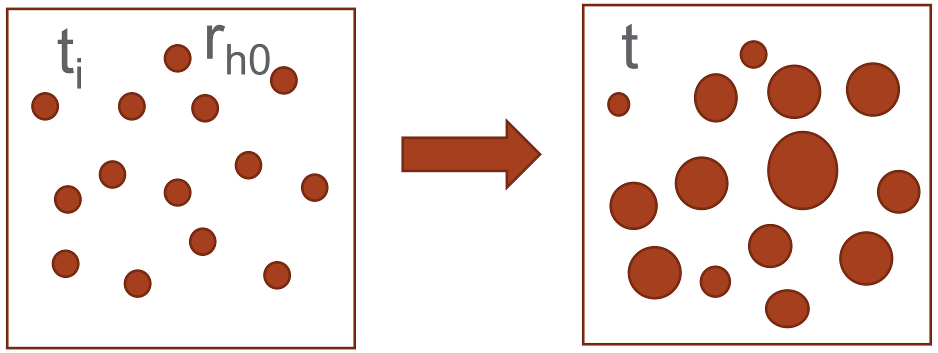
Young–Laplace equation relates the pressure jump across halo surface to halo radius or curvature;

$$\Delta P_h = \frac{2S_{th}}{r_h} = \frac{S_{eh}}{A_h r_h} \approx 0.1\bar{\rho}_h v_{cir}^2$$

$$S_{th} = \alpha_{st} G \rho_{sur}^2 r_h \propto r_h^{-1}$$ ➡ Halo surface mass density: $\rho_{sur} \sim r_h^{-1}$

Mass cascade (fast mass accretion) leads to finite halo surface energy, surface tension, surface mass density, and an effective potential exponent $n_e \sim -1.3$, [confirmed by N-body simulation].

Halo size evolution from theory of mass cascade



1D Random walk of halos in mass space:

$$\frac{\partial m_h(t)}{\partial t} = \frac{m_p \xi(t)}{\tau_g(m_h)} = \sqrt{2D_p(m_h)}\varsigma(t)$$

1D Random walk of halos in size space (**Geometric Brownian** motion):

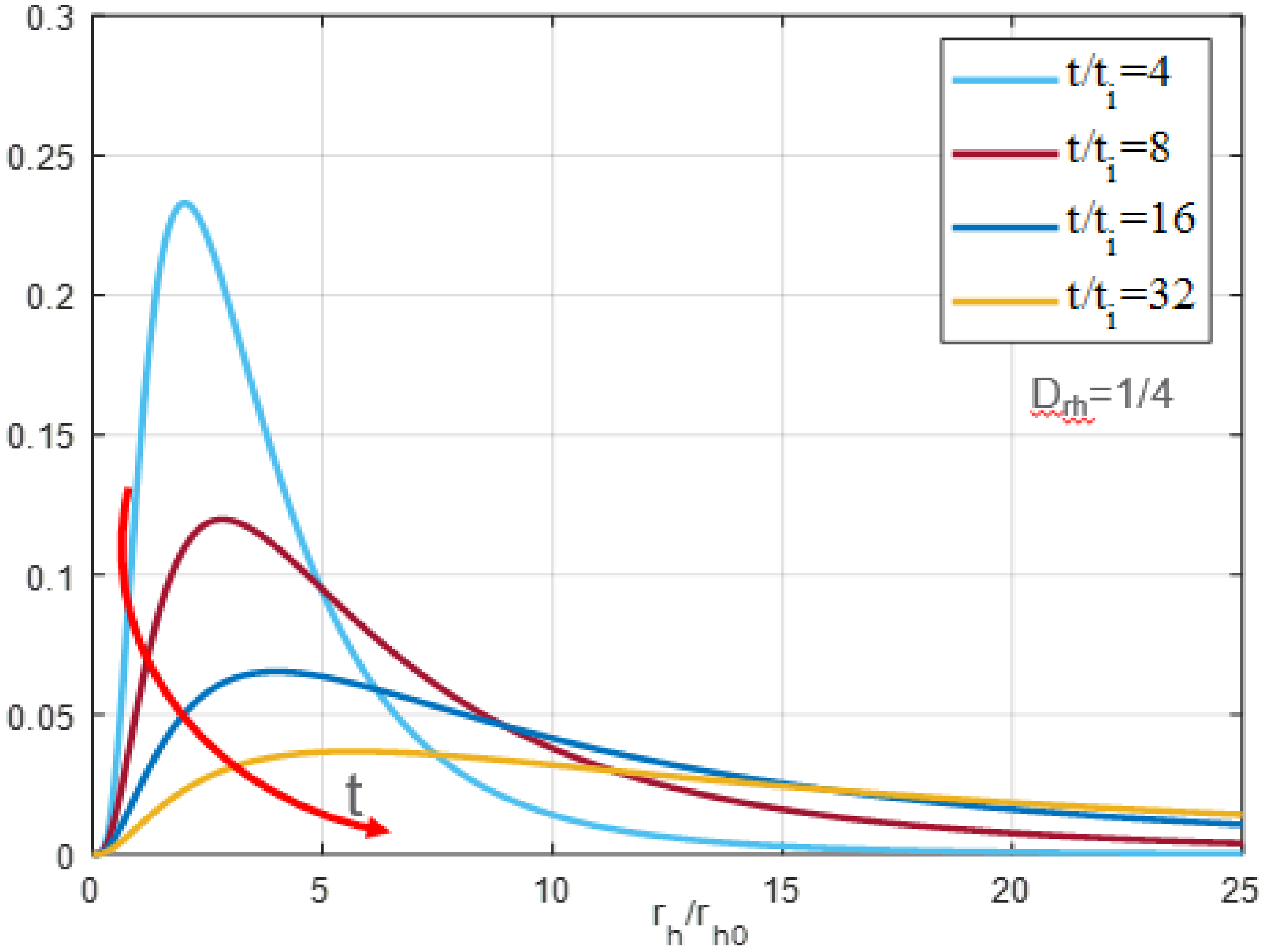
$$\frac{dr_h(t)}{dt} = \frac{3}{2}Hr_h(t) + Hr_h(t)\xi_{rh}(t)$$

Covariance:

$$\left\langle \xi_{rh}(t)\xi_{rh}(t')\right\rangle = 2D_{rh}\,\delta(t-t')/H$$

Solution leads to a **lognormal** probability distribution of halo size:

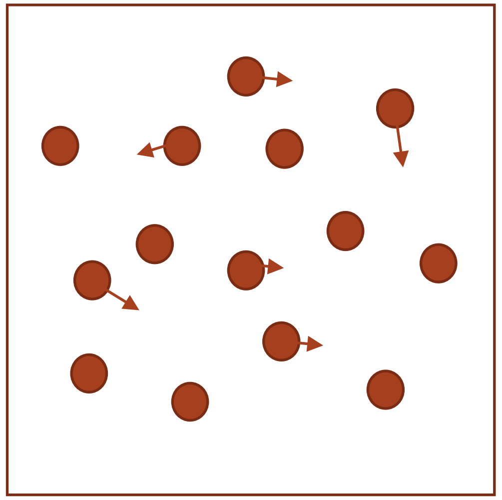
$$P_{rh}(r_h,t) = \frac{1}{r_h\sqrt{8\pi D_{rh}\ln(t/t_i)/3}}\exp\left\{-\frac{\left(\ln(r_h/r_{h0}) - (1-2D_{rh}/3)\ln(t/t_i)\right)^2}{8D_{rh}\ln(t/t_i)/3}\right\}$$



Particle distribution in halos: a review of Brownian motion

Quick review of standard Brownian motion in viscous liquid:

A spherical particle of radius a_B moving at a constant velocity u_h in a fluid of viscosity η_B subject to a force F_B. Local steady-state velocity u_h can be determined by the driving force F_B, i.e., the gradient of the osmotic pressure $\Pi_B = \rho_B k_B T$, which is a localized short-range force.



Current velocity from stokes law:

$$u_h = \frac{F_B}{6\pi\eta_B a_B} = -\frac{1}{6\pi\eta_B a_B}\cdot\frac{1}{\rho_B}\frac{\partial \Pi_B}{\partial x} = -\frac{\mu_B}{\rho_B}\frac{\partial(\rho_B k_B T)}{\partial x}$$

Osmotic velocity from diffusion flux:

$$u_h^* = D_B \frac{\partial \ln \rho_B}{\partial x}$$

A simple closure:

$$u_h = -u_h^*$$

The Einstein relation:

$$D_B = \mu_B k_B T$$

Stochastic equations for Brownian motion (forward and backward):

$$\frac{dr_t}{dt} = \left[u_h(x_t) + u_h^*(x_t)\right] + \sqrt{2D_B}\,\xi(t)$$

$$\frac{dr_t}{dt} = \left[u_h(x_t) - u_h^*(x_t)\right] + \sqrt{2D_B}\,\xi^*(t)$$

Fokker-Planck equations (forward and backward):

$$\frac{\partial P_r(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\left(u_h(x) + u_h^*(x)\right)P_r\right] + D_B\frac{\partial^2 P_r}{\partial x^2}$$

$$\frac{\partial P_r(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\left(u_h(x) - u_h^*(x)\right)P_r\right] - D_B\frac{\partial^2 P_r}{\partial x^2}$$

Diffusion equation for density distribution:

$$\frac{\partial P_r(x,t)}{\partial t} = D_B\frac{\partial^2 P_r}{\partial x^2}$$

$$u_h^* = -u_h = D_B\frac{\partial \ln P_r}{\partial x}$$

Particle distribution in halos: formulation

Brownian motion of particle in halos with stochastically (lognormal) growing size:

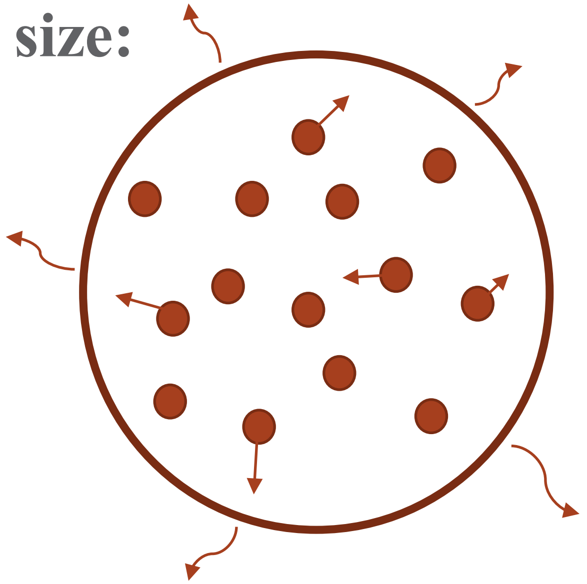
Stochastic equations for Brownian motion (forward and backward):

$$\frac{dr_t}{dt} = \underbrace{\frac{r_s(t)}{t}\left[u_h(x_t) + u_h^*(x_t)\right]}_{1} + \underbrace{\sigma(x_t)\, r_s(t)\, H \xi_{rh}(t)}_{2}$$

$$\frac{dr_t}{dt} = \frac{r_s(t)}{t}\left[u_h(x_t) - u_h^*(x_t)\right] + \sigma(x_t)\, r_s(t)\, H \xi_{rh}^*(t)$$

Radial flow   Osmotic flow

Multiplicative noise (dependent on r_t itself) due to random varying halo size!!



- Due to long-range interaction, $u_h \neq -u_h^*$
- Key is to find a simple closure to close equation! (an example in ref.)

Fokker-Planck equations (forward and backward):

$$\frac{\partial P_r(r,t)}{\partial t} = -\frac{r_s(t)}{t}\frac{\partial}{\partial r}\left[\left(u_h(x) + u_h^*(x)\right)P_r\right] + r_s^2(t) H D_{rh} \frac{\partial^2}{\partial r^2}\left(\sigma^2(x) P_r\right)$$

$$\frac{\partial P_r(r,t)}{\partial t} = -\frac{r_s(t)}{t}\frac{\partial}{\partial r}\left[\left(u_h(x) - u_h^*(x)\right)P_r\right] - r_s^2(t) H D_{rh} \frac{\partial^2}{\partial r^2}\left(\sigma^2(x) P_r\right)$$

$$\boxed{\begin{gathered} x\frac{\partial P_r}{\partial x} = \frac{\partial}{\partial x}\left[u_h(x) P_r\right] \\ u_h^*(x) = d_r \sigma^2(x) \frac{\partial}{\partial x} \ln\left[\sigma^2(x) P_r(x)\right] \end{gathered}}$$

Exact relation between current and osmotic velocities:

$$u_h^*(x) = \frac{d_r \sigma^2(x)}{x - u_h(x)}\frac{\partial u_h}{\partial x} + d_r \frac{\partial \sigma^2(x)}{\partial x}$$

With $\sigma(x_t) \sim x_t$ expected

Particle distribution in halos: halo density profile

To derive halo density, adopting a simple model of osmotic velocity :

$$\boxed{u_h^*(x) = \gamma_r x - \beta_r x^{1+\alpha_r}}$$

Two-parameter particle distribution function:

$$P_r(x) = \frac{b_r^{a_r}}{\Gamma(a_r)(a_r - b_r)} \exp\left(-b_r x^{\frac{1}{a_r - b_r}}\right) x^{\frac{b_r}{a_r - b_r}}$$

Three-parameter halo density profile:

$$\rho_h(x) = \frac{m_h P_r(x)}{4\pi r_s^3 x^2} = \rho_s e^{b_r} \exp\left(-b_r x^{\frac{1}{a_r - b_r}}\right) x^{\frac{3b_r - 2a_r}{a_r - b_r}}$$

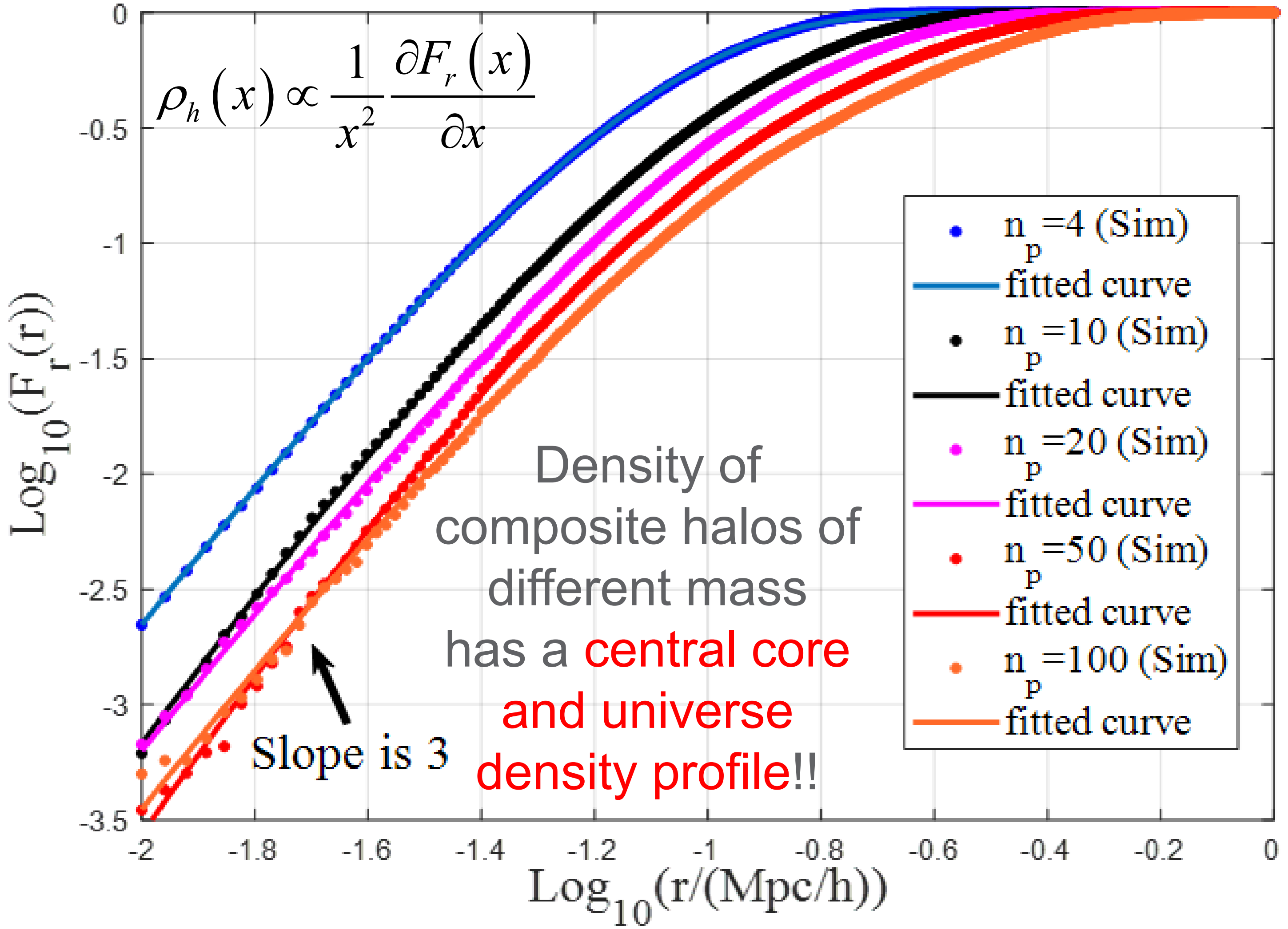
$$a_r / b_r = 3/2$$

Two-parameter Einasto:

$$\rho_h(r) = \rho_s e^{2/\alpha} \exp\left[-\frac{2}{\alpha} x^{\alpha}\right] = \rho_s e^{2/\alpha} \exp\left[-\frac{2}{\alpha}\left(\frac{r}{r_s}\right)^{\alpha}\right]$$

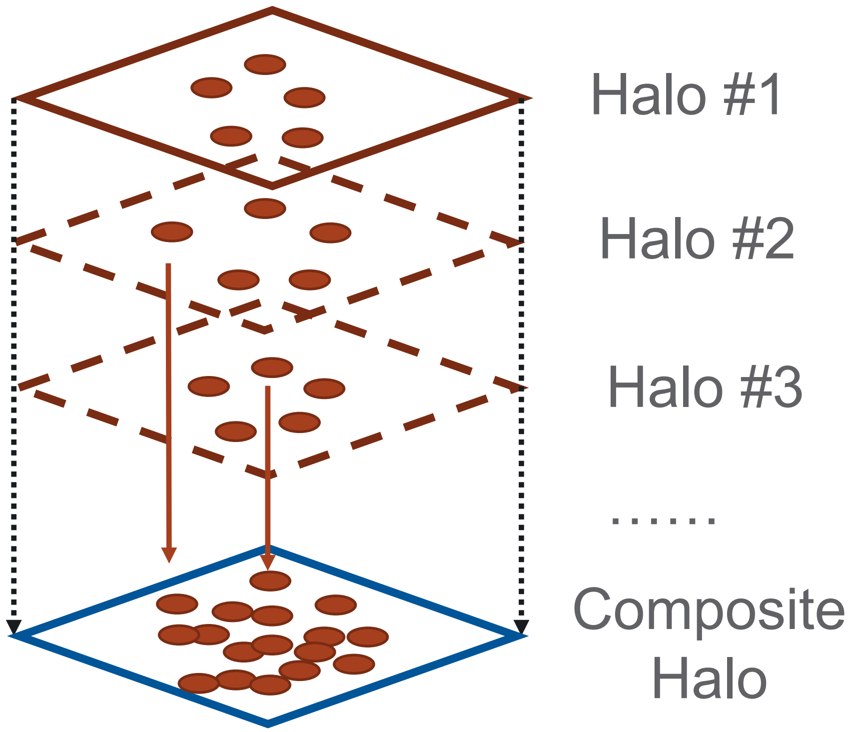
Two-parameter cumulative distribution function:

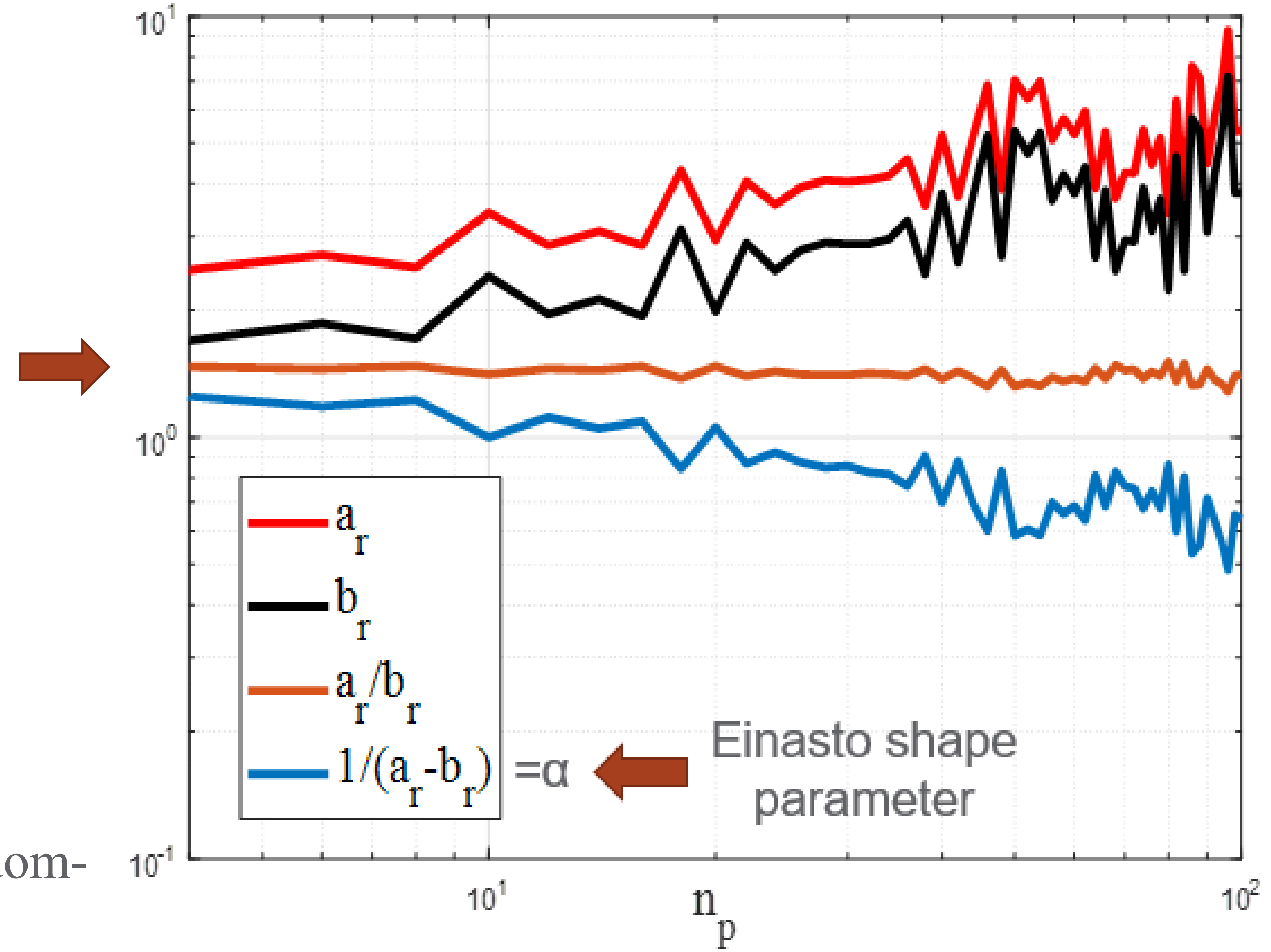
$$F_r\left(x = \frac{r}{r_s}\right) = \int_0^x P_r(y)\,dy = \frac{m_r}{m_h} = 1 - \frac{\Gamma\left(a_r, b_r x^{1/(a_r - b_r)}\right)}{\Gamma(a_r)} = \frac{\gamma\left(a_r, b_r x^{1/(a_r - b_r)}\right)}{\Gamma(a_r)}$$



Particle distribution in halos and halo density profile

Constructing composite halo for a halo group including all halos of the same mass:





- Composite halo reflects complete statistics of particle distribution resulting from particle random-walk in dynamic halos;
- All composite halos have a central core (no cusp)
- The density profile of composition halo (α=[1.2 0.7]) can be different from individual halo (α≈0.2);

- Fitted a_r/b_r=3/2 for all size of halo groups (implies an Einasto profile)

Equation of state for relative pressure and density

$$\rho_h(r) = \rho_s e^{2/\alpha} \exp\left(-b_r x^{\frac{2}{b_r}}\right) \quad \text{with} \quad b_r = 2/\alpha$$

For small x (halo center)

$$\rho_h(x) \approx \rho_h(0)\left(1 - b_r x^{\frac{2}{b_r}}\right)$$

Parabolic pressure at halo center:

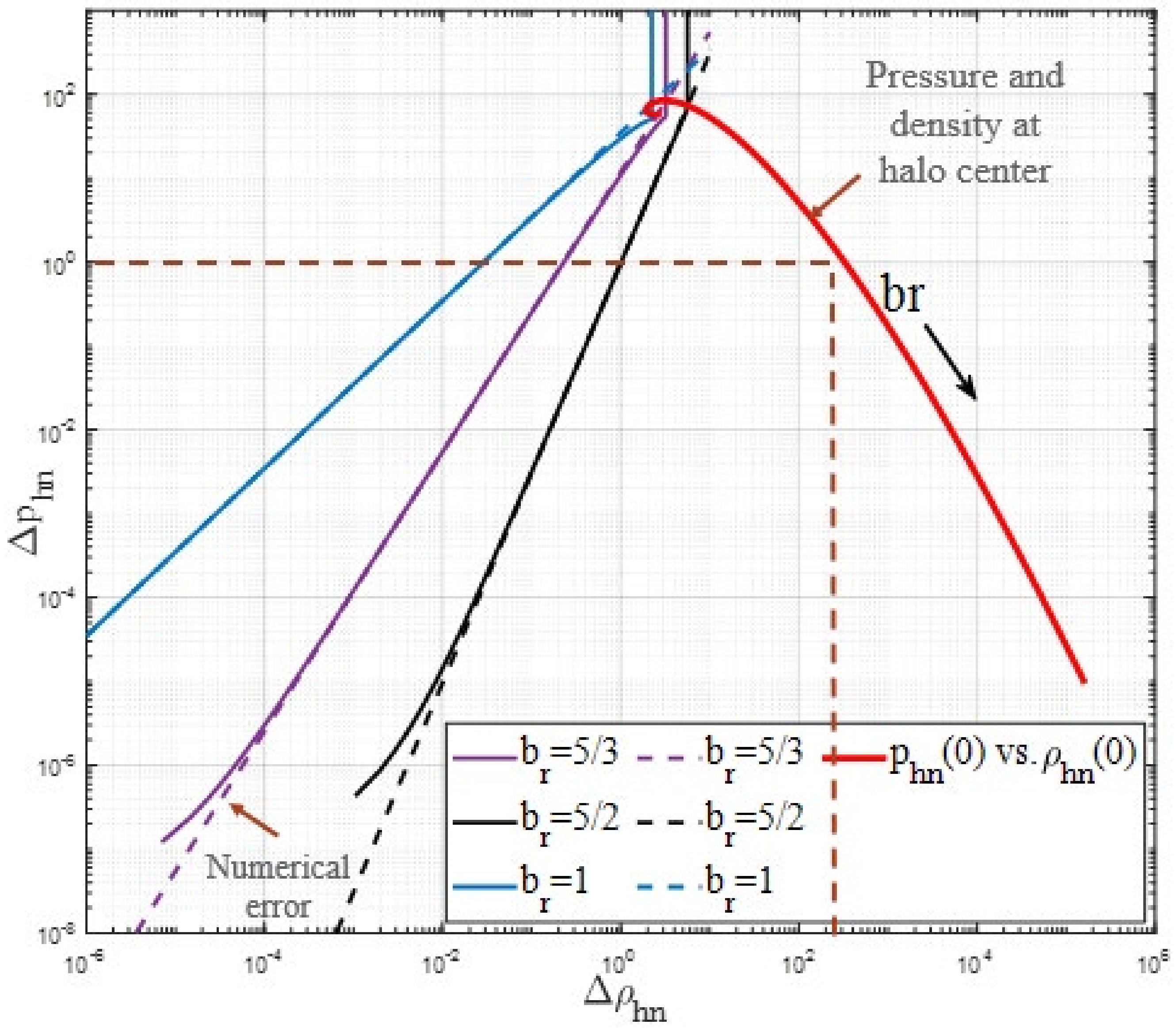
$$p_h(x) = \rho_h(x)\sigma_r^2(x) = p_h(x=0) - \frac{1}{2}\frac{\rho_h^2(0)v_{cir}^2}{\bar{\rho}_h c^2}x^2$$

Cancel x in both Equations:

$$\left[p_h(0) - p_h(x)\right] = \frac{\left[\rho_h(0)\right]^{2-b_r} v_{cir}^2}{2(b_r)^{b_r}\bar{\rho}_h c^2}\left[\rho_h(0) - \rho_h(x)\right]^{b_r}$$

Equation of state (EoS) for relative pressure and relative density (relative to the center of halo):

$$\Delta p_h = K_s\left(\Delta\rho_h\right)^{b_r}$$



- EoS is good for entire range of relative P and ρ
- Why? might because of halo grows from center

Summary and key words

Radial flow & scale radius	Halo surface energy/tension	Current velocity	Mean flow& random motion
Deformation parameter α_h	Deformation rate parameter γ_h	Osmotic velocity	Limiting concentration
Angle of incidence	Random walk	Fokker-Planck	Equation of state

- Mass cascade induced nonzero radial flow (outwards and inwards).
- Self-similar solution to relate halo density profile with radial flow.
- Radial flow leads to an extra length scale (the scale radius r_s).
- Limiting halo concentration c=3.5 for fast growing halos at their early stage, with a Hubble flow at halo center leading to a central core.
- Composite halos from N-body simulation always have a central core.
- Radial flow enhances velocity dispersion in outer region.
- Radial flow leads to a nonzero halo surface energy/tension.
- Random walk of halo size is a geometric Brownian process with log-normal distribution
- Random walk of particles in halo with varying size leads to analytical particle probability distribution (i.e. the halo density profile).
- Equation of state for relative pressure and relative density (relative to halo center)

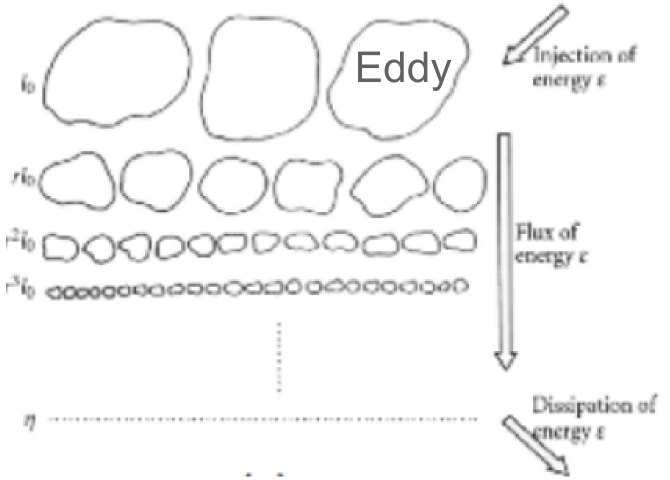
Energy cascade in dark matter flow

Xu Z., 2021, arXiv:2110.13885v1 [astro-ph.GA]
https://doi.org/10.48550/arXiv.2110.13885

Introduction

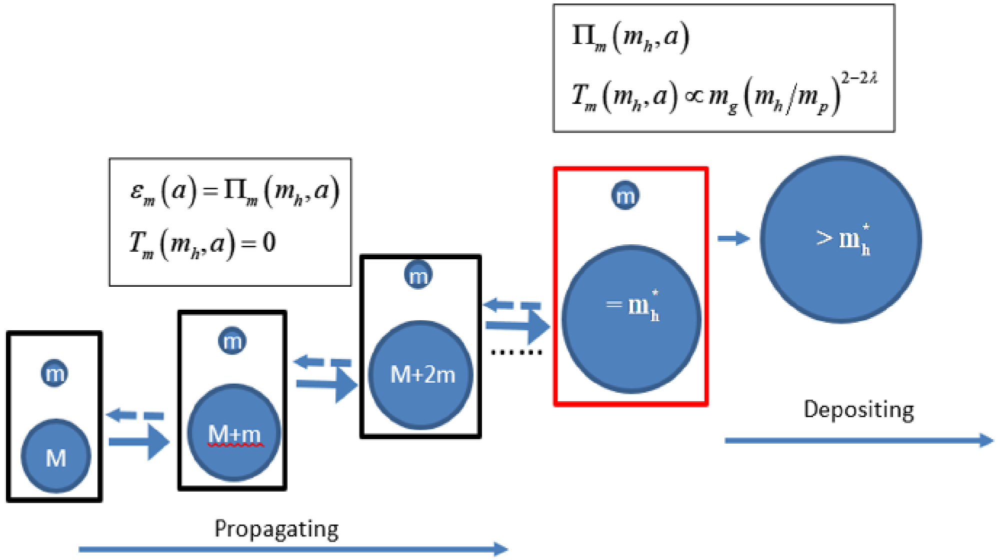
Review: In hydrodynamic turbulence, "Energy cascade" involves the energy transfer from large eddies to small eddies with a scale-independent rate of energy cascade (direct cascade). No mass cascade!
Vortex stretching is a major mechanism for energy cascade in turbulence.

"Big whorls have little whorls, That feed on their velocity;
And little whorls have lesser whorls, And so on to viscosity."



"Eddy" is not a well-defined object in turbulence literature. However, "halo" are well-defined dynamically growing and rotating objects with nonuniform density, whose abundance and internal structure have been extensively studied over several decades.

"Little halos have big halos, That feed on their mass;
And big halos have greater halos, And so on to growth"



- Goal 1: Identify and formulate kinetic/potential energy cascade
- Goal 2: Identify a constant scale-independent rate of energy cascade
- Goal 3: Explore the effect of halo shape on energy cascade

Decomposition of kinetic energy

Decompose particle velocity into halo velocity and velocity fluctuation ("Reynolds decomposition")

$$\mathbf{v}_p = \mathbf{v}_h + \mathbf{v}_p'$$

Similarly, decompose velocity dispersion into halo velocity dispersion and halo virial dispersion

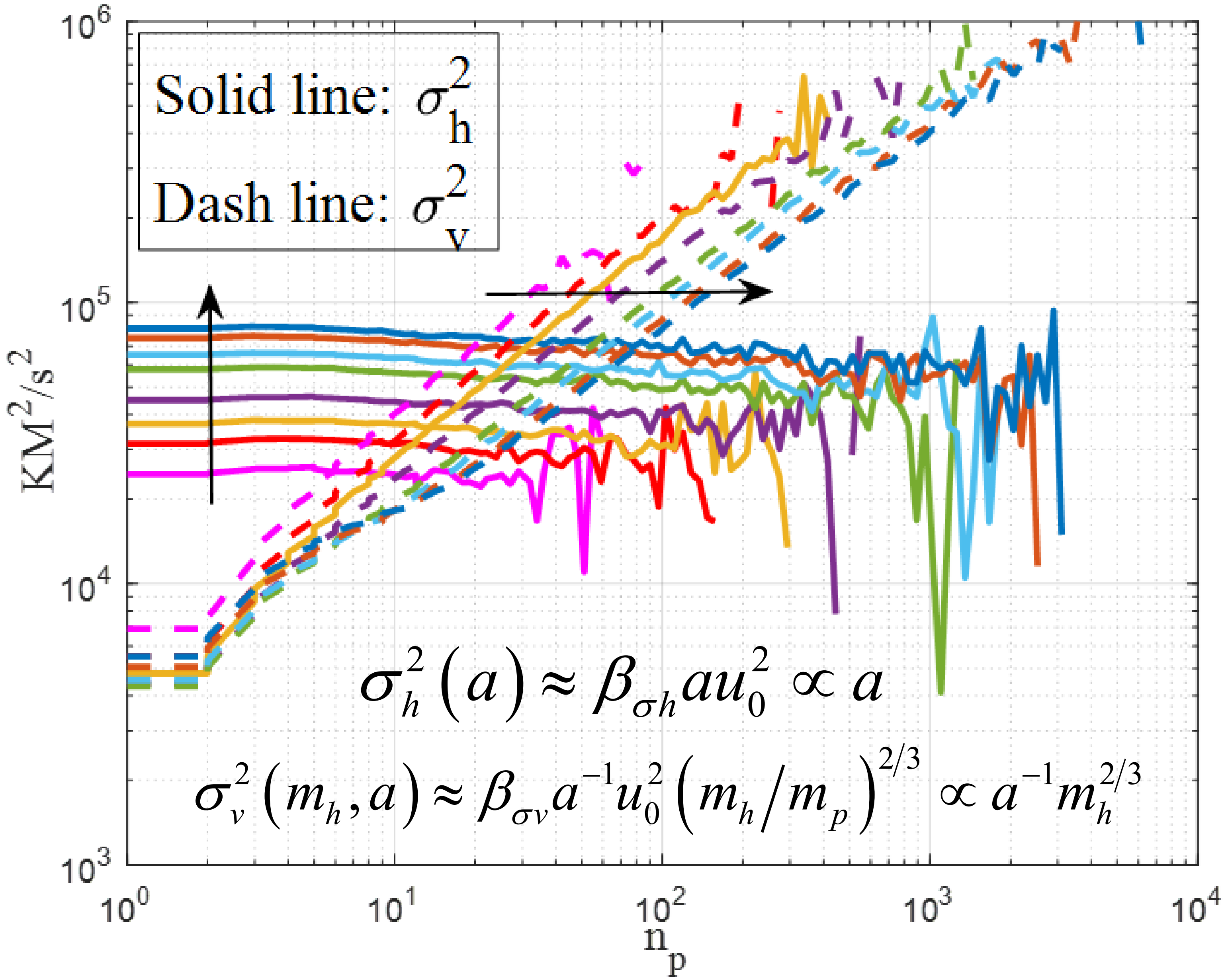
$$\sigma^2 = \sigma_h^2 + \sigma_v^2$$

Halo group temperature | Halo temperature

$$\sigma_h^2 = \mathrm{var}\left(\mathbf{v_h}\right)$$

Halo group temperature is independent of halo size

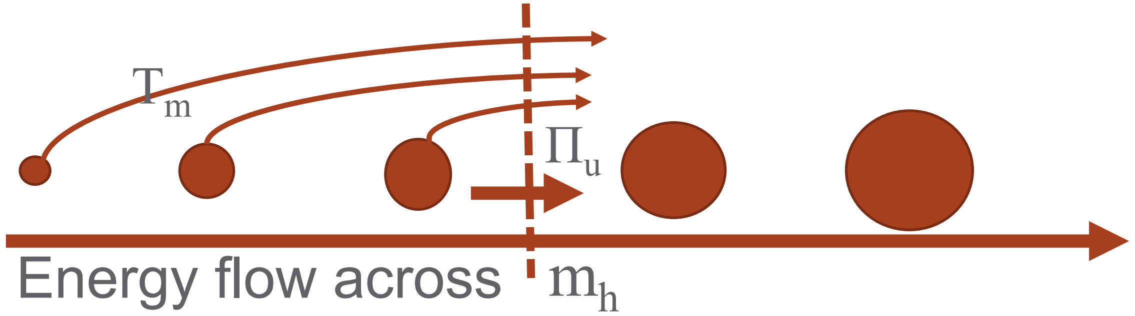
$$\sigma_v^2 = \mathrm{var}\left(\mathbf{v}_p'\right) \propto m_h^{2/3}$$



Variation with halo size for redshifts z = 0, 0.1, 0.3, 0.5, 1.0, 1.5, 2.0, and 3.0

(Kinetic) energy flux functions

Mass flux function: total mass flux from all halos below m_h



Mass transfer function: rate of mass transfer for halos of mass m_h

$$\Pi_m(m_h,a) = -\frac{\partial}{\partial t}\left[M_h(a)\int_{m_h}^{\infty} f_M(m,a)\,dm\right] = \int_0^{m_h} T_m(m,a)\,dm$$

$$T_m(m_h,a) = \frac{\partial \Pi_m(m_h,a)}{\partial m_h} = \frac{\partial m_g(m_h,a)}{m_p \partial t}$$

Energy flux function for halo kinetic energy σ_h^2:

$$\Pi_{kh}(m_h,a) = -\int_{m_h}^{\infty} T_m(m,a)\sigma_h^2(m,a)\,dm \approx \Pi_m(m_h,a)\left\langle \sigma_h^2 \right\rangle$$

Energy flux function for virial kinetic energy σ_v^2:

$$\Pi_{kv}(m_h,a) = -\int_{m_h}^{\infty} T_m(m,a)\sigma_v^2(m,a)\,dm \neq \Pi_m(m_h,a)\left\langle \sigma_v^2 \right\rangle$$

Mean (specific) halo kinetic energy:

$$\left\langle \sigma_h^2 \right\rangle = \int_0^{\infty} f_M(m_h,m_h^*)\sigma_h^2(m_h,a)\,dm_h \propto a$$

Mean (specific) virial kinetic energy:

$$\left\langle \sigma_v^2 \right\rangle = \int_0^{\infty} f_M(m_h,m_h^*)\sigma_v^2(m_h,a)\,dm_h \propto a$$

Equipartition requires: $\left\langle \sigma_h^2 \right\rangle \approx \left\langle \sigma_v^2 \right\rangle = \frac{1}{2}\sigma^2$

Total mass of all halos: $M_h(a)$

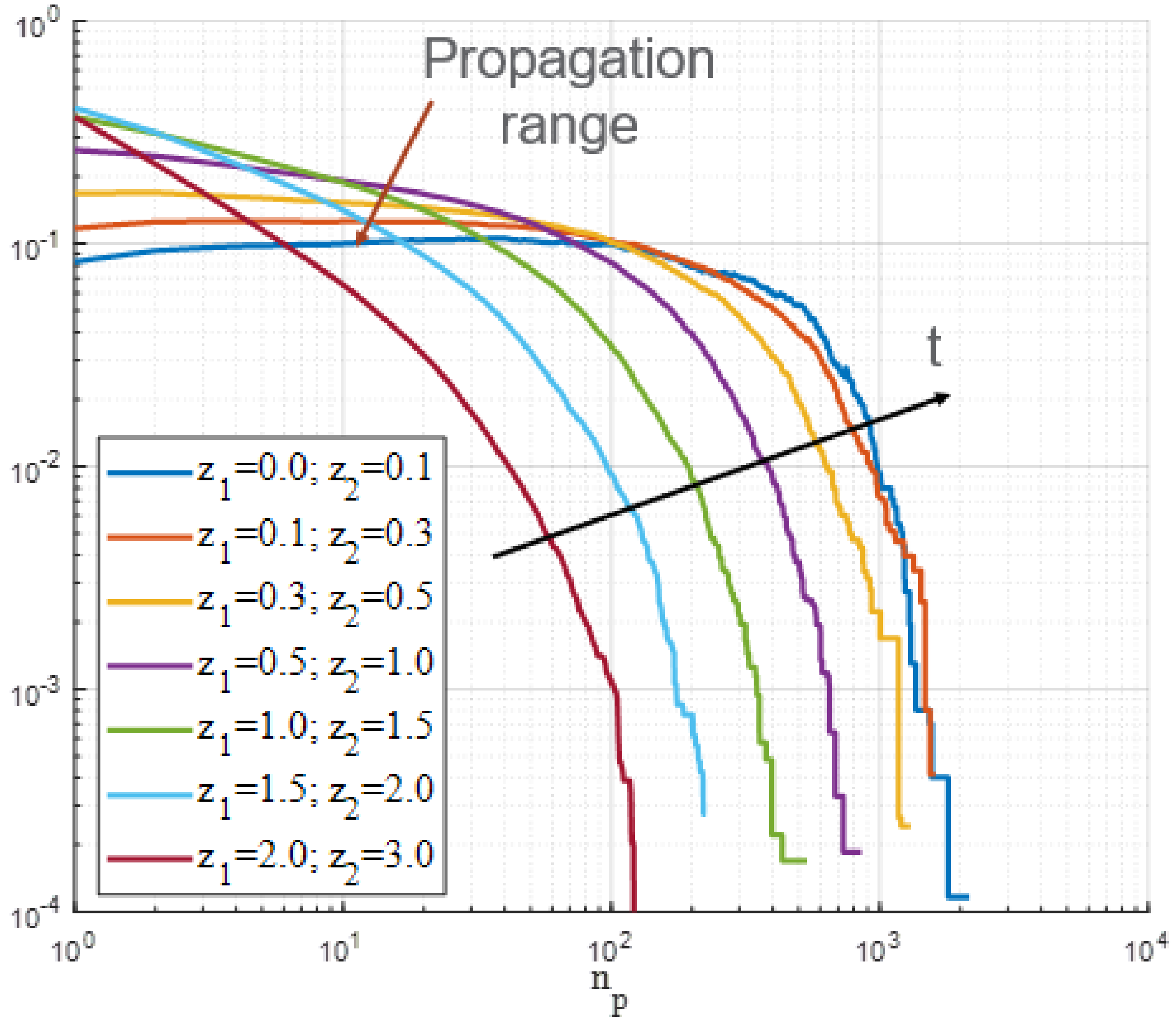
Halo mass: m_h

Halo mass function: $f_M(m,a)$

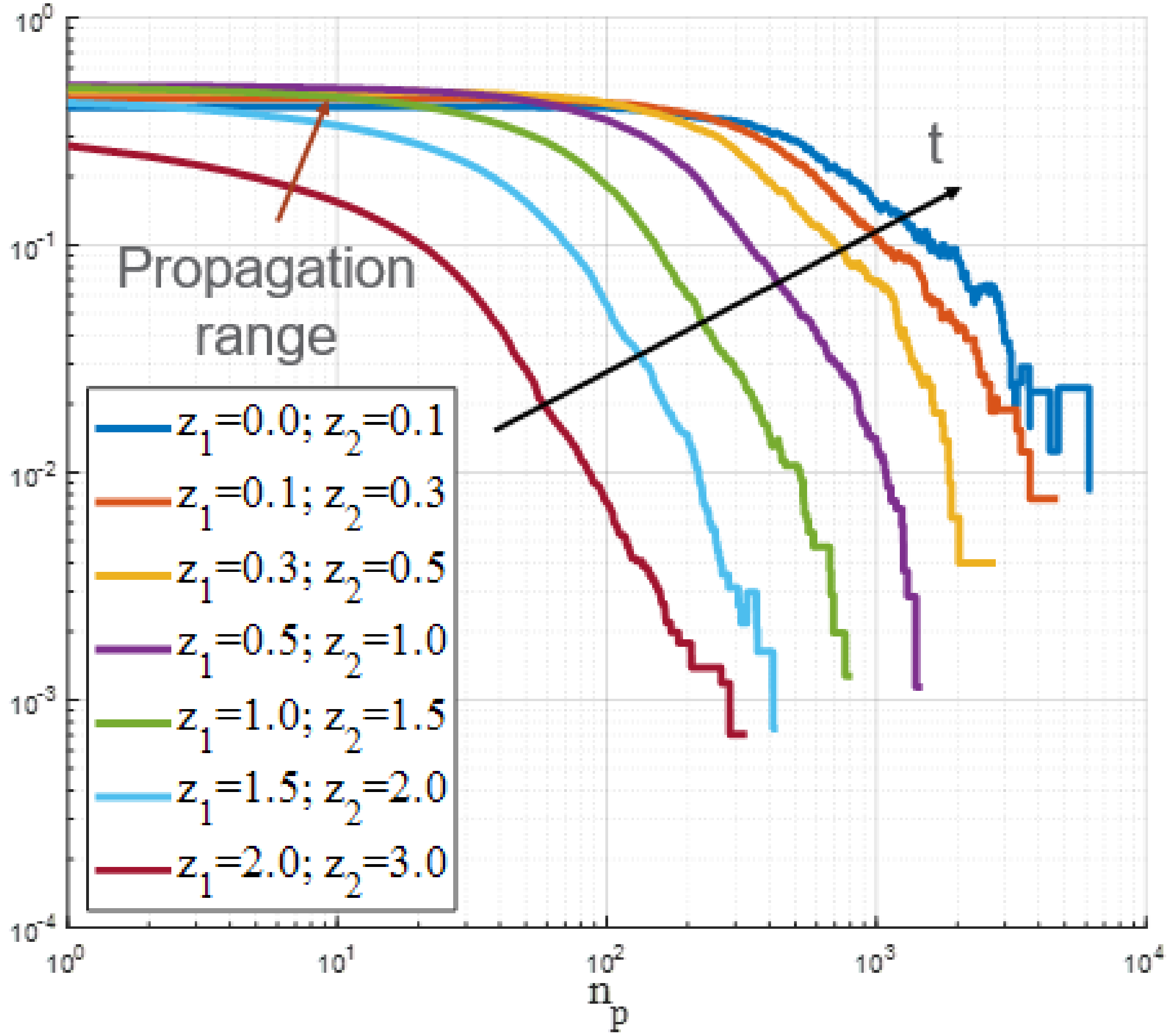
Dispersion of all particles: u^2

- **Direct** energy cascade in hydrodynamic turbulence through the change of vortex shape
- In dark matter flow, **inverse** energy cascade is facilitated by the inverse mass cascade through mass transfer function T_m

(Kinetic) energy flux functions π_{kh} and π_{kv}



The variation of energy flux function π_{kv} with the size of halo groups.



The variation of energy flux function π_{kh} with the size of halo groups.

(Potential) energy flux functions

Decompose particle potential into inter-halo potential (due to interaction with particles from other halos) and intra-halo potential (due to interaction with particles in the same halo):

$$\phi = \phi_h + \phi_v$$

Inter-halo potential | Intra-halo potential

Inter-halo potential is relatively independent of halo size

The virial ratios:

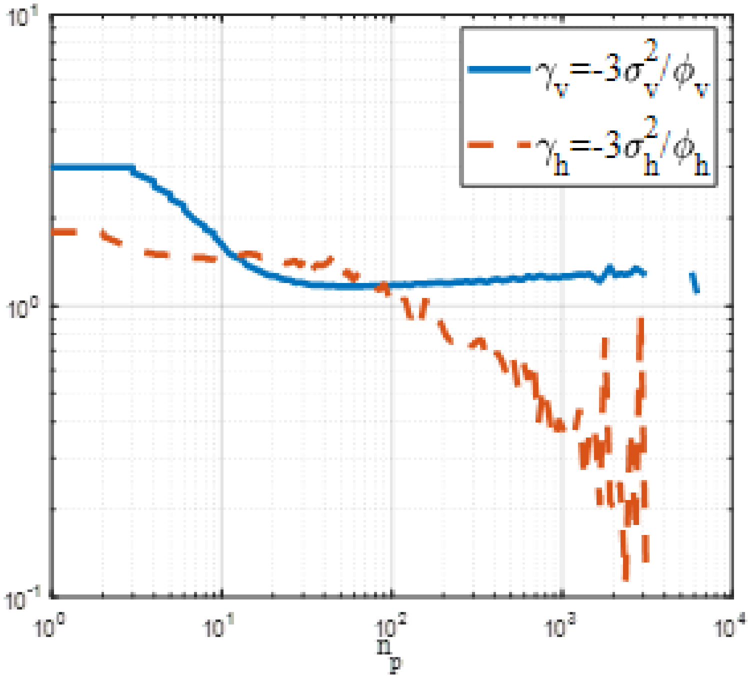
Intra-halo: $\gamma_v = -3\sigma_v^2 / \phi_v$

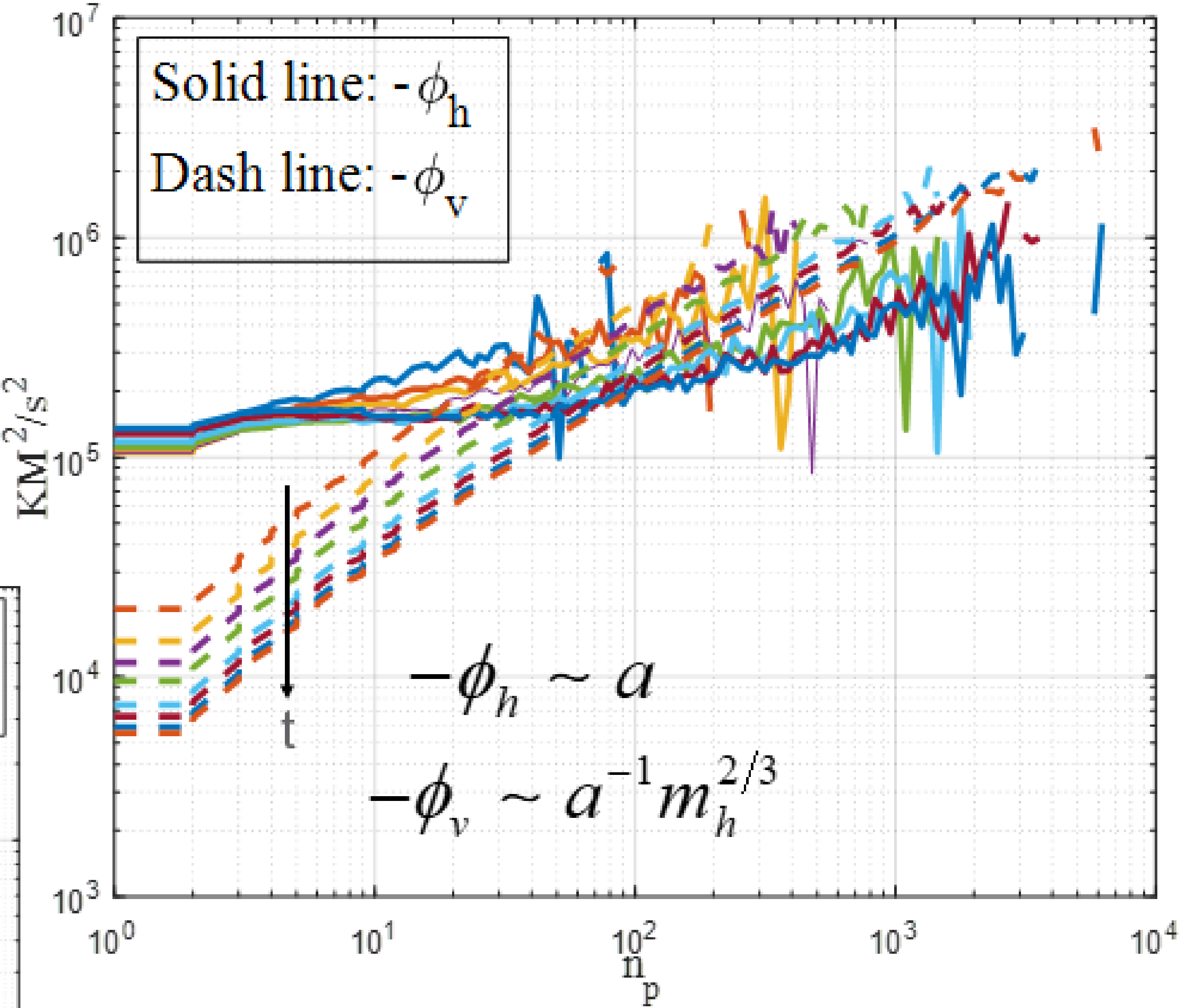
Inter-halo: $\gamma_h = -3\sigma_h^2 / \phi_h$

For large halos: $\gamma_v \approx 1.3$

due to halo surface energy

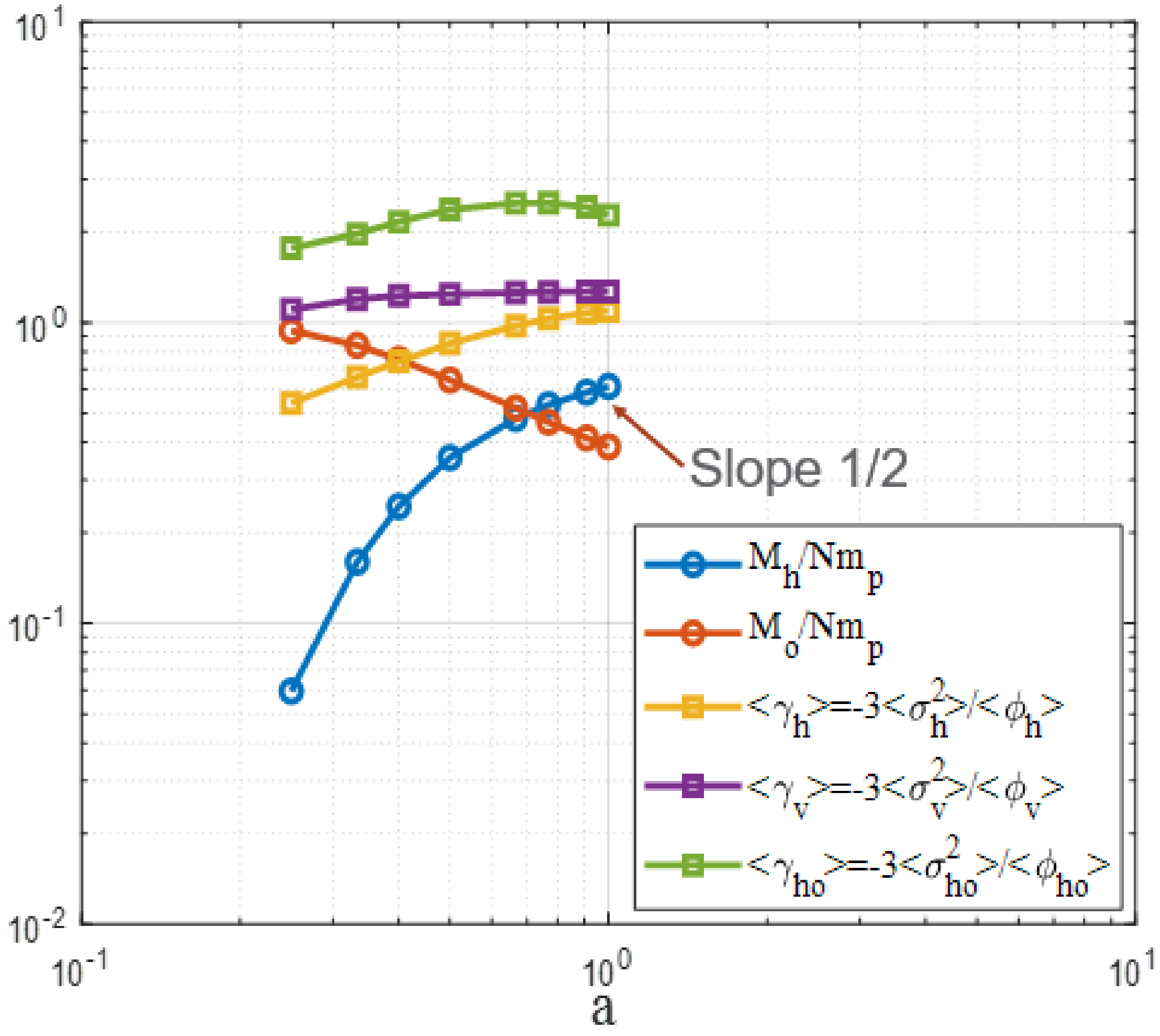
Direct cascade for potential energy from large to small





Variation with halo size for redshifts z = 0, 0.1, 0.3, 0.5, 1.0, 1.5, 2.0, and 3.0

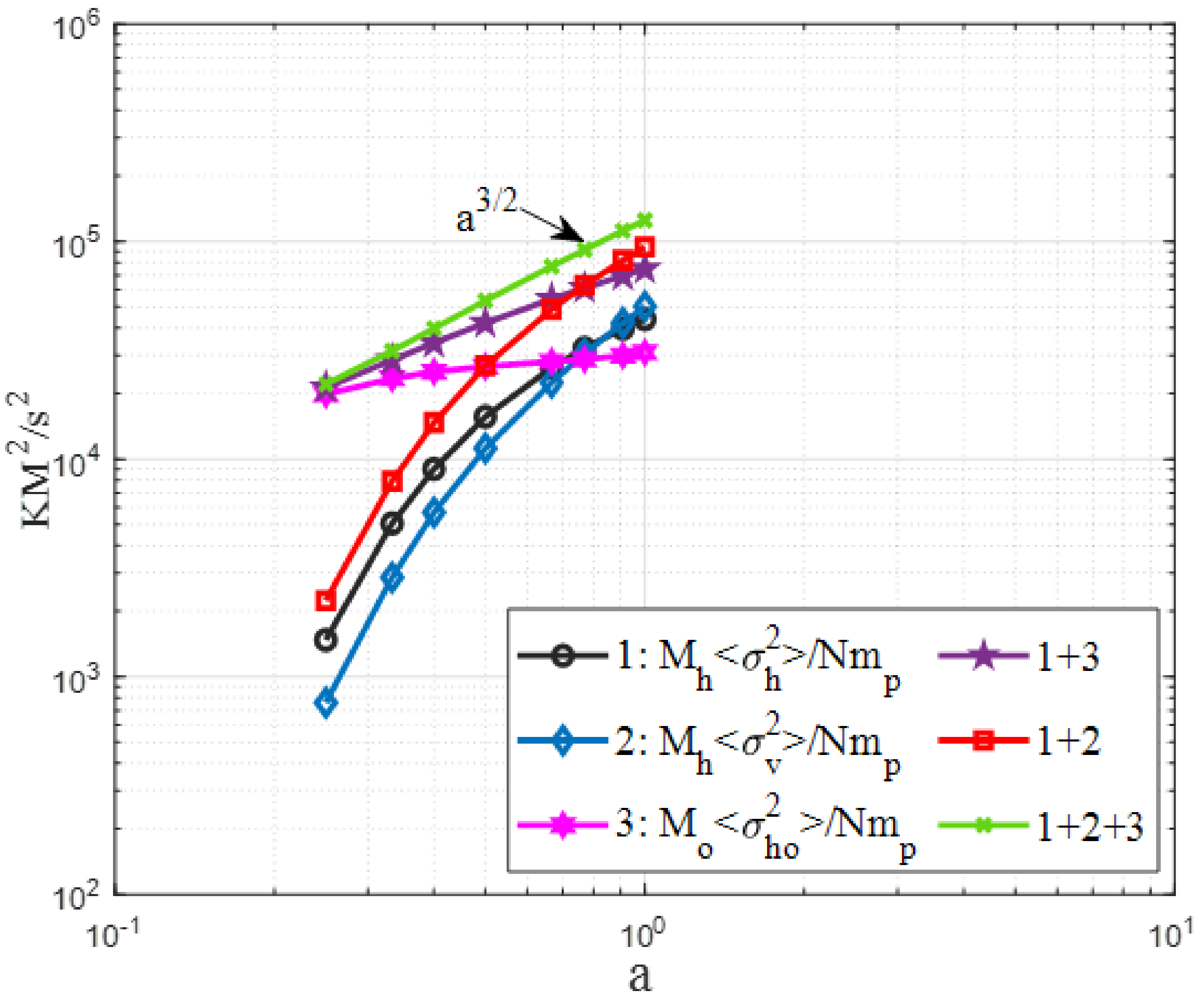
Redshift evolution of halo mass and virial ratio



The variation of total halo mass M_h, out-of-halo mass M_o and virial ratios with scale factor *a*.

- Mass flux from out-of-halo to halos sustains the total halo mass growing as $M_h(a) \sim a^{1/2}$, as predicted from mass cascade.
- ~ 60% of total mass are in halos and ~40% in out-of-halo (single merges)
- For the motion of halos, virial ratio (yellow) takes longer time to reach equilibrium due to weak gravity between halos.
- For motion in halos, virial equilibrium is established much faster with virial ratio ≈1.3 (yellow).
- Virial ratio≈2 (green) for out-of-halo particles(single mergers). The out-of-halo sub-system is energy conserved (no virilization), i.e. KE+PE = 0.

Redshift evolution of kinetic energies



Variation of three kinetic energies for halo and out-of-halo particles with scale factor *a*

- Total total kinetic energy of entire N-body system (green line: 1+2+3) grows ∝ t.
- Total kinetic energy in out-of-halo sub-system (magenta: 3) is time-invariant.
- The total kinetic energy of halo sub-system (red: 1+2) becomes dominant over out-of-halo sub-system grows ∝ t.

$$\left\langle \sigma_h^2 \right\rangle \approx \left\langle \sigma_v^2 \right\rangle = \frac{1}{2}\sigma^2$$

- A cross-over can be found at around a=0.5.
- A constant and scale-independent rate of energy cascade can be identified:

$$\varepsilon_u = -\frac{3}{2}\frac{u^2}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} = -\frac{9}{4}H_0 u_0^2 \approx -4.6\times10^{-7}\,\frac{m^2}{s^3}$$

Rate of mass and kinetic energy cascade

The rate of mass cascade:

$$\varepsilon_m = \Pi_m\left(m_h \to 0, a\right) = -\frac{1}{2}M_h\left(a\right)H \propto a^{-1}$$

The rate of cascade of halo kinetic energy σ_h^2:

$$\varepsilon_{kh} = \Pi_{kh}\left(m_h \to 0, a\right) = \varepsilon_m\left\langle\sigma_h^2\right\rangle = -\frac{1}{2}M_h\left(a\right)H\left\langle\sigma_h^2\right\rangle \propto a^0$$

The rate of cascade of virial kinetic energy σ_v^2 :

$$\varepsilon_{kv} = \Pi_{kv}\left(m_h \to 0, a\right) = -\frac{5}{2}M_h\left(a\right)H\left\langle\sigma_v^2\right\rangle \propto a^0$$

The rate of cascade of total kinetic energy:

$$\varepsilon_u = \frac{3}{2}\frac{\left(\varepsilon_{kh} + \varepsilon_{kv}\right)}{M_h\left(a\right)}\frac{M_h\left(a\right)}{M_{tot}} = -\frac{9}{4}H\sigma^2\frac{M_h\left(a\right)}{M_{tot}} \approx \frac{3}{2}\frac{u^2}{t}$$

$$\left\langle\sigma_h^2\right\rangle \approx \left\langle\sigma_v^2\right\rangle = \frac{1}{2}\left\langle\sigma^2\right\rangle \propto a^1$$

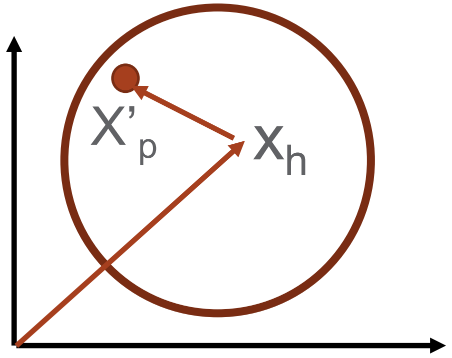
- Total mass in N-body system: M_{tot}
- Total halo mass in all halos: M_h
- Total mass in out-of-halo: M_{oh}
- One-dimensional velocity dispersion in N-body system: u^2
- One-dimensional velocity dispersion in all halos: $<\sigma^2>$
- One-dimensional halo velocity dispersion in all halos: $<\sigma_h^2>$
- One-dimensional halo virial dispersion in all halos: $<\sigma_v^2>$
- Hubble parameter: H
- Physical time: t

Inverse cascade of halo radial and rotational kinetic energy

Decompose halo particle position and velocity

$$\mathbf{x}_p = \mathbf{x}_h + \mathbf{x}'_p$$

$$\mathbf{u}_p = \mathbf{u}_h + \mathbf{u}'_p$$



Define the mean square radius r_g:

$$r_g = \sqrt{\sum_{p=1}^{n_p} \left|\mathbf{x}'_p\right|^2 \Big/ n_p}$$

Angular momentum:

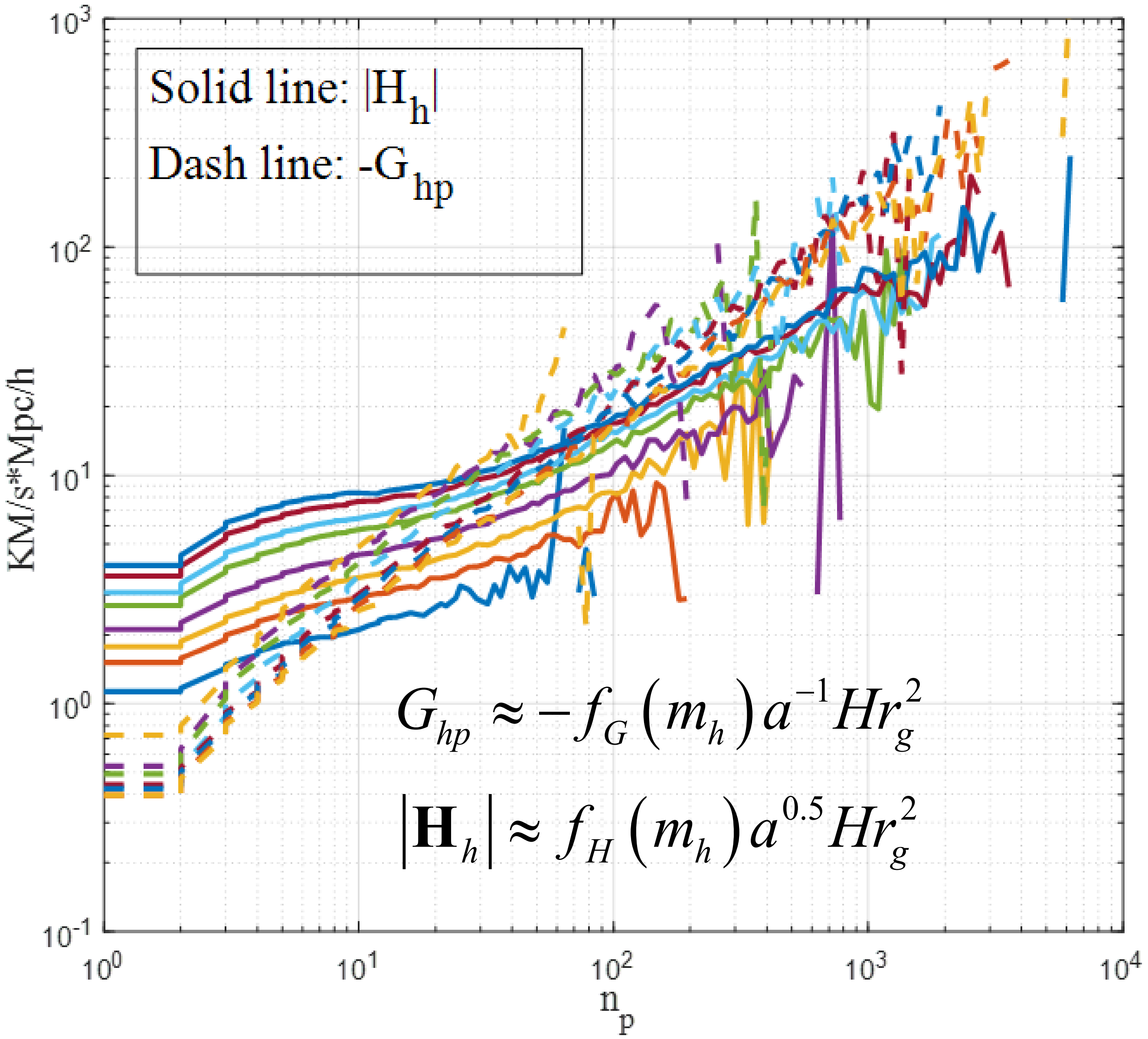
$$\mathbf{H}_h = \frac{1}{n_p} \sum_{i=1}^{n_p} \left(\mathbf{x}'_p \times \mathbf{u}'_p\right)$$

(peculiar) virial quantity (radial momentum):

$$G_{hp} = \frac{1}{n_p} \sum_{i=1}^{n_p} \left(\mathbf{x}'_p \cdot \mathbf{u}'_p\right)$$

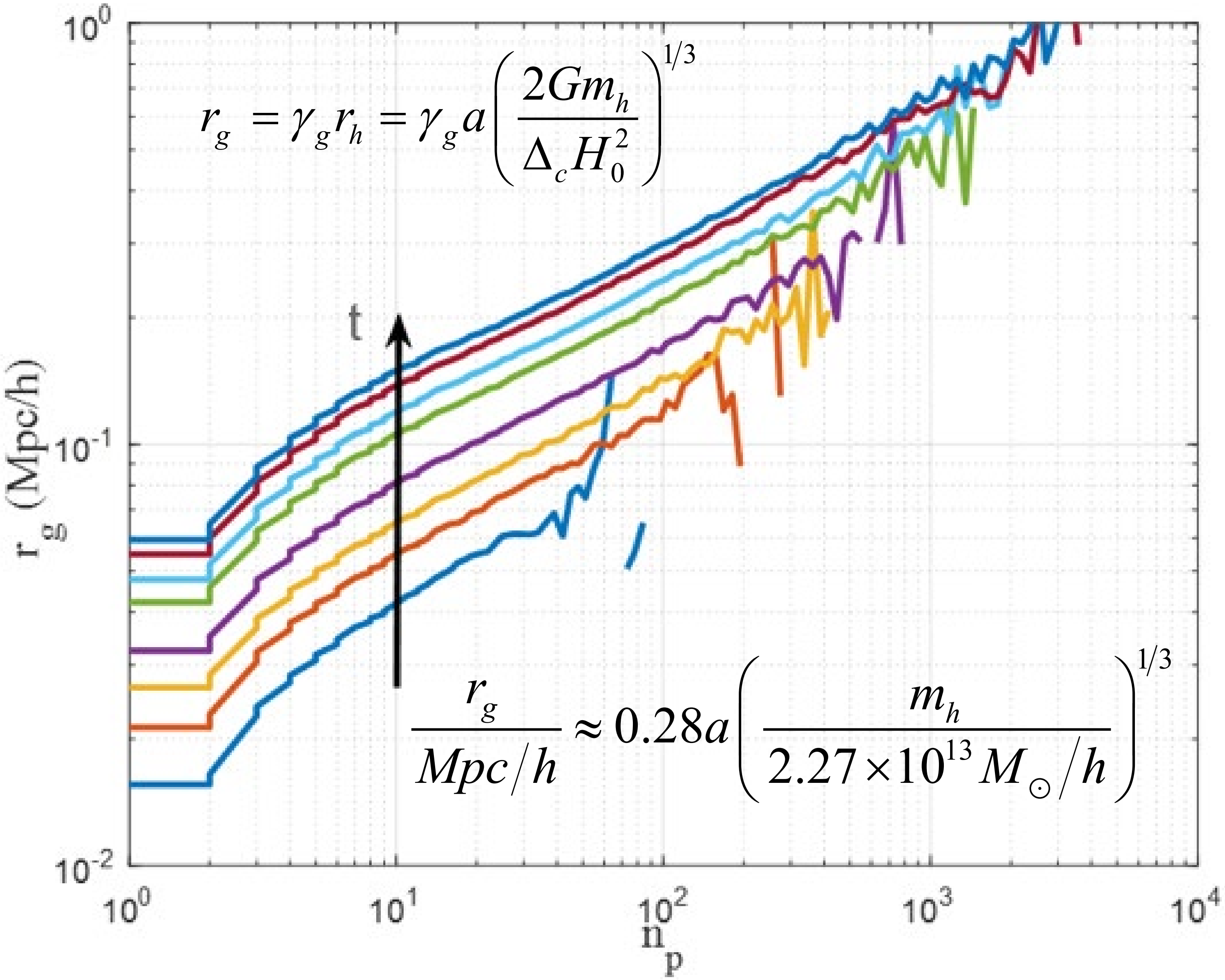
$$\gamma_G = \frac{-G_{hp}\left(m_h, a\right) a^{3/2}}{\left|\mathbf{H}_h\left(m_h, a\right)\right|} = \frac{f_G\left(m_h\right)}{f_H\left(m_h\right)}$$

(Next slides)

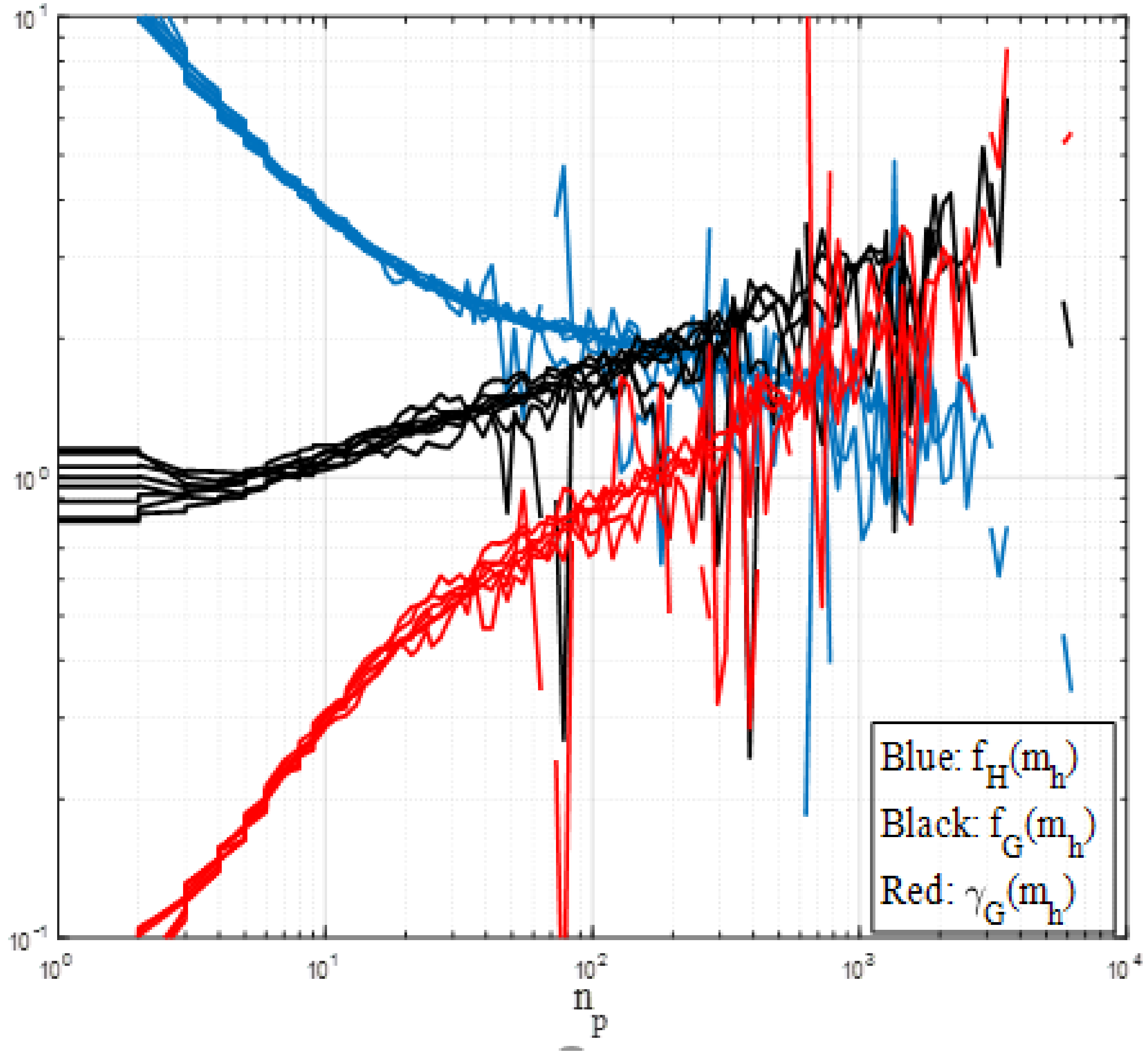


Variation with halo size for different redshifts
z = 0, 0.1, 0.3, 0.5, 1.0, 1.5, 2.0, and 3.0.

Modeling halo angular and radial momentum

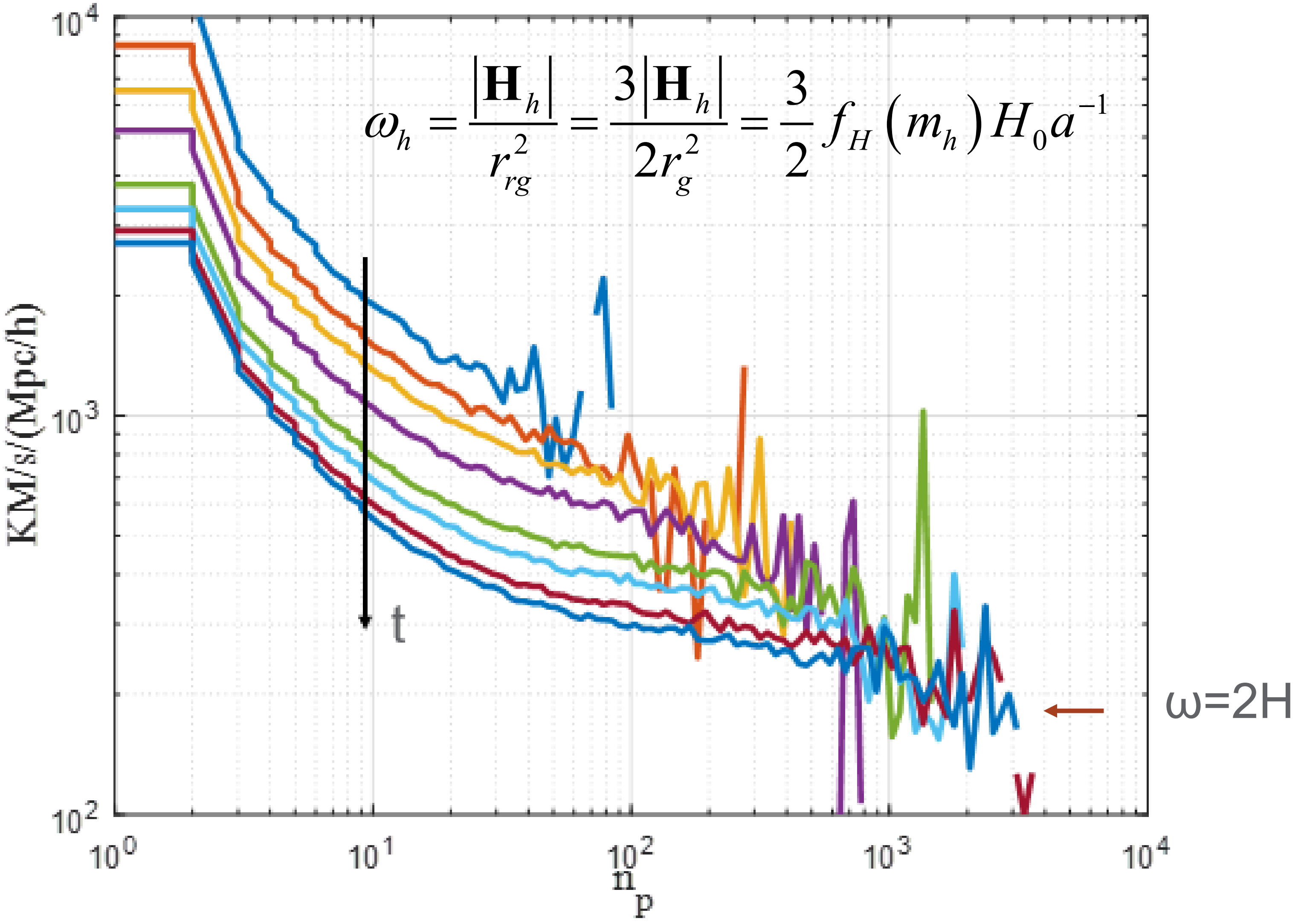


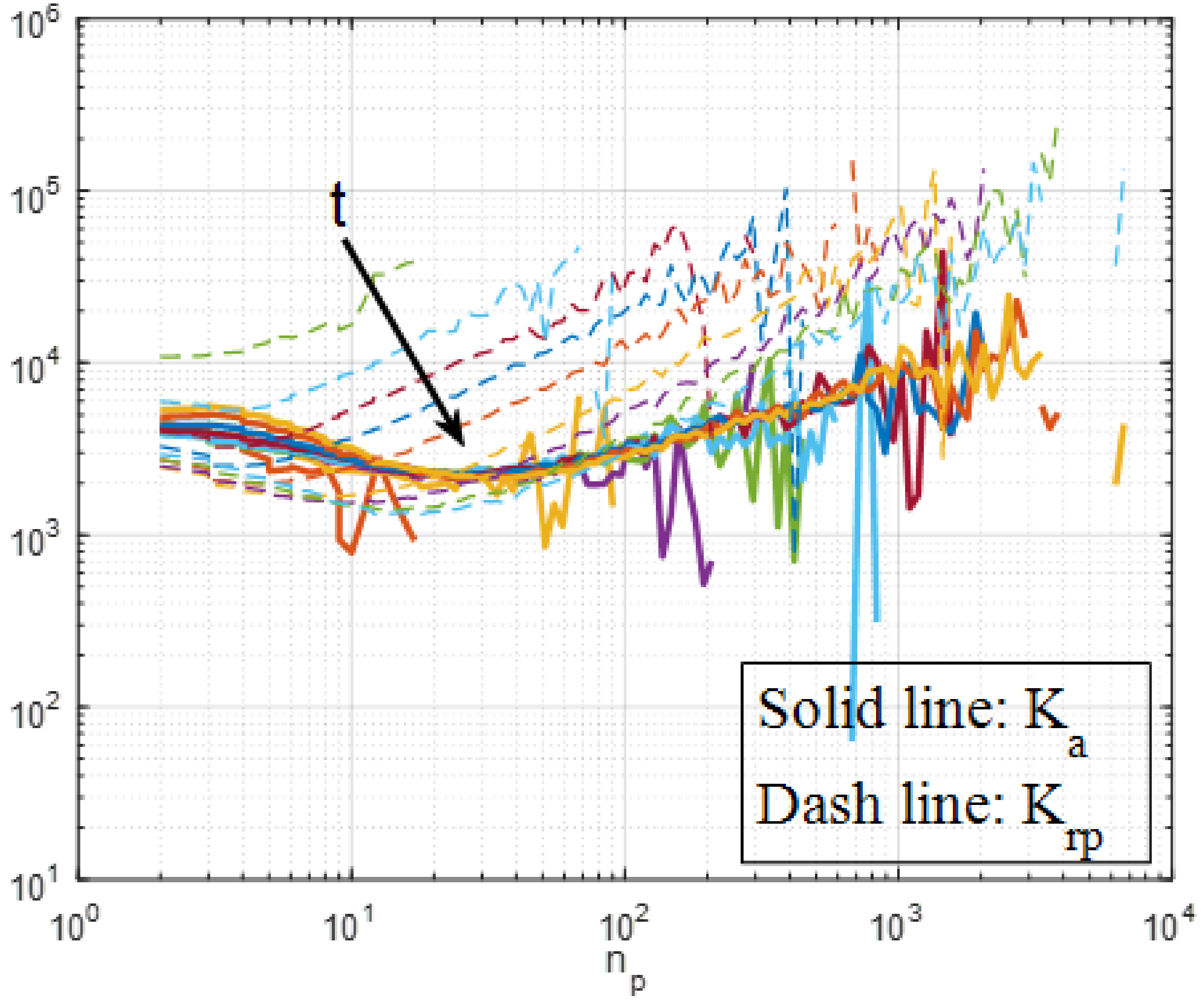
The variation of mean square radius r_g



The variation of two coefficients f_G, f_H and ratio γ_G

Halo angular velocity and kinetic energy from coherent motion (mean flow)





The variation of halo angular velocity, rotational kinetic energy and radial kinetic energy

$$K_a = \frac{1}{2}|\mathbf{H}_h|\omega_h = \frac{3}{4}\left(|\mathbf{H}_h|/r_g\right)^2 = \frac{3}{4}\gamma_g^2\left[f_H(m_h)\right]^2\left[\frac{2Gm_hH_0}{\Delta_c}\right]^{2/3}$$

$$K_{rp} = \frac{1}{2}\left(G_{hp}/r_g\right)^2 = \frac{1}{2}\gamma_g^2 a^{-3}\left[f_G(m_h)\right]^2\left[\frac{2Gm_hH_0}{\Delta_c}\right]^{2/3}$$

The effect of halo shape on energy cascade

Vortex Stretching (shape changing) responsible for energy cascade in turbulence.

What about the shape change of halo?

Assuming ellipsoid shape, 3x3 inertia tensor for every halo:

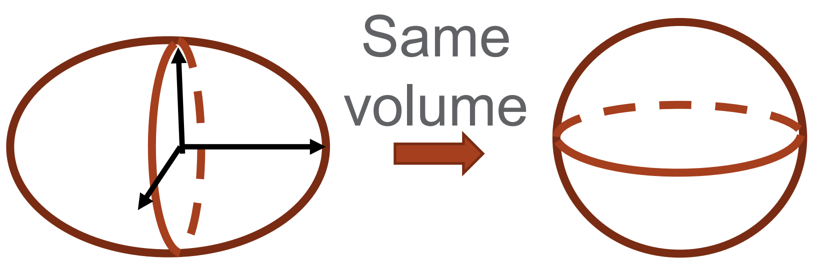
$$I_{ij} = \sum_{p=1}^{n_p} x'_{p,i} x'_{p,j}$$ ➡ Three eigenvalues (length of semimajor axis) $$r_{\lambda 1} \le r_{\lambda 2} \le r_{\lambda 3}$$

Mean square radius:

$$r_g^2 = r_{\lambda 1}^2 + r_{\lambda 2}^2 + r_{\lambda 3}^2$$

Define two critical ratios:

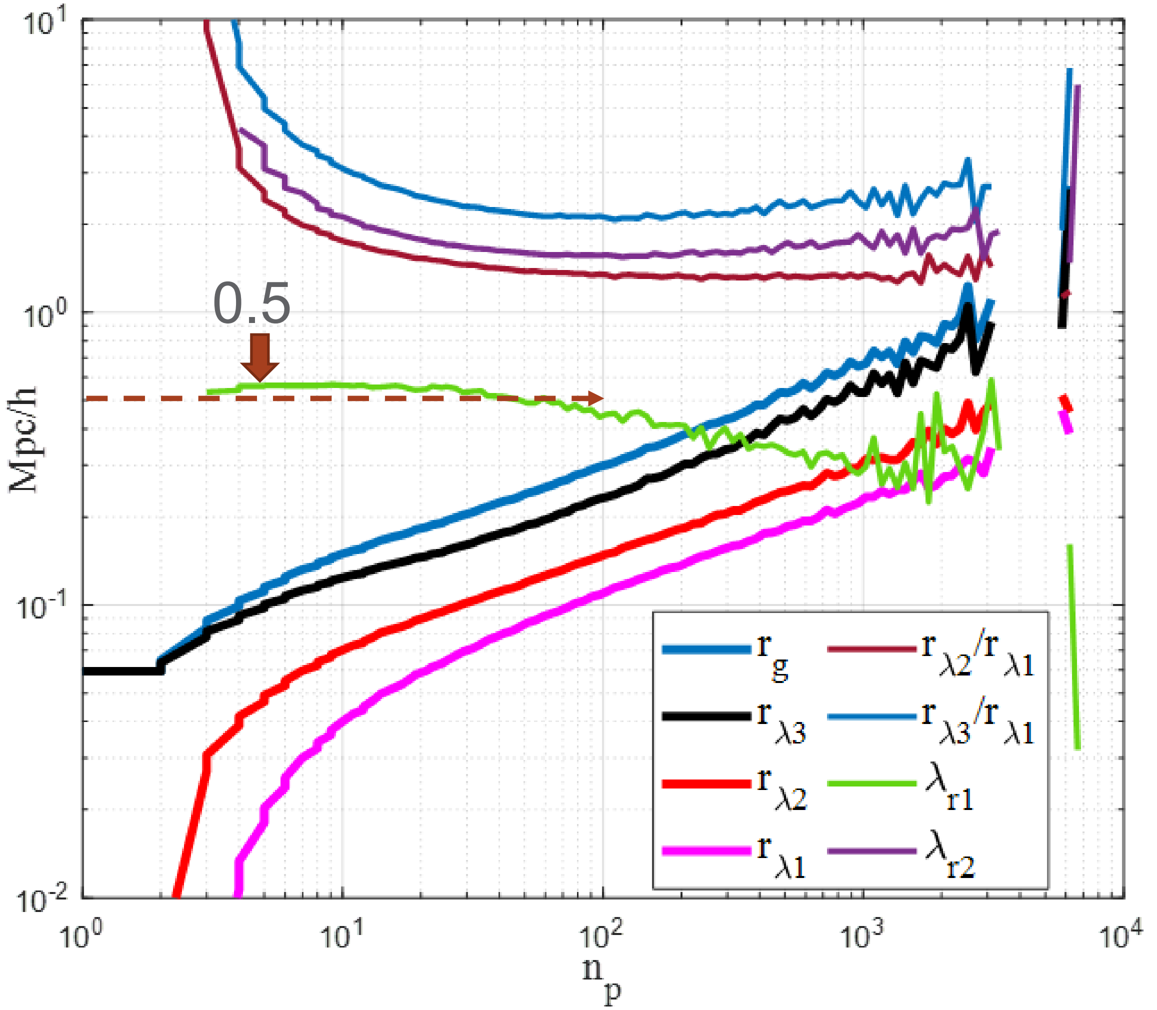
$$\lambda_{r1} = \frac{r_{\lambda 2} - r_{\lambda 1}}{r_{\lambda 3} - r_{\lambda 2}}$$



Moment of inertia for ellipsoid

$$\lambda_{r2} = \frac{r_{\lambda 3}^2 + r_{\lambda 2}^2}{2\left(r_{\lambda 1} r_{\lambda 2} r_{\lambda 3}\right)^{2/3}}$$

Moment of inertia for sphere



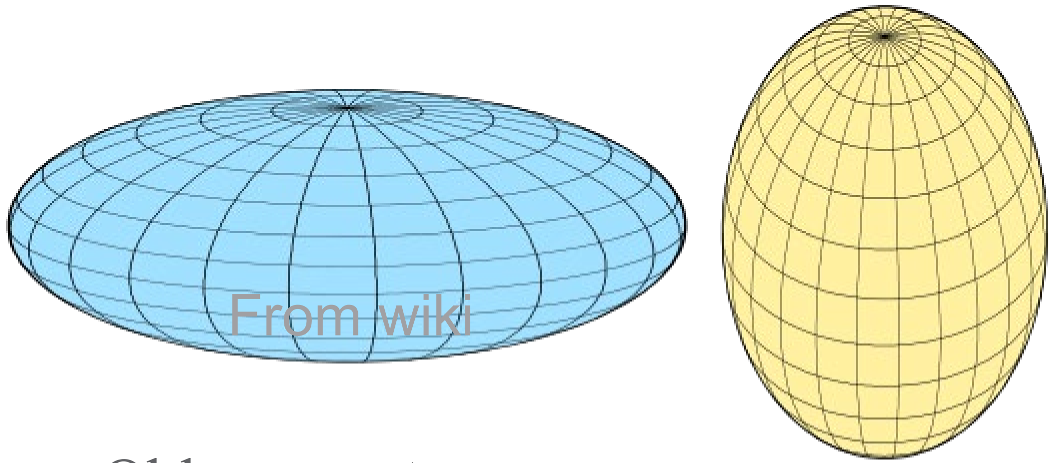
$\lambda_{r1} \approx 0.5$ for small halos, a unique path of shape evolution (green);

$\lambda_{r2} = 1$ for sphere;

Simulated halos: $\lambda_{r2} = [1.55, 2]$

Change of halo shape should not play a significant role in energy cascade.

Various halo shape parameters



Oblate: $r_{\lambda 1} < r_{\lambda 2} = r_{\lambda 3}$ Prolate: $r_{\lambda 1} = r_{\lambda 2} < r_{\lambda 3}$

Triaxiality parameter:

$$h_t = \frac{r_{\lambda 3}^2 - r_{\lambda 2}^2}{r_{\lambda 3}^2 - r_{\lambda 1}^2}$$

$h_t = 1$ prolate

$h_t = 0$ oblate

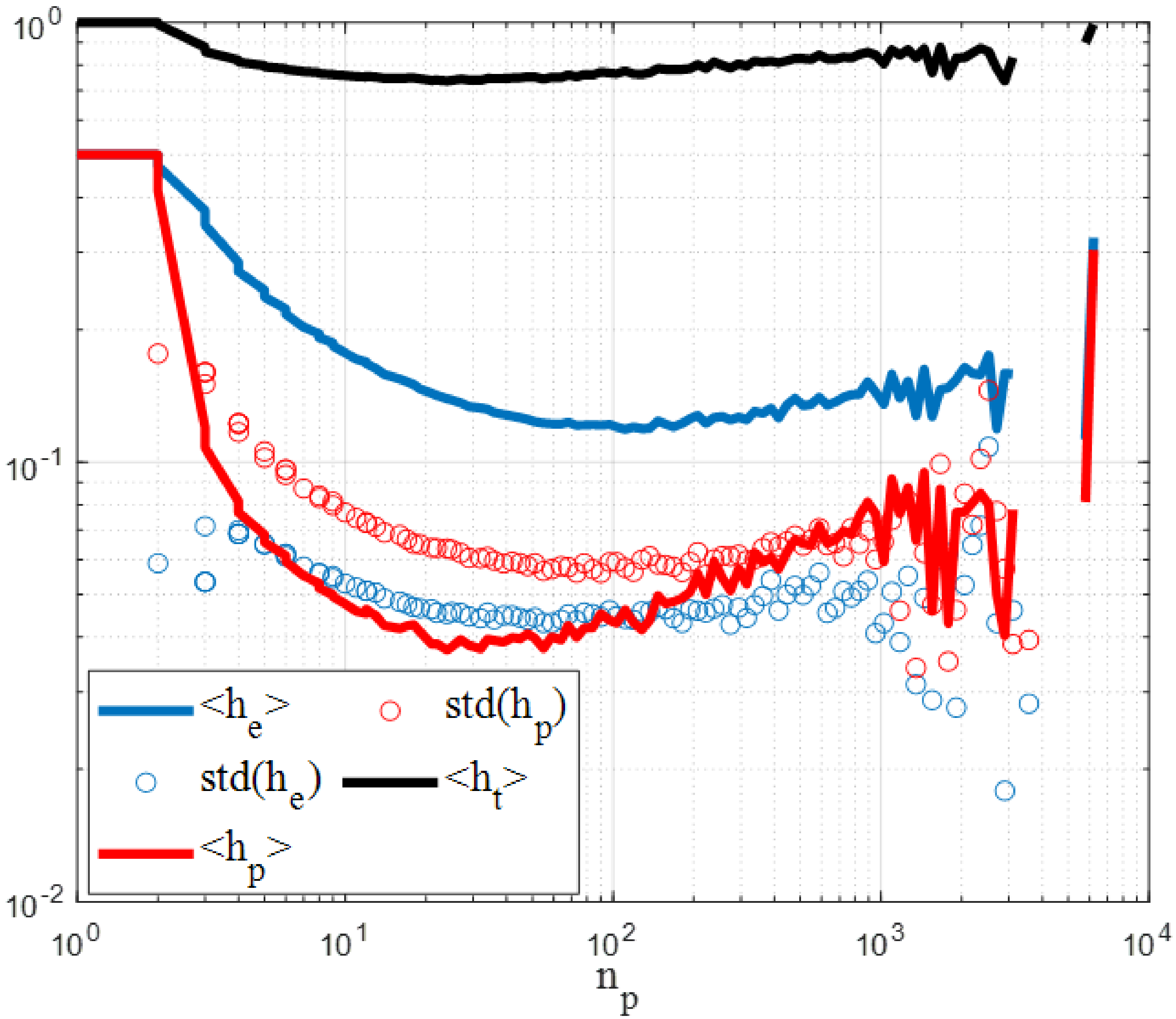
Ellipticity & prolateness parameters:

$$h_e = \frac{r_{\lambda 3} - r_{\lambda 1}}{2\left(r_{\lambda 1} + r_{\lambda 2} + r_{\lambda 3}\right)}$$

$h_p = -h_e$ oblate

$$h_p = \frac{r_{\lambda 3} - 2r_{\lambda 2} + r_{\lambda 1}}{2\left(r_{\lambda 1} + r_{\lambda 2} + r_{\lambda 3}\right)}$$

$h_p = h_e$ prolate



The variation of halo shape parameters with halo size at z=0

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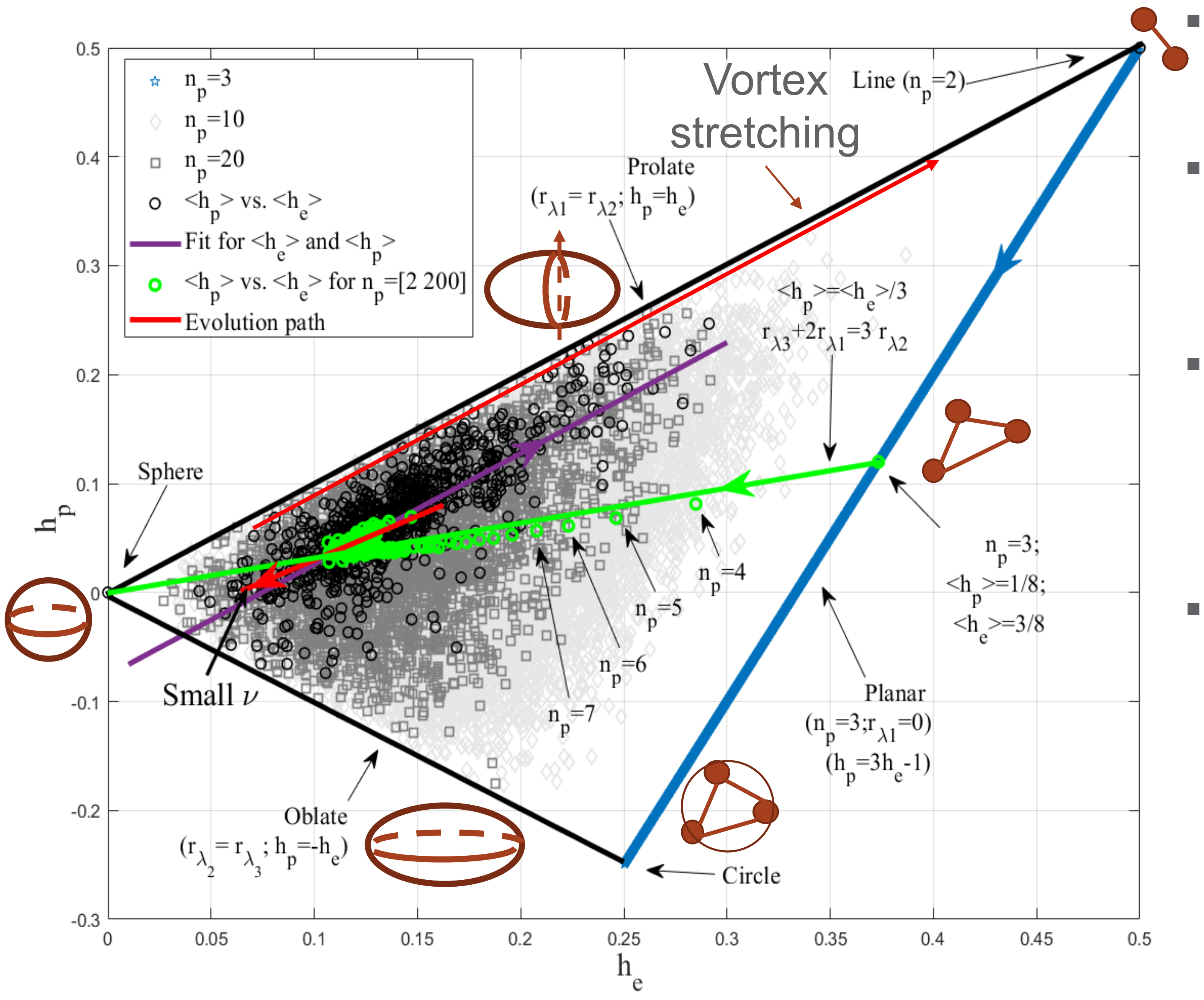
Two-dimension h_e-h_p mapping of halo shape



- All three-body halos have planar structure (blue line) with mean values of 1/8 and 3/8.
- The mean shape parameters for all halo groups (black circles). Green circles highlight the halos in range of n_p=[3 200]. Halos are more prolate.
- With increasing size, the shape of halos evolves consistently toward sphere along a unique path (green line) before a “V” turn. Path required λ_{r1}=0.5.
- Red line with arrow pointing to low peak height indicates the evolution path of simulated halo shape from early stage (ν=5) to late stage (ν=0.5).

$$h_e = 0.098\log_{10}\nu + 0.094$$

$$h_p = 0.079\log_{10}\nu + 0.025$$

Peak height: $\nu = \delta_{cr}/\sigma(m_h, z)$ [5 to 0.5]

Summary and keywords

Inverse energy cascade	Direct energy cascade	Halo inertia tensor
Energy flux function	Energy transfer function	Halo mean square radius
Prolate & oblate	Ellipticity & prolateness	Halo moment of inertia
Halo virial/velocity dispersion	Intra- and inter-halo potential	Halo radial & angular momentum

- Establish connections of energy cascade in turbulence and dark matter flow
- Direct energy cascade in hydrodynamic turbulence is facilitated by the vortex stretching (shape changing) along its axis of rotation
- Inverse cascade of kinetic energy from small to large mass scales in dark matter flow
- Direct cascade of potential energy from large to small mass scales
- A constant scale-independent rate of energy cascade $\varepsilon_u \sim a^0$ and *a* is scale factor
- Energy cascade in dark matter flow is mostly facilitated by the mass cascade of halos
- The shape change of halos does not play the major role.
- A unique evolution path of halo shape that gradually approaches spherical shape with increasing halo size

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The mean flow, velocity dispersion, energy transfer and evolution of rotating & growing dark matter halos

Xu Z., 2022, arXiv:2201.12665 [astro-ph.GA]
https://doi.org/10.48550/arXiv.2201.12665

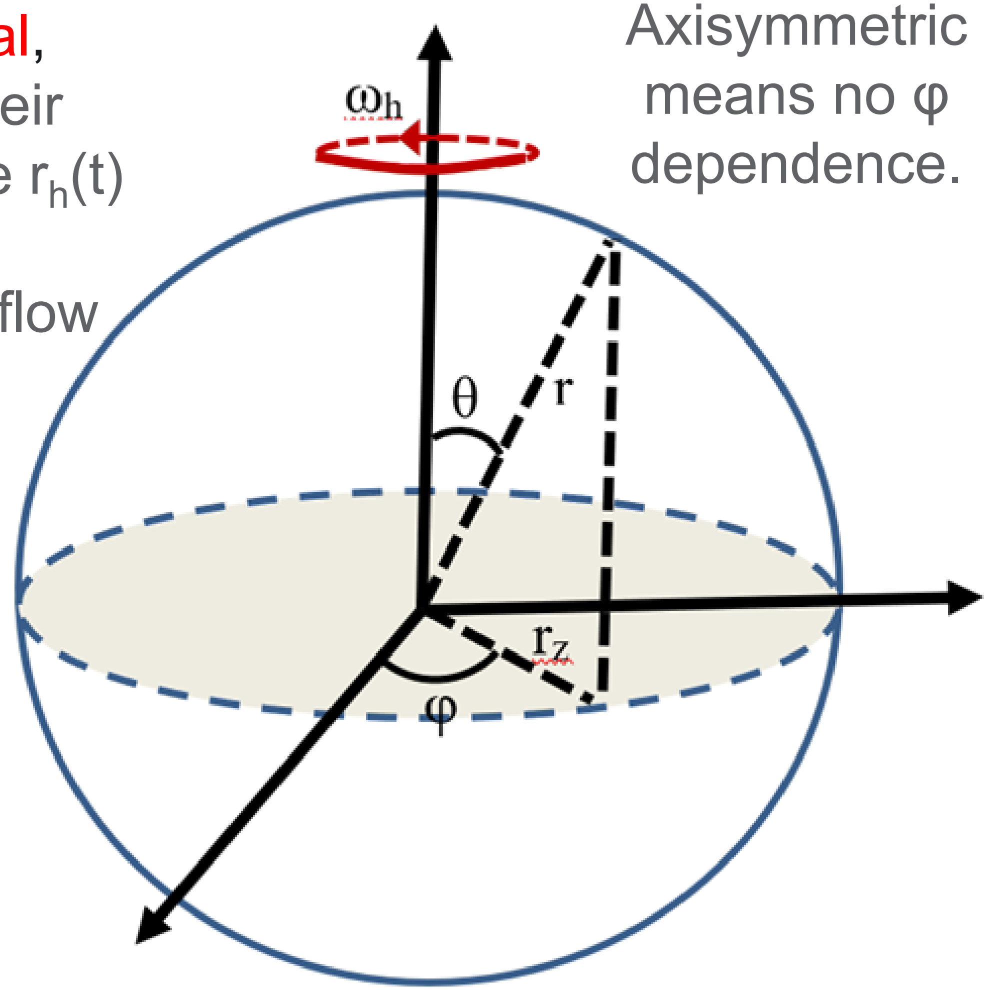
Introduction

Review: In hydrodynamic turbulence, "Reynolds stress" facilitates the **one-way** energy exchange from coherent (mean) flow to random fluctuation (turbulence) and enhances system entropy.

Existing study of halos mostly focus on the **spherical non-rotating non-growing** halos with a vanishing radial flow (fully virialized halos with slow mass accretion in their late stage).

- Goal 1: Explore solutions of mean flow and dispersions for spherical, axisymmetric, growing and rotating halos (fast mass accretion in their early stage) with an effective angular velocity $\omega_h(t)$ and varying size $r_h(t)$
- Goal 2: Explore the transition of halos from early to late stage
- Goal 3: Explore the role of halos in energy transfer between mean flow and random fluctuation.

Axisymmetric means no φ dependence.



Density:	$\rho_h = \rho_h(r,t)$	Potential:	$\phi_r = \phi_r(r,t)$
Radial flow:	$u_r = u_r(r,t)$		$\sigma_{rr}^2 = \sigma_{rr}^2(r,\theta,t)$
The polar flow (meridional flow) :	$u_\theta = u_\theta(r,\theta,t)$		$\sigma_{\theta\theta}^2 = \sigma_{\theta\theta}^2(r,\theta,t)$
azimuthal flow (zonal flow):	$u_\varphi = u_\varphi(r,\theta,t)$		$\sigma_{\varphi\varphi}^2 = \sigma_{\varphi\varphi}^2(r,\theta,t)$

Reduced equations and anisotropic parameter

The continuity equation reduces to:

$$\frac{\partial \rho_h}{\partial t}+\frac{1}{r^2}\frac{\partial\left(r^2\rho_h u_r\right)}{\partial r}=0$$

Six equations and 8 Variables; need extra closures to solve;

Observations of flow on rotating sphere strongly suggest that as the rotation rate increases, the azimuthal flow becomes dominant and the polar flow may be neglected.

With $u_\theta \approx 0$ and $\sigma_{r\theta}^2=\sigma_{r\varphi}^2=\sigma_{\varphi\theta}^2=0$

The full momentum equations (Jeans' equation) reduces to

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{1}{\rho_h}\frac{\partial\left(\rho_h\sigma_{rr}^2\right)}{\partial r}+\frac{2}{r}\sigma_{rr}^2\underbrace{\left(1-\frac{\sigma_{\theta\theta}^2+\sigma_{\varphi\varphi}^2+u_\varphi^2}{2\sigma_{rr}^2}\right)}_{1}+\frac{\partial \phi_r}{\partial r}=0$$

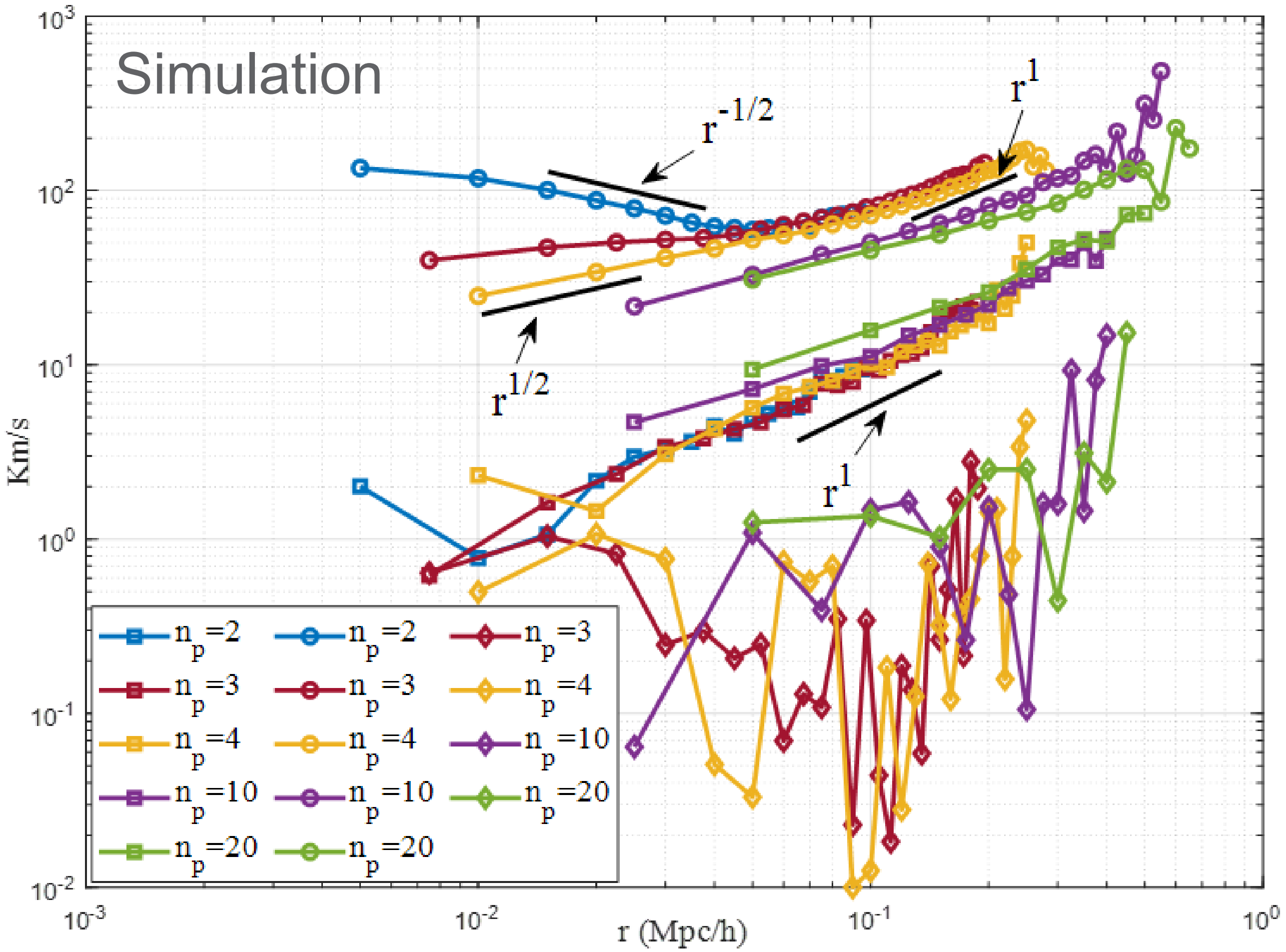
For θ: $$u_\varphi^2=\sigma_{\theta\theta}^2-\sigma_{\varphi\varphi}^2+\frac{\sin\theta}{\cos\theta}\frac{\partial\sigma_{\theta\theta}^2}{\partial\theta}$$

For φ: $$\frac{\partial u_\varphi}{\partial t}+u_r\frac{\partial u_\varphi}{\partial r}+\frac{u_r u_\varphi}{r}=0$$

$$\boxed{\frac{\partial \phi_r}{\partial r}=\frac{Gm_r(r,t)}{r^2}\qquad \rho_h=\frac{1}{4\pi r^2}\frac{\partial m_r(r,t)}{\partial r}}$$

Anisotropic parameter should include effect of u_φ or centripetal force:

$$\beta_h=1-\frac{\sigma_{\theta\theta}^2+\sigma_{\varphi\varphi}^2}{2\sigma_{rr}^2}\ \text{Old}\qquad \boxed{\beta_{h1}=1-\frac{\sigma_{\theta\theta}^2+\sigma_{\varphi\varphi}^2+u_\varphi^2}{2\sigma_{rr}^2}\ \text{New}}$$



Circle: u_φ ; Square: radial flow u_{rp} ; Diamond: u_θ

Polar flow can be neglected

Evolution of halo angular momentum

From continuity and momentum equations:

$$\frac{\partial\left(\rho_h u_\varphi\right)}{\partial t}+\frac{1}{r^2}\frac{\partial\left[\left(\rho_h u_\varphi\right)u_r r^2\right]}{\partial r}+\frac{u_r}{r}\left(\rho_h u_\varphi\right)=0$$

The halo angular momentum is:

$$\bar{H}_h=\int_0^{r_h}2\pi r^3\rho_h(r)\left(\int_0^\pi u_\varphi \sin^2\theta d\theta\right)dr$$

Time evolution of angular momentum:

$$\frac{\partial \bar{H}_h}{\partial t}=2\pi r_h^3\rho_h\left(r_h\right)\int_0^\pi u_\varphi\left(r_h,\theta\right)\sin^2\theta d\theta\left(\frac{\partial r_h}{\partial t}-u_r\left(r_h\right)\right)$$

The halo angular momentum is conserved only if $\frac{\partial r_h}{\partial t}=u_r\left(r_h\right)$

However, for growing halos $\partial r_h/\partial t>0$, $u_r\left(r_h\right)\le 0$ $\Rightarrow$ $\frac{\partial \bar{H}_h}{\partial t}>0$

- In hydrodynamic turbulence, angular momentum is conserved during vortex stretching.
- In dark matter flow, halo angular momentum is not conserved and always increasing with time.
- The Tidal Torque Theory (TTT) relates the angular momentum to the misalignment between the tidal shear field and halo shape.
- TTT predicts a linear increase with time t for halo with a fixed given mass $\bar{H}_h \sim t$
- A growing halo may obtain its momentum through continuous mass acquisition and $\bar{H}_h \sim t^2$

Evolution of halo rotational kinetic energy

From continuity and momentum equations:

$$\underbrace{\frac{\partial\left(\rho_h u_\varphi^2\right)}{\partial t}}_{derivative}+\underbrace{\frac{1}{r^2}\frac{\partial\left[\left(\rho_h u_\varphi^2\right)u_r r^2\right]}{\partial r}}_{advection}+\underbrace{2\boxed{\rho_h u_\varphi^2}\frac{u_r}{r}}_{production}=0$$

Fictitious stress

gradient

The halo rotational kinetic energy is obtained by integration:

$$\bar{K}_a=\frac{1}{2}\int_0^{r_h}2\pi r^2\int_0^{\pi}\left(\rho_h u_\varphi^2\right)\sin\theta d\theta dr$$

Time evolution of rotational kinetic energy:

$$\frac{\partial \bar{K}_a}{\partial t}=\underbrace{\pi r_h^2\rho_h\left(r_h\right)\int_0^{\pi}u_\varphi^2\left(r_h,\theta\right)\sin\theta d\theta\left(\frac{\partial r_h}{\partial t}-u_r\left(r_h\right)\right)}_{1}-\underbrace{\int_0^{r_h}2\pi r^2\frac{u_r}{r}\rho_h\left(\int_0^{\pi}u_\varphi^2\sin\theta d\theta\right)dr}_{2}$$

1: surface contribution from mass cascade
2: bulk cont. from energy transfer

- In hydrodynamic turbulence, the "Reynolds" stress facilitates the one-way energy exchange from coherent (mean) flow to random fluctuation and enhances entropy.
- In dark matter flow, the production term describes the fictitious stress acting on the gradient of mean radial flow to facilitate the energy transfer between mean azimuthal flow and random fluctuation.
- Since u_r is positive in core region and negative in outer region, the energy transfer is two-way, i.e. energy is drawn from random motion to mean flow in outer region and from mean flow to random motion in core region.
- However, for entire halo, there is a net transfer from mean flow to random flow to enhance the halo entropy.

General solutions for rotating, and growing halos

Key: decomposition of velocity dispersion:

Introduce reduced spatial/temporal coordinate: $x(r,t)=\frac{r}{r_s(t)}=\frac{cr}{r_h(t)}$

$$\sigma_{\theta\theta}^2(r,\theta,t)=\underbrace{\sigma_{r0}^2(r,t)}_{1}+\underbrace{\alpha_\varphi(r,t)u_\varphi^2(r,\theta,t)}_{2}$$

1: Axial-dispersion 2: Spin-dispersion

$$\sigma_{\varphi\varphi}^2(r,\theta,t)=\sigma_{r0}^2(r,t)+\beta_\varphi(r,t)u_\varphi^2(r,\theta,t)$$

$$\sigma_{rr}^2(r,\theta,t)=\sigma_{r0}^2(r,t)+\gamma_\varphi(r,t)u_\varphi^2(r,\theta,t)$$

Separation of variables: $u_\varphi(r,\theta,t)=\omega_h(t)r_s(t)F_\varphi(x)K_\varphi(\theta)$

$$\boxed{\sigma_{rr}^2(r,\theta=0,t)=\sigma_{\theta\theta}^2(r,\theta=0,t)=\sigma_{\varphi\varphi}^2(r,\theta=0,t)=\sigma_{r0}^2(r,t)}$$

- Spin causes velocity anisotropy; Velocity dispersions can be expressed as a function of azimuthal flow u_φ.
- Velocity dispersion is expected to be isotropic for non-rotating halos with a spherical symmetry.
- For spherical halos with a finite spin, velocity dispersions are only isotropic along the axis of rotation (θ=0)

Momentum equation for θ:

$$\frac{\partial u_\varphi}{\partial t}+u_r\frac{\partial u_\varphi}{\partial r}+\frac{u_r u_\varphi}{r}=0$$

Halo spin

Mass cascade

$$\frac{\partial \ln F_\varphi}{\partial \ln x}=\frac{u_h(x)+x\left(\frac{\partial \ln \omega_h}{\partial \ln t}+\frac{\partial \ln r_s}{\partial \ln t}\right)}{x\frac{\partial \ln r_s}{\partial \ln t}-u_h(x)}$$

Radial flow

Momentum equation for φ:

$$u_\varphi^2=\sigma_{\theta\theta}^2-\sigma_{\varphi\varphi}^2+\frac{\sin\theta}{\cos\theta}\frac{\partial\sigma_{\theta\theta}^2}{\partial\theta}$$

$$K_\varphi(\theta)=(\sin\theta)^{\alpha_\theta}$$

with an angular exponent

$$\alpha_\theta=\frac{1+\beta_\varphi-\alpha_\varphi}{2\alpha_\varphi}$$

General solutions for rotating, and growing halos

Momentum equation for r:

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{1}{\rho_h}\frac{\partial\left(\rho_h\sigma_{rr}^2\right)}{\partial r}+\frac{2}{r}\sigma_{rr}^2\left(1-\frac{\sigma_{\theta\theta}^2+\sigma_{\varphi\varphi}^2+u_\varphi^2}{2\sigma_{rr}^2}\right)+\frac{\partial\phi_r}{\partial r}=0$$

Equation for axial-dispersion:

$$\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{1}{\rho_h}\frac{\partial\left(\boxed{\rho_h\sigma_{r0}^2}\right)}{\partial r}+\frac{\partial\phi_r}{\partial r}+F_a\left(r,t\right)=0$$

and

Equation for spin-dispersion:

$$\frac{\partial\ln\left(\boxed{\gamma_\varphi u_\varphi^2}\right)}{\partial\ln x}+\frac{\partial\ln\rho_h}{\partial\ln x}+2-2\alpha_a=\frac{rF_a\left(r,t\right)}{\gamma_\varphi u_\varphi^2}$$

The coupling function reflects the coupling between axial-dispersion and spin-dispersion

Two anisotropy parameters are related:

$$\beta_{h1}=\frac{1-\alpha_a}{1+\sigma_{r0}^2/\left(\gamma_\varphi u_\varphi^2\right)}$$

$$\alpha_a=\frac{\left(\alpha_\varphi+\beta_\varphi+1\right)}{2\gamma_\varphi}$$

For virialized "small" halos with slow mass accretion (late stage), the axial- and spin-dispersions are decoupled. Axial-dispersion is dominant to balance gravity.

$$F_a\left(r,t\right)=0 \quad \text{and} \quad \sigma_{r0}^2 \gg \gamma_\varphi u_\varphi^2 \Rightarrow \beta_{h1}\approx 0$$

For "large" halos with fast mass accretion (early stage), the axial- and spin-dispersions are decoupled. Spin-dispersion is dominant to balance gravity.

$$F_a\left(r,t\right)\approx-\frac{\partial\phi_r}{\partial r} \quad \text{and} \quad \sigma_{r0}^2 \ll \gamma_\varphi u_\varphi^2 \Rightarrow \beta_{h1}\approx 1-\alpha_a$$

Two limiting situations: "small" and "large" halos

We still require a clear definition of "small" and "large" halos.

Enclose mass within radius r

$$m_r(r,t) = m_h(t)\frac{F(x)}{F(c)}$$

Halo density

$$\rho_h(r,t) = \frac{m_h(t)}{4\pi r_s^3}\frac{F'(x)}{x^2 F(c)}$$

The ratio of core mass to halo mass:

$$C_F = \frac{F(1)}{F(c)} = \frac{m_r(r_s,t)}{m_h(t)}$$

Peak height:

$$\nu = \delta_{cr}/\sigma(m_h, z)$$

From spherical collapse model

$$\delta_{cr} \approx 1.68$$

σ is (root mean square) fluctuation of the smoothed density

At same redshift z, large halos has higher ν

Properties of "large" halos:

- Early stage of halo life with high peak height ν
- Extremely fast mass accretion
- A growing core with scale radius r_s~t
- Growing halo size r_h~t and halo mass m_h~t
- Constant halo concentration c≈3.5 (limiting c)

Properties of "small" halos:

- Late stage of halo life with low peak height ν
- Extremely slow mass accretion
- A stable core, constant scale radius $r_{s,}$ and constant core-to-halo mass ratio C_F
- Increasing concentration c~$t^{2/3}$~*a* and m_h~F(c)

Solutions for "small" halos at late stage

Coupling function: $F_a(r,t)=0$

Mean flow: $u_r = u_\theta = 0$

Velocity dispersions: $\sigma_{rr}^2 = \sigma_{\varphi\varphi}^2 = \sigma_{\theta\theta}^2 + u_\phi^2$

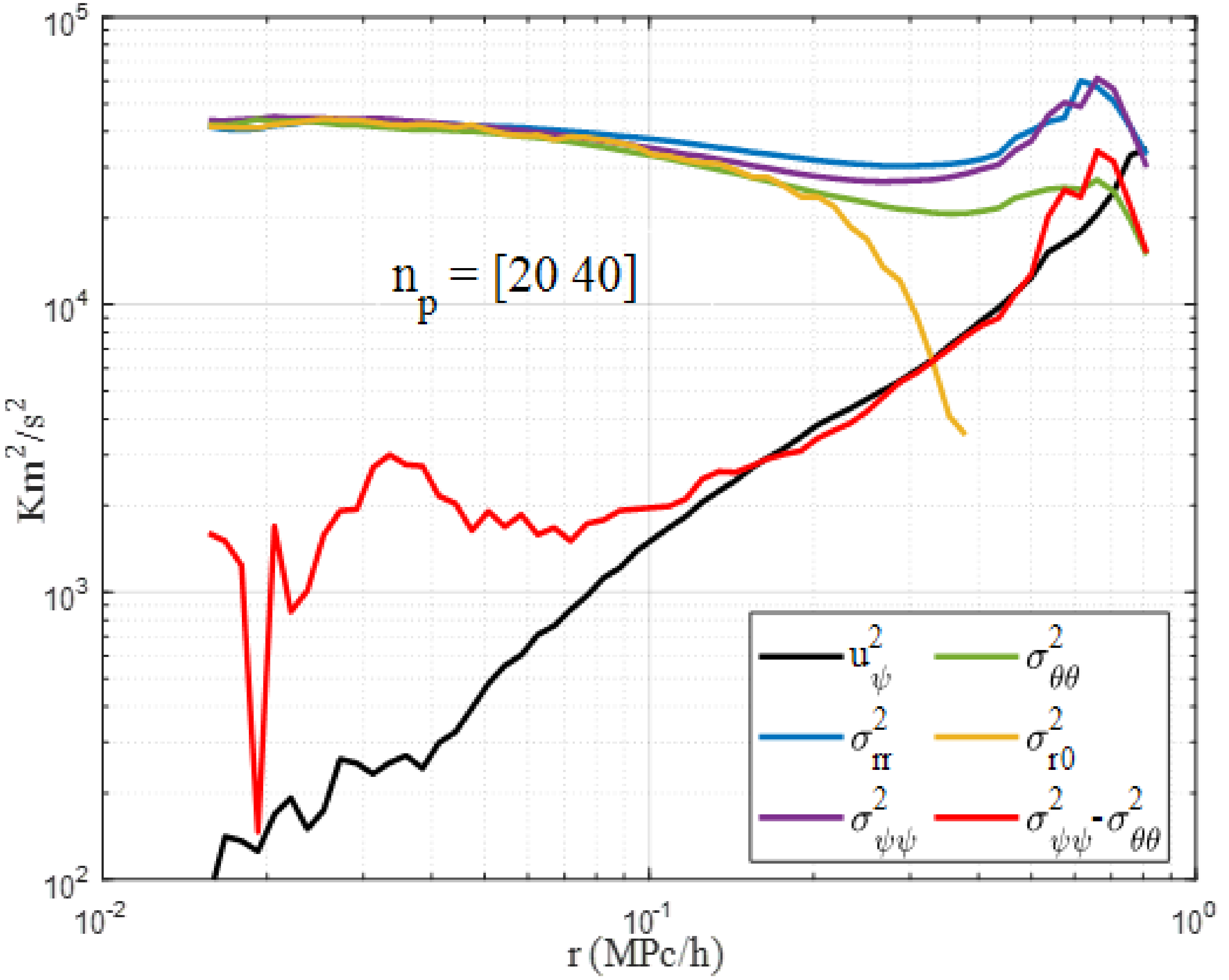
Anisotropy parameters : $\alpha_a = 1 \quad \beta_{h1} = 0$

Angular exponent : $\alpha_\theta = 1$

$1+\alpha_\varphi = \beta_\varphi = \gamma_\varphi \qquad \alpha_\varphi = 1 \quad \beta_\varphi = \gamma_\varphi = 2$

Properties of "small" halos (continued):

- Virialized and bound with vanishing radial flow
- Incompressible (proper velocity) with $\nabla \cdot \mathbf{v} = 0$
- More spherical and isotropic
- Axial-dispersion dominant over spin-dispersion
- Azimuthal flow u_φ strongly dependent on polar angle θ
- Negligible surface energy



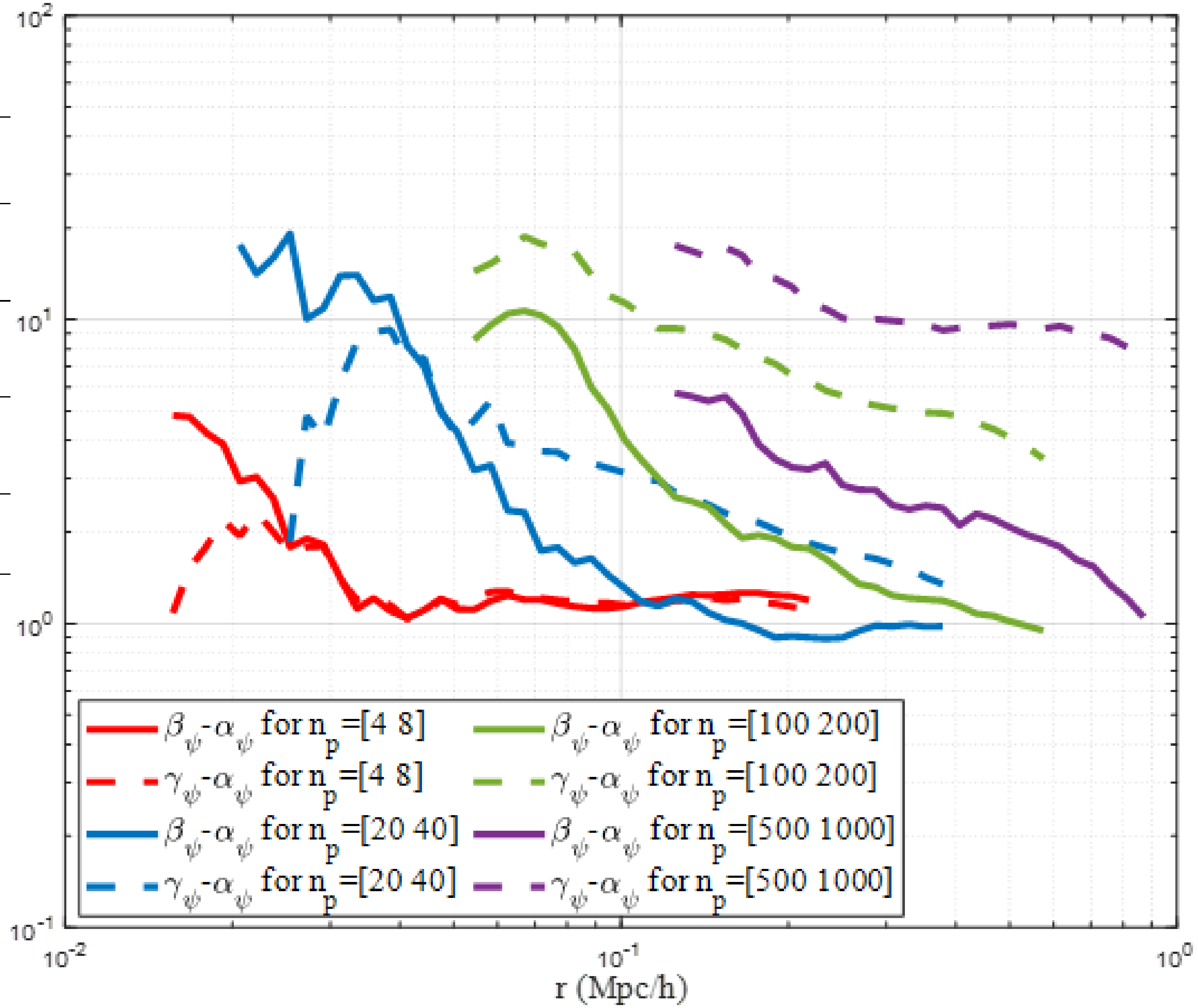
The variation of mean flow and velocity dispersions from N-body simulation

Energy equipartition along three directions

Table 2. Velocity dispersions for rotating and non-rotating halos

		Radial (r)	Azimuthal (φ)	Polar (θ)
Rotating halo (Eq. (9))	Random	$\sigma_{rr}^2 = \sigma_{r0}^2 + 2u_\varphi^2$	$\sigma_{\varphi\varphi}^2 = \sigma_{r0}^2 + 2u_\varphi^2$	$\sigma_{\theta\theta}^2 = \sigma_{r0}^2 + u_\varphi^2$
	Mean flow	0	u_φ^2	0
Non-rotating halo (Eq. (50))	Random	$\sigma_{rr}^2 = \sigma_r^2 = \sigma_{r0}^2$	$\sigma_{\varphi\varphi}^2 = \sigma_{r0}^2$	$\sigma_{\theta\theta}^2 = \sigma_{r0}^2$
	Mean flow	0	0	0

- Due to finite spin, kinetic energy is not equipartitioned along each direction with the greatest energy along the azimuthal direction and the smallest along the polar direction.
- Different from usual objects, halos are hotter with faster spin due to energy transfer between mean flow and random motion.



The variation of dispersion parameters α_φ, β_φ, and γ_φ

Solutions for “large” halos at early stage

Coupling function:

$$F_a(r,t) \approx -\frac{\partial \phi_r}{\partial r}$$

Mean flow:

$$u_\theta = 0$$

Velocity dispersions:

$$\sigma_{rr}^2 = \sigma_{\varphi\varphi}^2 = \sigma_{\theta\theta}^2 + u_\phi^2$$

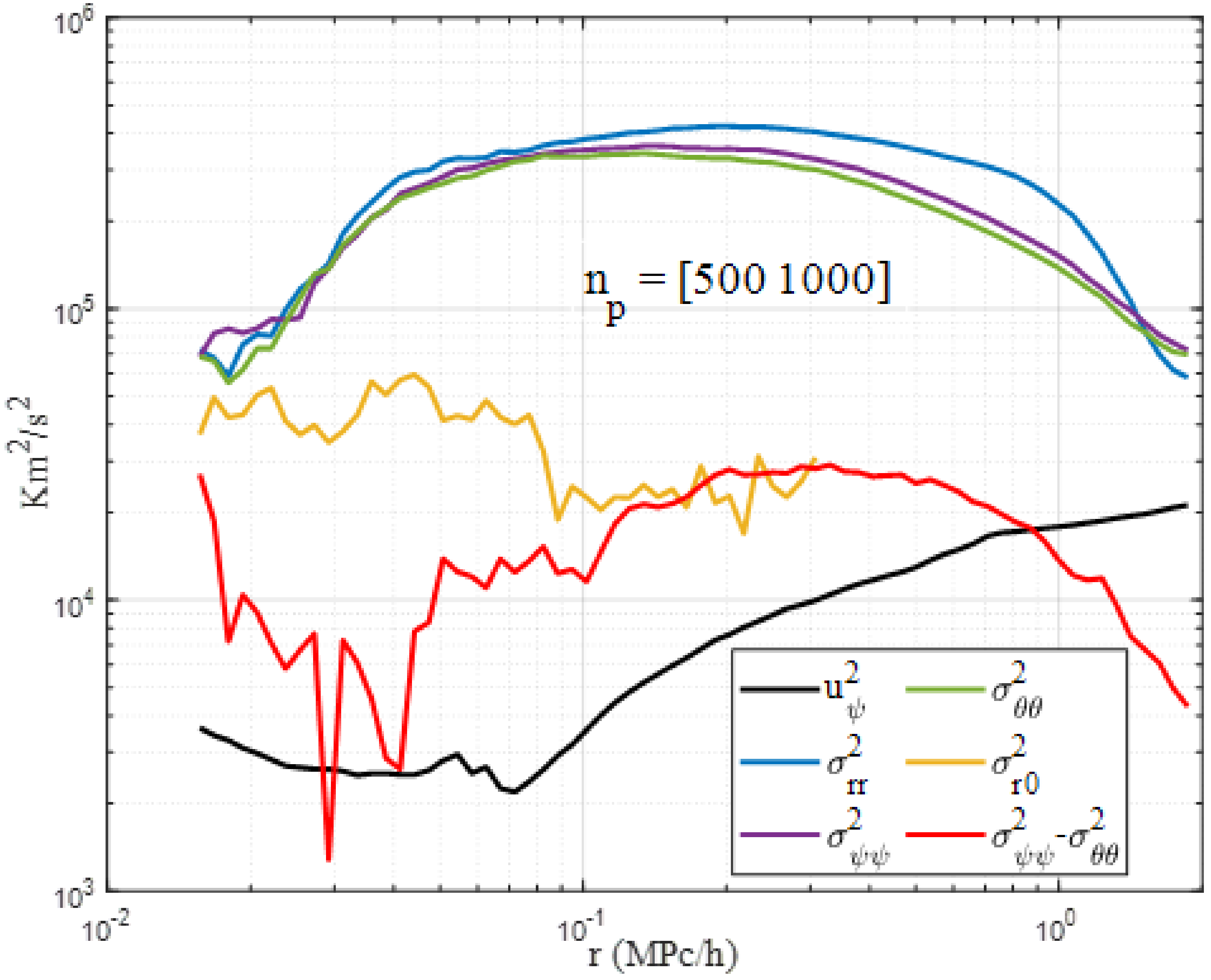
Anisotropy parameters : $\beta_{h1} \approx \beta_h$

Angular exponent : $\alpha_\theta \ll 1$

$$\beta_\varphi = \alpha_\varphi + 1 \qquad \alpha_\varphi \gg 1 \qquad \gamma_\varphi \approx \alpha_\varphi + 10$$

Properties of “large” halos (continued):

- Non-virialized with non-zero self-similar radial flow
- Spin-dispersion dominant over axial-dispersion
- Azimuthal flow u_φ is less dependent on polar angle θ
- Non-zero surface energy



The variation of mean flow and velocity dispersions from N-body simulation

Solutions for “large” halos at early stage

Radial flow: $u_h(x) = u_r(r)\dfrac{t}{r_s(t)} = x - \dfrac{F(x)}{F'(x)}$

Azimuthal flow: $u_\varphi(r,\theta,t) = u_\varphi(x,\theta) = \alpha_f \omega_h(t) r_s(t)\dfrac{F(x)}{x}$

Axial-dispersion: $\sigma_{r0}^2(x) = \dfrac{v_{cir}^2 x^2}{4\pi^2 c^2 F'(x)}\left\{\left.\dfrac{F^2(x)}{x^2 F'(x)}\right|_x^\infty - \int_x^\infty \left[\dfrac{2F(x)}{x^2} - \dfrac{2F^2(x)}{F'(x)x^3}\right]dx\right\}$

Angular velocity: $\omega_h = \left(\dfrac{3}{2\alpha_h} - \dfrac{1}{2}\right)\dfrac{c^2}{F(c)\alpha_f} H \propto t^{-1}$ and $\alpha_f = \dfrac{16c^2}{3\pi F(c)}\gamma_g^2$

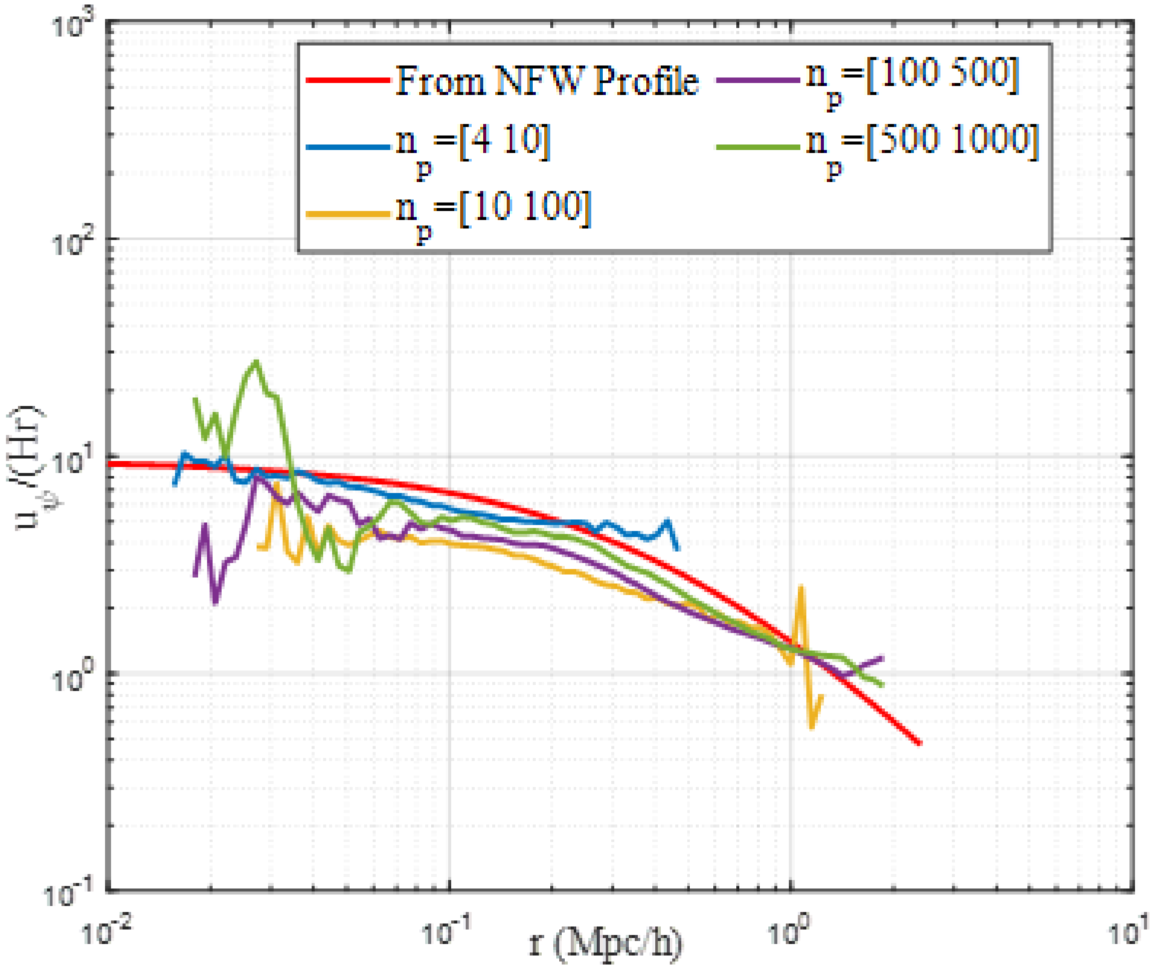
Dispersion parameter: $\gamma_\varphi(x) = \dfrac{x^4}{F^2(x)F'(x)}\left[18\int_x^\infty \dfrac{F^2(y)F'(y)}{y^5}dy + \lambda_f \int_x^\infty \dfrac{F(y)F'(y)}{y^4}dy\right]$

$$\lambda_f = \frac{9\pi^2 F(c)}{\left(3/(2\alpha_h) - 1/2\right)^2 c}$$

$\beta_\varphi = \alpha_\varphi + 1$ and $\gamma_\varphi \approx \alpha_\varphi + 10$

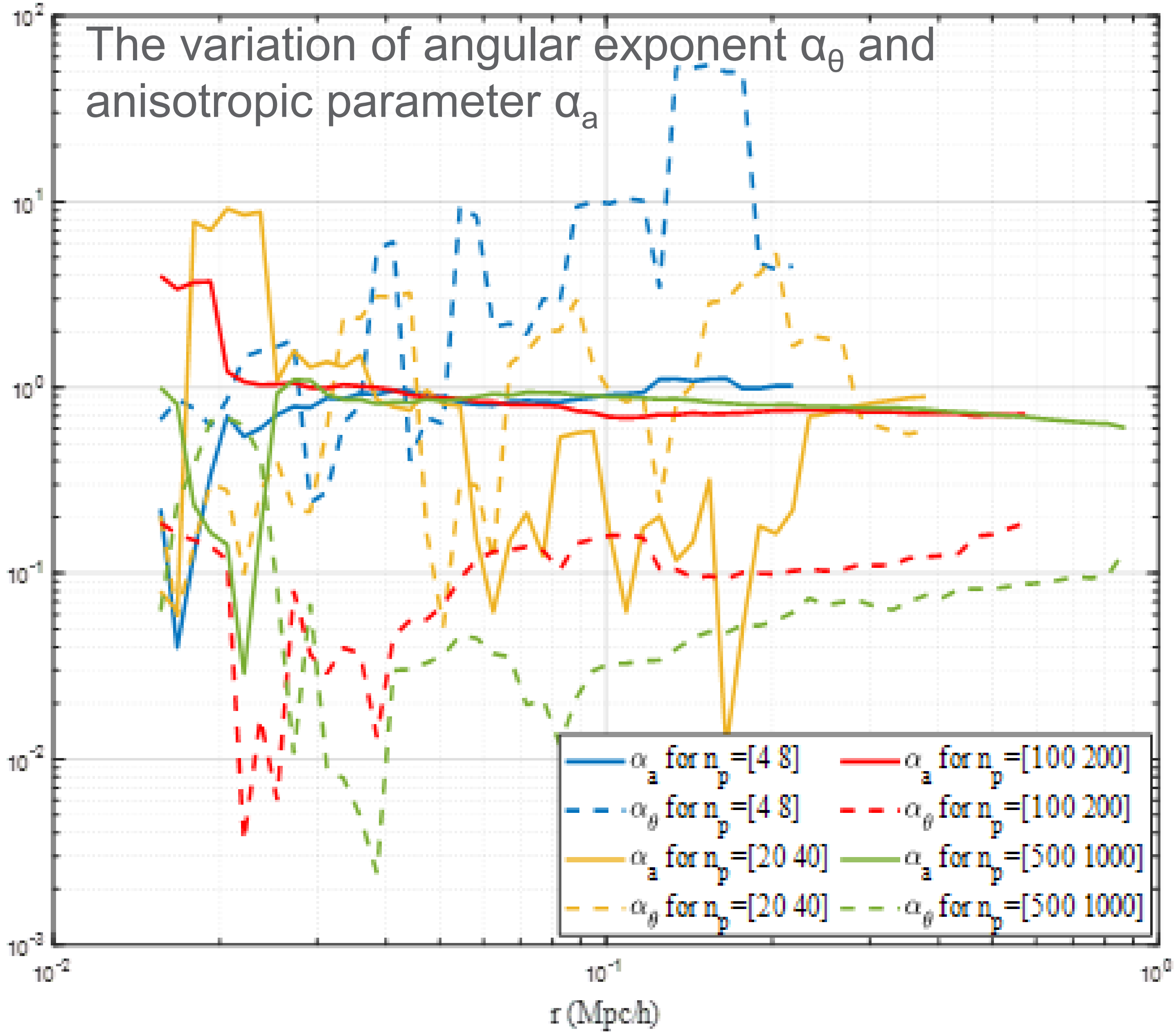
Deformation parameter: α_h

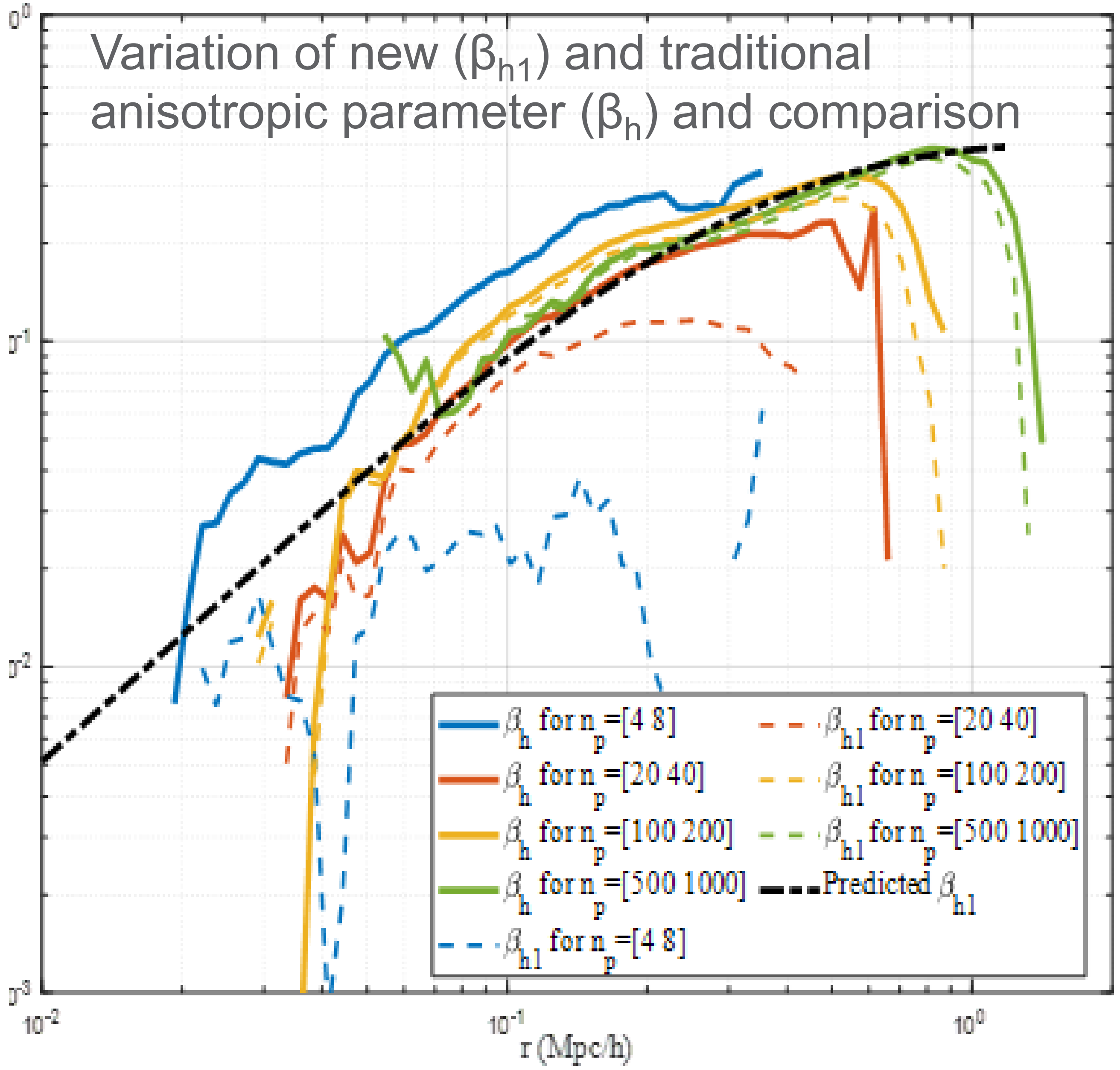
Anisotropic parameters: $\alpha_a = \dfrac{\left(\alpha_\varphi + \beta_\varphi + 1\right)}{2\gamma_\varphi}$ $\beta_{h1} = \dfrac{1-\alpha_a}{1+\sigma_{r0}^2/\left(\gamma_\varphi u_\varphi^2\right)}$



The variation of azimuthal flow from N-body simulation and comparison

Angular exponent and anisotropic parameters





Halo momentum and energy in terms of F(x)

Mean square radius:

$$r_g^2 = \frac{1}{m_h}\int_0^{r_h} 4\pi r^2 \rho_h(r) r^2 dr = r_h^2\left[1 - \frac{2}{c^2 F(c)}\int_0^c xF(x)dx\right] = \gamma_g^2 r_h^2$$

(physical) radial linear momentum:

$$L_h = \frac{1}{m_h}\int_0^{r_h} 4\pi r^2 \rho_h u_r dr = \frac{3}{2}\left(1 - \frac{2}{cF(c)}\int_0^c F(x)dx\right)Hr_h$$

(peculiar) radial linear momentum:

$$L_{hp} = \frac{1}{m_h}\int_0^{r_h} 4\pi r^2 \rho_h u_{rp} dr = \frac{1}{2}\left(1 - \frac{4}{cF(c)}\int_0^c F(x)dx\right)Hr_h$$

(physical) virial quantity:

$$G_h = \frac{1}{m_h}\int_0^{r_h} 4\pi r^3 \rho_h u_r dr = \frac{3}{2}\left[1 - \frac{3}{c^2 F(c)}\int_0^c xF(x)dx\right]Hr_h^2$$

(peculiar) virial quantity:

$$G_{hp} = \frac{1}{m_h}\int_0^{r_h} 4\pi r^3 \rho_h u_{rp} dr = \frac{1}{2}\left[1 - \frac{5}{c^2 F(c)}\int_0^c xF(x)dx\right]Hr_h^2$$

Angular momentum:

$$H_h = \left(\frac{1}{\alpha_h} - \frac{1}{3}\right)\frac{c^2}{F(c)\alpha_f}\left(G_h - G_{hp}\right) = \left(\frac{1}{\alpha_h} - \frac{1}{3}\right)\frac{c^2}{F(c)\alpha_f}Hr_g^2$$

Moment of inertia:

$$I_\omega = \frac{2}{3}m_h r_g^2$$

Angular momentum:

$$H_h = \frac{2}{3}\omega_h r_g^2$$

Specific momentum tensor:

$$\frac{1}{m_h}\int_V \mathbf{x} \otimes \mathbf{u}_p \rho_h dV = \begin{bmatrix} G_{hp}/3 & -H_h/2 & 0 \\ H_h/2 & G_{hp}/3 & 0 \\ 0 & 0 & G_{hp}/3 \end{bmatrix}$$

(physical) radial kinetic energy:

$$K_r = \frac{1}{2m_h}\int_0^{r_h} u_r^2(r,a) 4\pi r^2 \rho_h(r,a) dr$$

(peculiar) radial kinetic energy:

$$K_{rp} = \frac{1}{2m_h}\int_0^{r_h} u_{rp}^2 4\pi r^2 \rho_h(r,a) dr$$

Rotational kinetic energy:

$$K_a = \frac{1}{m_h}\int_0^{r_h} 2\pi r^3 \rho_h(r)\left(\int_0^\pi \frac{1}{2}u_\varphi^2 \sin\theta d\theta\right)dr$$

Halo spin parameters in terms of F(x)

Two definitions of spin parameters:

$$\lambda_p = \frac{H_h \left|E_h\right|^{1/2}}{Gm_h} \quad \text{and} \quad \lambda_p^{'} = \frac{H_h}{\sqrt{2} v_{cir} r_h}$$

Mean square radius:

$$r_g = \gamma_g r_h = \gamma_g a \left(\frac{2Gm_h}{\Delta_c H_0^2}\right)^{1/3}$$

Halo (specific) energy and angular momentum:

$$E_h = \Phi_h + K_h \quad \text{and} \quad H_h = \gamma_H H r_h^2$$

Virial dispersion:

$$\sigma_v^2 = -\Phi_h \frac{\gamma_v}{3} = \frac{1}{3}\gamma_\Phi \gamma_v \left(\frac{\Delta_c}{2}\right)^{1/3} \left(Gm_h H_0\right)^{2/3} a^{-1}$$

Halo (specific) potential energy:

$$\Phi_h = -\gamma_\Phi \frac{Gm_h}{r_h} = -\frac{1}{m_h}\int_0^{r_h} 4\pi r^2 \rho_h(r,a) \frac{Gm_r}{r} dr = -\frac{1}{2}\gamma_\Phi \Delta_c H^2 r_h^2 \quad \text{and} \quad \gamma_\Phi = \left(\frac{c}{F^2(c)}\int_0^c \frac{F(x)F^{'}(x)}{x}dx\right) \approx 1$$

Halo (specific) kinetic energy and rotational kinetic energy:

$$K_h = 3/2\,\sigma_v^2 = (n_e/2)\Phi_h \quad \text{and} \quad K_a \approx \frac{1}{2}\left|\mathbf{H}_h\right|\omega_h = \frac{3}{4}\left(\left|\mathbf{H}_h\right|/r_g\right)^2$$

Circular velocity:

$$v_{cir} = \sqrt{\Delta_c/2} H r_h = 3\pi H r_h$$

$$\lambda_p = \gamma_\Phi \gamma_g \sqrt{\frac{4}{3}\left(1+\frac{n_e}{2}\right)\frac{K_a}{\left|\Phi_h\right|}} = \frac{2}{3}\gamma_\Phi \gamma_g \sqrt{\gamma_v \left(1-\frac{\gamma_v}{2}\right)\frac{K_a}{\sigma_v^2}} \Rightarrow \lambda_p = \frac{\gamma_H}{3\pi}\sqrt{\gamma_\Phi \left(1+\frac{n_e}{2}\right)} \approx 0.031$$

$$\lambda_p^{'} = \gamma_g \sqrt{\frac{2\gamma_\Phi K_a}{3\left|\Phi_h\right|}} = \frac{1}{3}\gamma_g \sqrt{2\gamma_\Phi \gamma_v \frac{K_a}{\sigma_v^2}} \Rightarrow \lambda_p^{'} = \frac{\gamma_H}{3\pi\sqrt{2}} \approx 0.038$$

Spin parameters reflects the ratio between rotational and virial kinetic energy

Energy, momentum and spin parameter for NFW and isothermal halos

Table 3. Relevant parameters for two different density profiles

Symbol	Physical meaning	Equation	Isothermal profile with $\alpha_\theta = 1$	NFW profile with $\alpha_\theta = 0$ and $c = 3.5$
$F(x)$	Function for density ρ_h	Eq. (33)	x/c	$\ln(1+x) - x/(1+x)$
α_h	Deformation parameter	Eq. (66)	1.0	0.833
γ_h	Deformation rate parameter	Eq. (69)	0	$1/2$
α_f	Constant for function $F_\varphi(x)$	Eq. (77)	$2c^2/3$	9.20
λ_f	Constant for equation for γ_φ	Eq. (92)	$9\pi^2/c$	10.895
γ_H	Coefficient for H_h	Eq. (106)	$1/3$	0.511
γ_Φ	Coefficient for potential Φ_h	Eq. (113)	1	0.936
γ_v	Virial ratio	Eq. (115)	1.5	1.3
$\gamma_g^2 = r_g^2/r_h^2$	Ratio of two halo sizes	Eq. (73)	$1/3$	0.3214
L_h	Specific radial momentum	Eq. (100)	0	0
L_{hp}	Peculiar radial momentum	Eq. (101)	$-Hr_h/2$	$-0.501Hr_h$

Table 3. Relevant parameters for two different density profiles

Symbol	Physical meaning	Equation	Isothermal profile with $\alpha_\theta = 1$	NFW profile with $\alpha_\theta = 0$ and $c = 3.5$
G_h	Specific virial quantity	Eq. (102)	0	$-0.027Hr_h^2$
G_{hp}	Peculiar virial quantity	Eq. (103)	$-Hr_h^2/3$	$-0.348Hr_h^2$
H_h	Specific angular momentum	Eq. (105)	$Hr_h^2/3$	$0.511Hr_h^2$
ω_h	Angular velocity	Eq. (81)	$1.5H$	$2.38H$
K_r	Radial kinetic energy	Eq. (108)	0	$0.0062H^2r_h^2$
K_{rp}	Peculiar radial kinetic energy	Eq. (109)	$H^2r_h^2/6$	$0.1937H^2r_h^2$
K_a	Rotational kinetic energy	Eq. (110)	$H^2r_h^2/3$	$0.7658H^2r_h^2$
Φ_h	Halo potential energy	Eq. (112)	$-9\pi^2H^2r_h^2$	$-8.424\pi^2H^2r_h^2$
λ_p	First halo spin parameter	Eq. (119)	0.018	0.031
$\lambda_p^{'}$	Second halo spin parameter	Eq. (119)	0.025	0.038

The energy transfer between mean flow and random flow in “large” high v halos

Two contributions for change of halo momentum /energy:
S1: Bulk contribution from internal exchange between mean flow and random flow
S2: Surface contribution from mass cascade

Example:

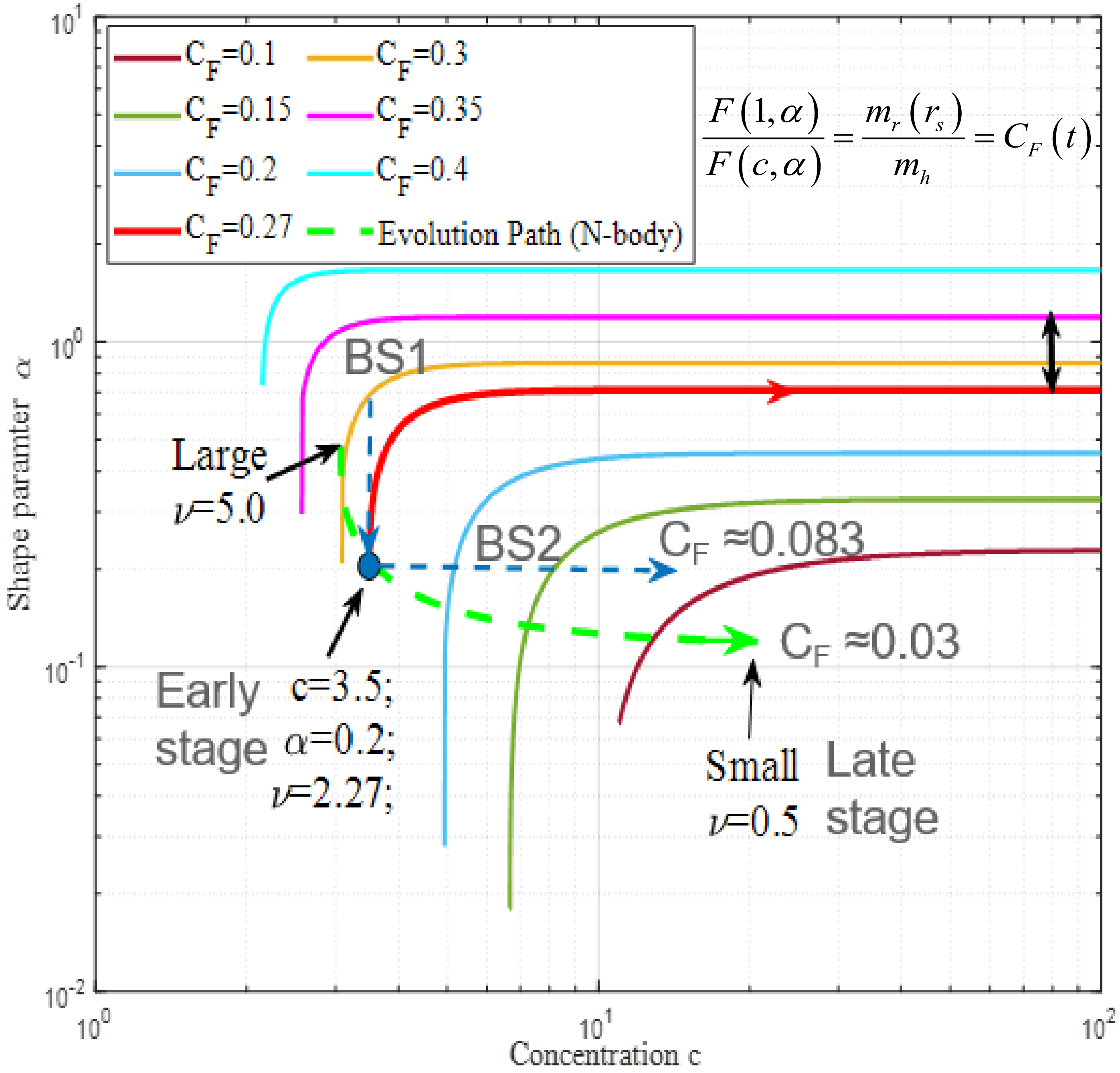
$$\frac{\partial \bar{L}_h}{\partial t} = \frac{m_h r_h}{t^2}\left[\underbrace{\left(1-\frac{1}{\alpha_h}\right)}_{S_2} + \underbrace{\left(\frac{1}{\alpha_h}-\frac{2}{cF(c)}\int_0^c F(x)\,dx\right)}_{S_1}\right]$$

- For angular momentum, all contributions from S2, i.e. mass cascade.
- For radial kinetic energy, two contributions are comparable.
- For rotational kinetic energy, contribution from S2 is dominant, i.e. mass cascade.
- In addition, local energy transfer can be two-way. S1<0 for entire halo, one-way net kinetic energy is transferred from mean flow to random motion to enhance halo entropy.

Table 4. The rate of change of halo momentum and energy for two different density profiles

Symbol	Physical meaning	Isothermal with $\alpha_\theta = 0$	NFW profile $\alpha_\theta = 0$; $c = 3.5$
$\partial \bar{L}_h/\partial t$	radial momentum	0	0
S_1	Bulk contribution	0	$0.2 m_h r_h/t^2$
S_2	Surface contribution	0	$-0.2 m_h r_h/t^2$
$\partial \bar{H}_h/\partial t$	angular momentum	$\frac{\pi}{4}\frac{m_h H r_h^2}{t}$	$\frac{\pi}{4}\frac{m_h H r_h^2}{t}\left(\frac{3}{2\alpha_h}-\frac{1}{2}\right)$
S_1	Bulk contribution	0	0
S_2	Surface contribution	$\frac{\pi}{4}\frac{m_h H r_h^2}{t}$	$\frac{\pi}{4}\frac{m_h H r_h^2}{t}\left(\frac{3}{2\alpha_h}-\frac{1}{2}\right)$
$\partial \bar{K}_r/\partial t$	radial kinetic energy	0	$0.0062 H^2 r_h^2 m_h/t$
S_1	Bulk contribution	0	$-0.0391 H^2 r_h^2 m_h/t$
S_2	Surface contribution	0	$0.0453 H^2 r_h^2 m_h/t$
$\partial \bar{K}_{rp}/\partial t$	peculiar radial kinetic energy	$H^2 r_h^2 m_h/(6t)$	$0.1937 H^2 r_h^2 m_h/t$
S_1	Bulk contribution	$-H^2 r_h^2 m_h/(3t)$	$-0.6525 H^2 r_h^2 m_h/t$
S_2	Surface contribution	$H^2 r_h^2 m_h/(2t)$	$0.8462 H^2 r_h^2 m_h/t$
$\partial \bar{K}_a/\partial t$	rotational kinetic energy	$H^2 r_h^2 m_h/(2t)$	$0.7661 H^2 r_h^2 m_h/t$
S_1	Bulk contribution	0	$\boxed{-0.0801 H^2 r_h^2 m_h/t}$
S_2	Surface contribution	$H^2 r_h^2/2$	$0.8462 H^2 r_h^2 m_h/t$

Halo relaxation (stretching) from early to late stages



- Two-parameter Einasto profile for relaxation
- The path of evolution in c-α space (shape parameter vs. concentration)
- Contour for constant core/halo mass ratio C_F
- Evolution path from N-body simulation (green)
- Simplified path for analytical calculation (blue)
  Blue segment 1 (BS1): constant c≈3.5
  Blue segment 2 (BS2): constant α≈0.2
- Path to composite halos with α≈0.7 (red) follows a constant C_F = 0.27; Adiabatic process
- Goal: explore the continuous variation of halo shape, density profile, mean flow, momentum, and energies during halo relaxation.

Decomposition of radial flow

Extend key function F(x) to two-parameter function F(x,α), where α is a shape parameter:

$$\rho_h(r,t)=\frac{1}{4\pi r^2}\frac{\partial m_r(r,a)}{\partial r}=\frac{m_h(t)}{4\pi r_s^3}\frac{F'(x,\alpha)}{x^2F(c,\alpha)}$$

Enclosed mass: $m_r(r,t)=m_h(t)\dfrac{F(x,\alpha)}{F(c,\alpha)}$

$$\frac{\partial\rho_h(r,a)}{\partial t}=\frac{1}{4\pi r^2}\frac{\partial^2 m_r(r,a)}{\partial r\partial t}$$

$$\frac{\partial m_r(r,a)}{\partial t}=-4\pi r^2 u_r(r,a)\rho_h(r,a)$$ (From continuity equation)

$$u_h=u_{hm}+u_{hc}+u_{h\alpha}$$

From mass cascade:

$$u_{hm}=x\frac{\partial\ln r_s}{\partial\ln t}-\frac{F(x,\alpha)}{F'(x,\alpha)}\frac{\partial\ln m_h}{\partial\ln t}$$

From conc. change:

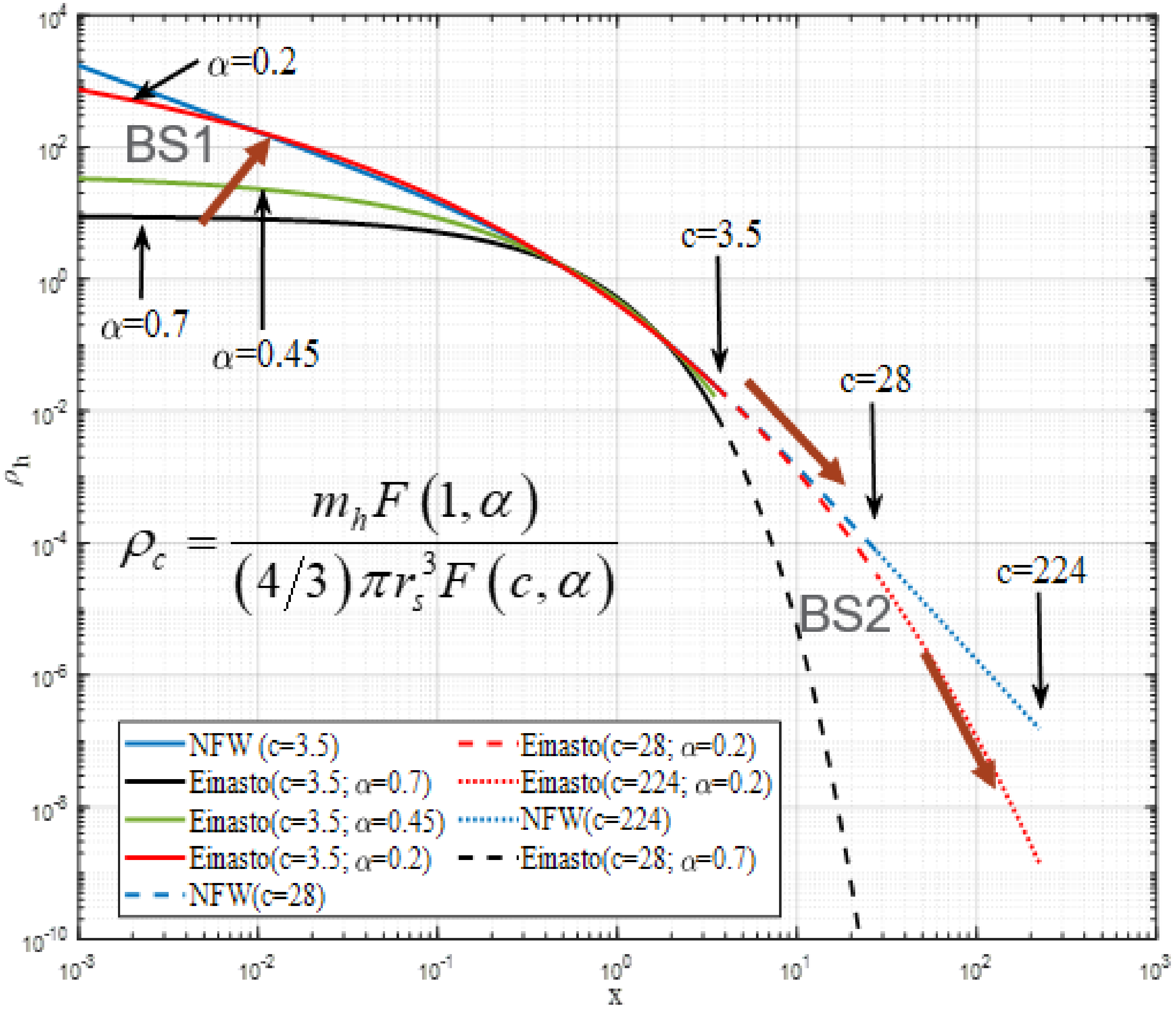
$$u_{hc}=\frac{\partial\ln c}{\partial\ln t}\frac{F(x,\alpha)}{F'(x,\alpha)}\frac{\partial\ln F(c,\alpha)}{\partial\ln c}$$

From shape change:

$$u_{h\alpha}=\frac{\partial\ln\alpha}{\partial\ln t}\frac{F(x,\alpha)}{F'(x,\alpha)}\left[\frac{\partial\ln F(c,\alpha)}{\partial\ln\alpha}-\frac{\partial\ln F(x,\alpha)}{\partial\ln\alpha}\right]$$

- Early stage “large” halos: u_{hc}=0 and $u_{h\alpha}$=0 radial flow from cascade u_{hm} is dominant;
- Late stage “small” halos: all three radial flows vanishes and u_h=0;
- For halo “relaxation” from early to late stage (BS2), we expect a constant r_s, constant α,m_h~F(c,α), $u_{h\alpha}$=0, and $u_{hm}+u_{hc}$=0

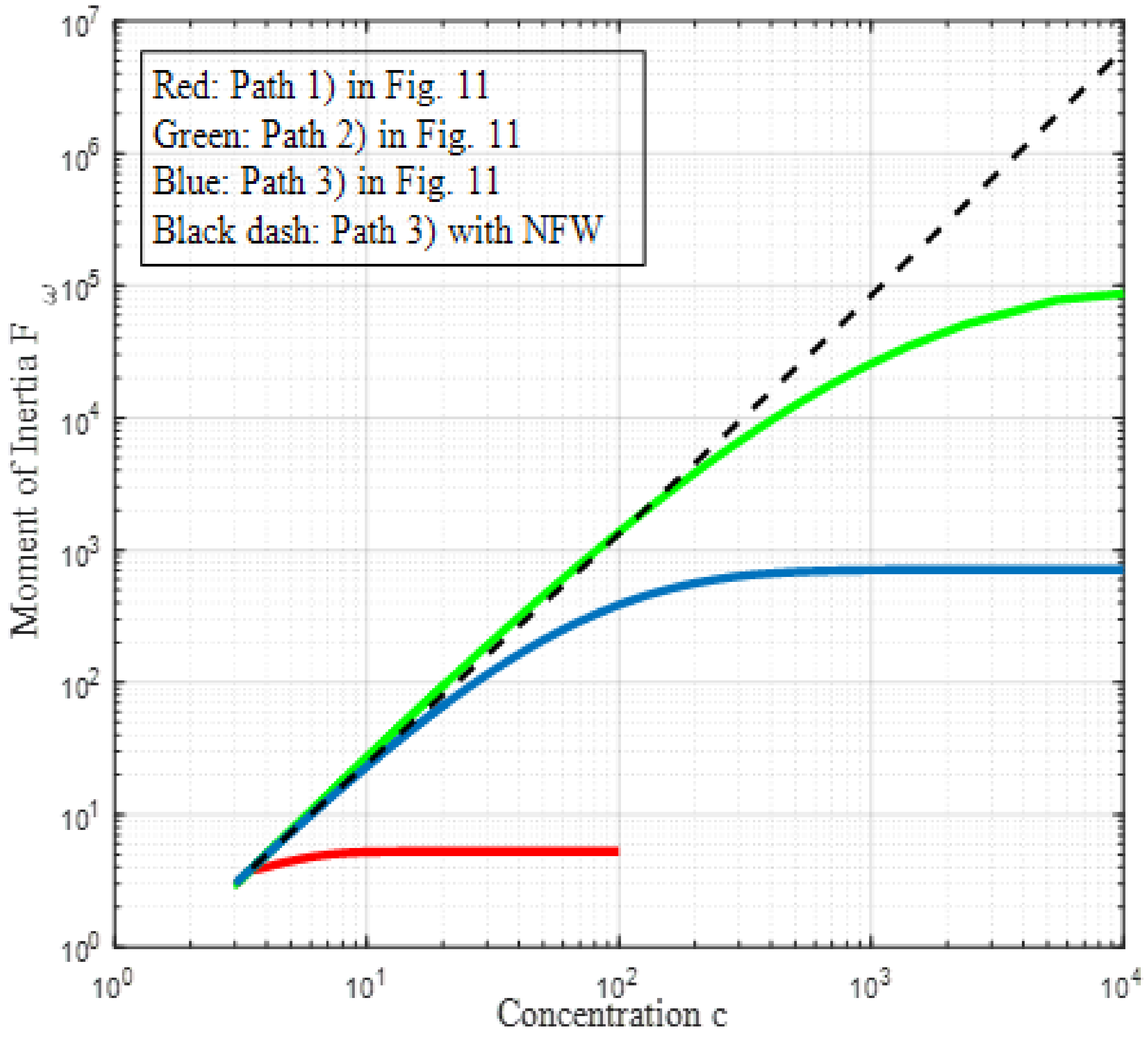
Density profile from early to late stages



Variation of halo density normalized by the average core density ρ_c (with $r<r_s$)

- During BS1 with constant c≈3.5 and constant C_F, decreasing α involves significant change of density in halo core, i.e. steeper density slope and increasing core mass.
- During BS2 with constant α≈0.2, increasing *c* involves a stable core (constant scale radius r_s, constant core mass, and core density ρ_c) and extending halo skirt ("halo stretching" vs. "vortex stretching" in turbulence).
- Vortex stretching: anisotropic, volume conserving, constant density, and decreasing momentum of inertia.
- Halo stretching: isotropic, increasing volume, varying density, and increasing momentum of inertia.

Moment of inertia from early to late stage



Variation of moment of inertia with concentration c

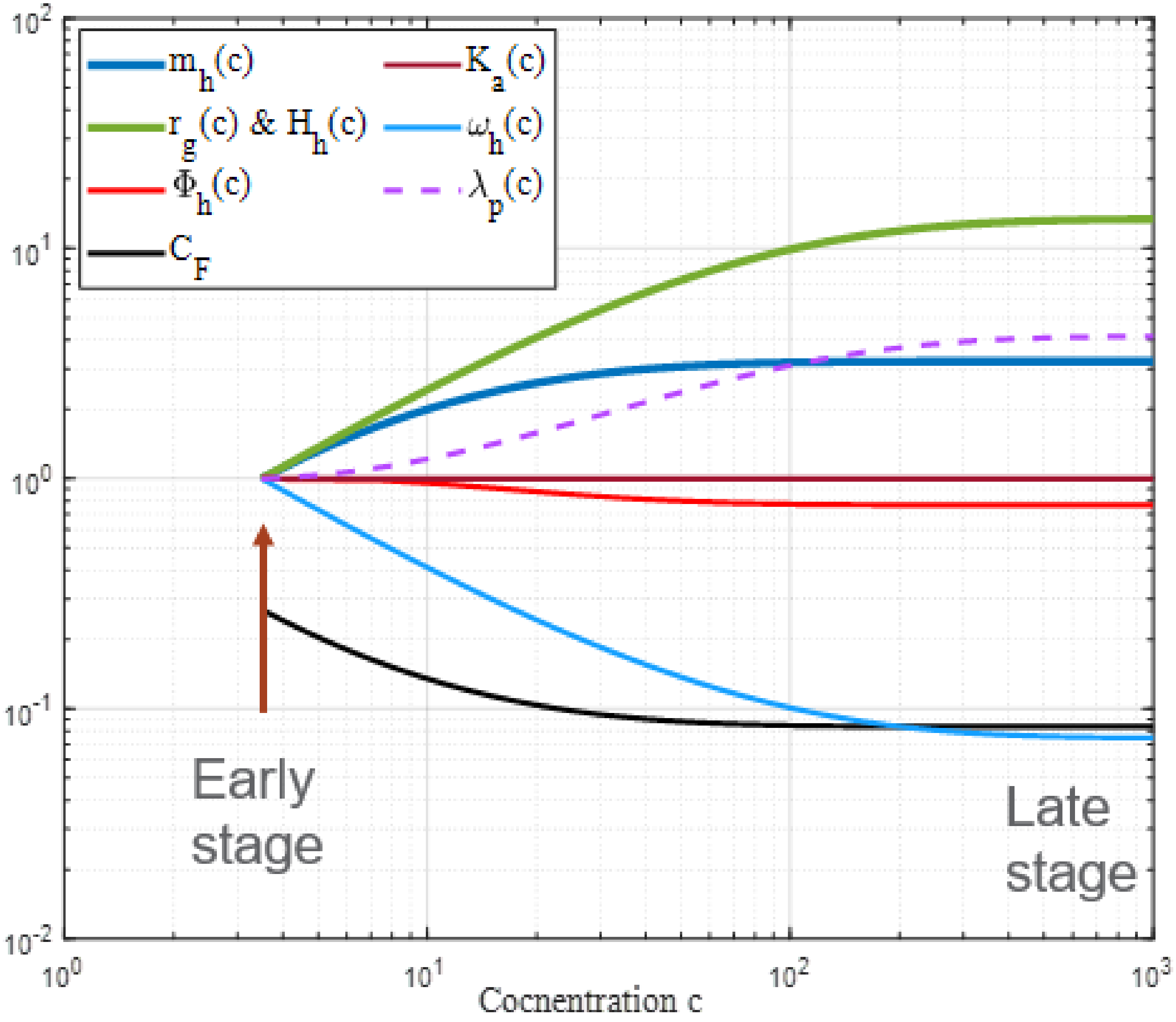
Moment of inertia: $I_\omega = \frac{2}{3} m_h r_g^2 = \frac{2}{3} m_h r_s^2 F_\omega(\alpha, c)$

Mean square radius: $r_g(c) = r_s \sqrt{F_\omega(c)}$

$$F_\omega(\alpha, c) = \left(\frac{\alpha}{2}\right)^{\frac{2}{\alpha}} \frac{\Gamma(5/\alpha) - \Gamma(5/\alpha, 2c^\alpha/\alpha)}{\Gamma(3/\alpha) - \Gamma(3/\alpha, 2c^\alpha/\alpha)}$$

- Red path is adiabatic with constant halo mass, with both angular momentum and rotational energy conserved.
- Green path from simulation shows significant increase in moment of inertia from halo “stretching”.
- Simplified blue path with constant r_s and core mass shows the increase in moment of inertia that plateaus at large c.

Variation of mass, moment, energy during relaxation



Variation of halo momentum and energies during halo relaxation

Specific rotational kinetic energy: $K_a = \frac{1}{2}|\mathbf{H}_h|\omega_h = \frac{3}{4}\left(|\mathbf{H}_h|/r_g\right)^2$

Specific angular momentum: $|\mathbf{H}_h| = \frac{2}{3}\omega_h r_g^2$

λ_p: spin parameter

For early stage "large" halos:

$\lambda_p \approx 0.031$ $m_h \propto t$ $|\mathbf{H_h}| \propto t$ $\boxed{K_a \propto t^0}$

$C_F = 0.27$ $r_g \propto t$ $\omega_h \propto t^{-1}$ $\Phi_h \propto t^0$

Spin-dispersion dominant

For late stage "small" halos:

$\lambda_p \approx 0.124$ $m_h \propto t^0$ $|\mathbf{H_h}| \propto t^0$ $\boxed{K_a \propto t^0}$

$C_F = 0.083$ $r_g \propto t^0$ $\omega_h \propto t^0$ $\Phi_h \propto t^0$

Axial-dispersion dominant

- Halo "relaxation" (via BS2): with constant α≈0.2, increasing c, constant r_s , core mass, and core density
- Specific rotational kinetic energy is relatively conserved
- $|\mathbf{H_h}| \propto r_g$ $\omega_h \propto r_g^{-1}$
- Spin-dispersion dominant to axial-dispersion dominant

Summary and keywords

Early stage "large" halos	Late stage "small" halos	Core mass ratio	Axial dispersion
Vortex stretching	Halo stretching	Fictitious stress	Spin dispersion
Path of halo evolution "relaxation"	Radial flow decomposition	Energy transfer	

- Review one-way energy transfer via vortex stretching in turbulence;
- Halos enable a two-way energy transfer between mean flow and random motion;
- Analytical solutions of mean flow, velocity dispersion, and anisotropy parameters for halos at their early stage and late stage using decomposition of velocity dispersion.
- "Early-stage" halos have their mass, size, kinetic/potential/rotational energy, and the specific angular momentum all increase linearly with time via continuous mass acquisition. Halo core spins faster than the outer region.
- "Late-stage" halos are more spherical in shape, incompressible, and isotropic. Due to finite halo spin, kinetic energy is not equipartitioned along each direction with the greatest energy along the azimuthal direction. Halos are hotter with faster spin.
- Identify the path of relaxation via halo stretching for halos relaxing from early to late stage involving continuous variation of shape, density profile, mean flow, momentum, and energy.
- Might extend to consider effect of black hole at halo center on radial flow

Maximum entropy distributions in dark matter flow

Xu Z., 2021, arXiv:2110.03126v1 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2110.03126

Maximum entropy distributions in kinetic theory of gases

Review on how to derive maximum entropy distributions (Boltzmann distribution)

Assume the distribution of one-dimensional gas molecule velocity is some unknown function X(v)

Two constraints on X(v), normalization and fixed mean kinetic energy:

$$\int_{-\infty}^{\infty} X(v)\,dv = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} X(v)\,\varepsilon(v)\,dv = \frac{3}{2}k_B T = \frac{3}{2}\sigma_0^2$$

Particle energy:

$$\varepsilon(v) = 3v^2/2$$

Write down the entropy functional with Lagrangian multiplier:

$$S[X(v)] = -\int_{-\infty}^{\infty} X(v)\ln X(v)\,dv + \lambda_1\left(\int_{-\infty}^{\infty} X(v)\,dv - 1\right) + \lambda_2\left(\int_{-\infty}^{\infty} X(v)\,\varepsilon(v)\,dv - \frac{3}{2}\sigma_0^2\right)$$

This is the key to be identified for dark matter flow

Taking the variation of the entropy functional with respect to distribution X:

$$\frac{\delta S(X(v))}{\delta X} = -\ln X(v) - 1 + \lambda_1 + \lambda_2\varepsilon(v) = 0 \;\Rightarrow\; X(v) \propto \exp\left(\lambda_2 v^2\right)$$

Boltzmann distribution

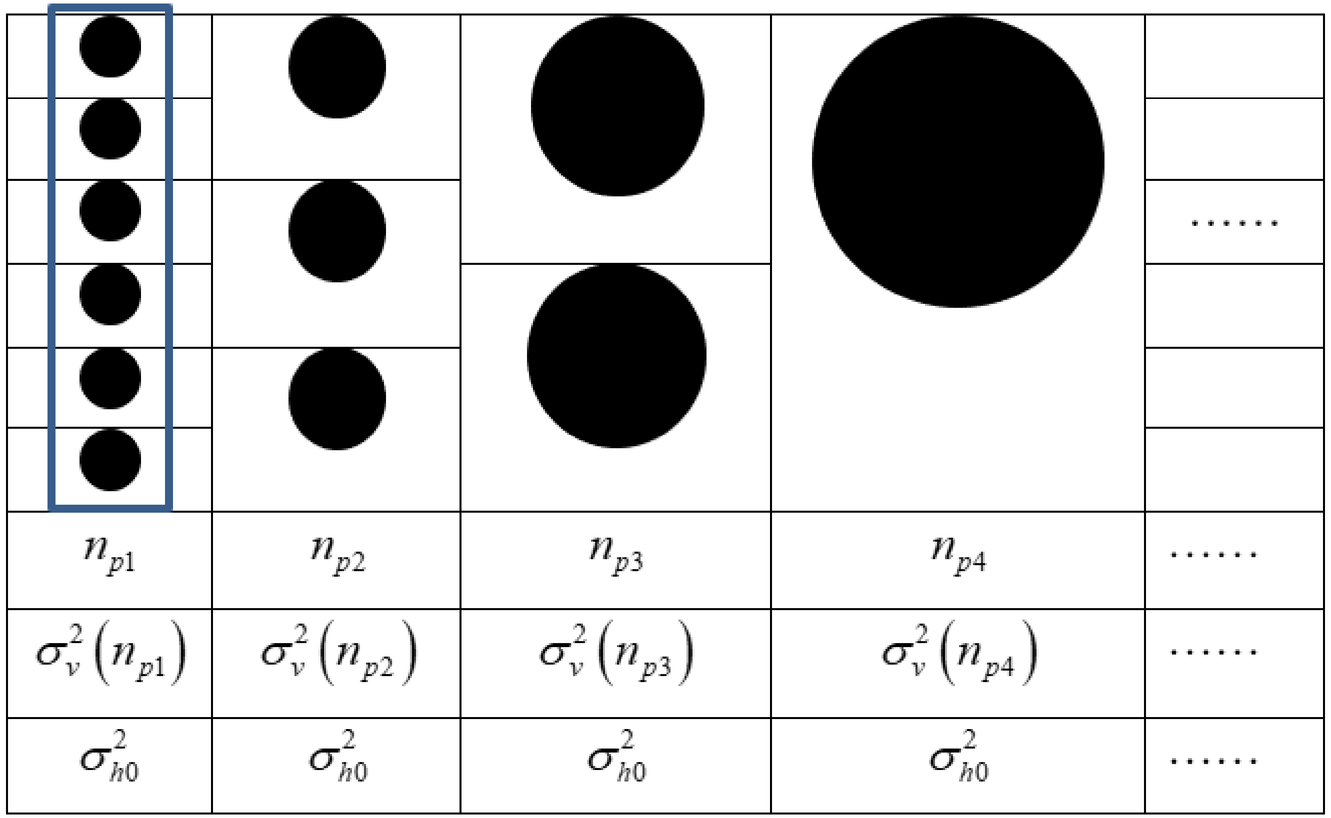
Distribution for particle energy:

$$E(\varepsilon) = 2\sqrt{\frac{\varepsilon}{\pi\sigma_0^2}}\frac{1}{\sigma_0^2}e^{-\varepsilon/\sigma_0^2}$$

Maxwell-Boltzmann distribution for speed:

$$Z(v) = \sqrt{\frac{2}{\pi}}\frac{v^2}{\sigma_0^3}e^{-v^2/2\sigma_0^2}$$

Maximum entropy distributions in dark matter flow



- Long-range and collisionless nature
- Identify all halos of different sizes at given *z*
- Group halos according to halo size n_p

Goal: maximum entropy distributions in DMF

Symbol	Physical meaning
X(v)	Distribution of one-dimensional particle velocity *v*
Z(v)	Distribution of particle speed
E(ε)	Distribution of particle energy ε
$H(\sigma_v^2)$	Distribution of particle virial dispersion σ_v^2 (halo mass function)

A general power-law for two-body potential:

$$V(r) \propto r^n$$

n=-1 for standard gravity

Velocity and dispersion decomposition

Decompose particle velocity into halo velocity and velocity fluctuation ("Reynolds decomposition")

$$\mathbf{v}_p = \mathbf{v}_h + \mathbf{v}_p'$$

Similarly, decompose velocity dispersion into halo velocity dispersion and halo virial dispersion

$$\sigma^2 = \sigma_h^2 + \sigma_v^2$$

Halo group temperature

Halo temperature

$$\sigma_h^2 = \mathrm{var}\left(\mathbf{v}_\mathbf{h}\right)$$

Halo group temperature is independent of halo size

$$\sigma_v^2 = \mathrm{var}\left(\mathbf{v}_p'\right) \propto m_h^{1+n/3}$$

Gaussian velocity distribution (Maxwell-Boltzmann statistics) is expected for all particles in the same halo group.

$$X(v) = \int_0^\infty \boxed{\frac{1}{\sqrt{2\pi\sigma}} e^{-v^2/2\sigma^2}} H\left(\sigma_v^2\right) d\sigma_v^2$$

weighted average

Boltzmann distribution

# of particles in halo group

$$Z(v) = \int_0^\infty \boxed{\sqrt{\frac{2}{\pi}} \frac{v^2}{\sigma^3} e^{-v^2/2\sigma^2}} H\left(\sigma_v^2\right) d\sigma_v^2$$

weighted average

Maxwell-Boltzmann distribution

# of particles in halo group

Relation between X and Z:

$$Z(v) = -2v \frac{\partial X}{\partial v}$$

Particle energy in dark matter flow

In a given halo group, from Virial Theorem:

$$2\langle KE\rangle_g - n\langle PE\rangle_g = 0$$

The specific kinetic energy of particle in that group:

$$\langle KE\rangle_g = (3/2)\sigma^2$$

The total specific energy of particle in group:

$$\varepsilon_h = \langle KE\rangle_g + \langle PE\rangle_g = \left(\frac{3}{2}+\frac{3}{n}\right)\sigma^2$$

Energy distribution with respect to particle speed v:

$$\langle \varepsilon_h\rangle = \int_0^\infty \varepsilon_v(v)\,dv$$

Mean particle energy
for entire system

Energy distribution with respect to particle speed

$$N\varepsilon_v(v)\,dv = \left(\underbrace{\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{v^2}{\sigma^3}e^{-v^2/2\sigma^2}dv}_{2}\,\underbrace{N\varepsilon_h(\sigma_v^2)H(\sigma_v^2)\,d\sigma_v^2}_{1}\right)$$

For entire system, energy of all particles with a speed of v:

$$\varepsilon_v(v)\,dv = 2\left(\frac{3}{2}+\frac{3}{n}\right)v^2X(v)\,dv$$

Energy per particle with a speed of v:

$$\varepsilon(v) = \frac{\varepsilon_v(v)\,dv}{Z(v)\,dv} = \frac{X(v)v^2}{Z(v)}\left(3+\frac{6}{n}\right)$$

$$Z(v) = -2v\frac{\partial X}{\partial v}$$

# of particles
with speed v

Energy per particle with a speed of v:

$$\varepsilon(v) = -\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

Maximum entropy distributions in dark matter flow

Deriving maximum entropy distributions in dark matter flow (X distribution)

Two constraints on X(v):

$$\int_{-\infty}^{\infty} X(v)\,dv = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} X(v)\,\varepsilon(v)\,dv = \frac{3}{2}k_B T = \frac{3}{2}\sigma_0^2$$

Particle energy:

$$\varepsilon(v) = -\frac{X(v)\,v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

Write down the entropy functional with Lagrangian multiplier:

$$S\left[X(v)\right] = -\int_{-\infty}^{\infty} X(v)\ln X(v)\,dv + \lambda_1\left(\int_{-\infty}^{\infty} X(v)\,dv - 1\right) + \lambda_2\left(\int_{-\infty}^{\infty} X(v)\,\varepsilon(v)\,dv - \frac{3}{2}\sigma_0^2\right)$$

This is the key

Taking the variation of the entropy functional with respect to X:

$$\frac{\delta S(X(v))}{\delta X} = -\ln X(v) - 1 + \lambda_1 + \lambda_2\varepsilon(v) = 0 \;\Rightarrow\; X(v) = \frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)}$$

The X distribution

E distribution for particle energy:

$$E(\varepsilon) = -\frac{2n}{3(n+2)}\frac{e^{-\gamma}\sqrt{\gamma^2-\alpha^2}}{\alpha K_1(\alpha)v_0^2}$$

Z distribution for speed:

$$Z(v) = \frac{1}{\alpha K_1(\alpha)}\cdot\frac{v^2}{v_0^3}\cdot\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{\sqrt{\alpha^2+(v/v_0)^2}}$$

Maximum entropy distributions in dark matter flow

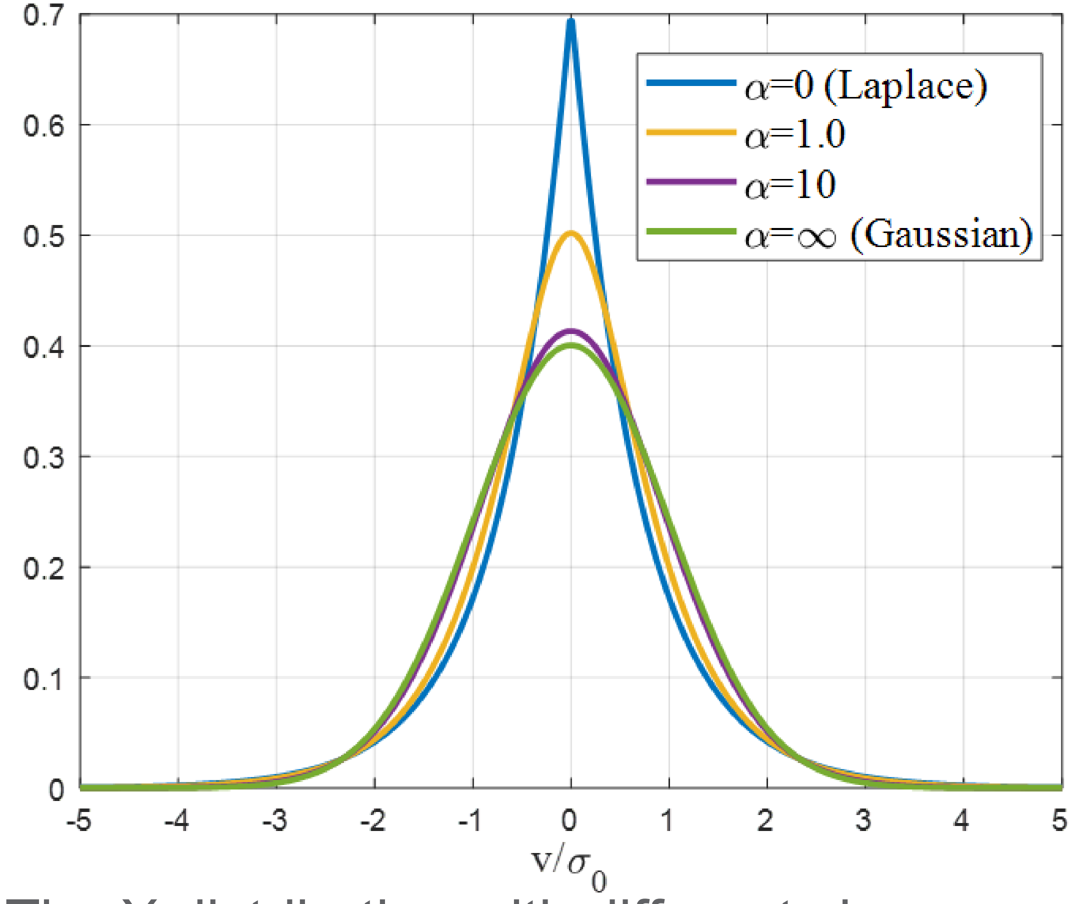
Gaussian core for $|v| \ll v_0$

$$X(v) = \frac{e^{-\alpha}}{2\alpha v_0 K_1(\alpha)} \exp\left(-\frac{v^2}{2\alpha v_0^2}\right)$$

Exponential wings for $|v| \gg v_0$

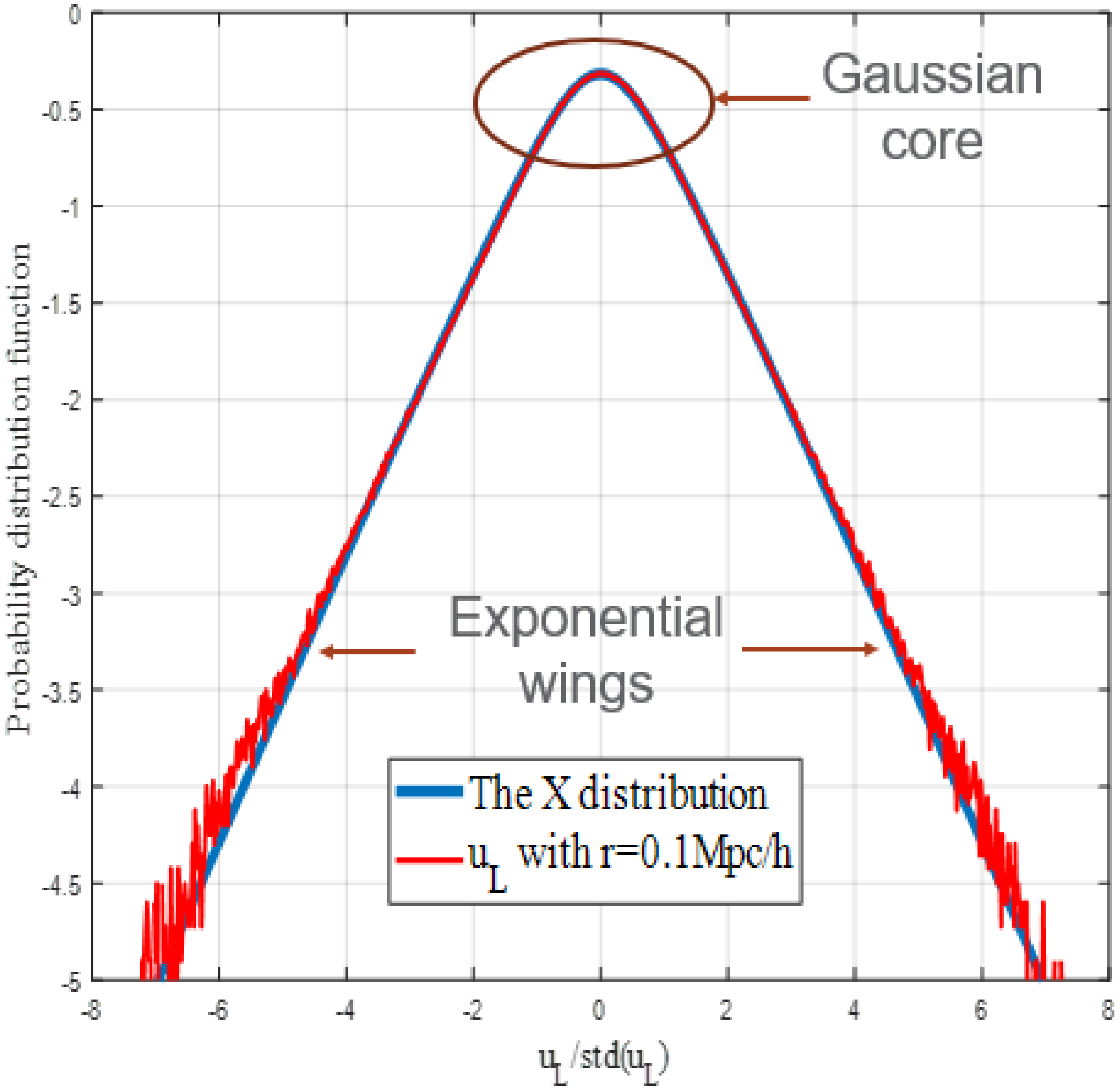
$$X(v) = \frac{1}{2\alpha v_0 K_1(\alpha)} \exp\left(-\frac{v}{v_0}\right)$$

Bessel function



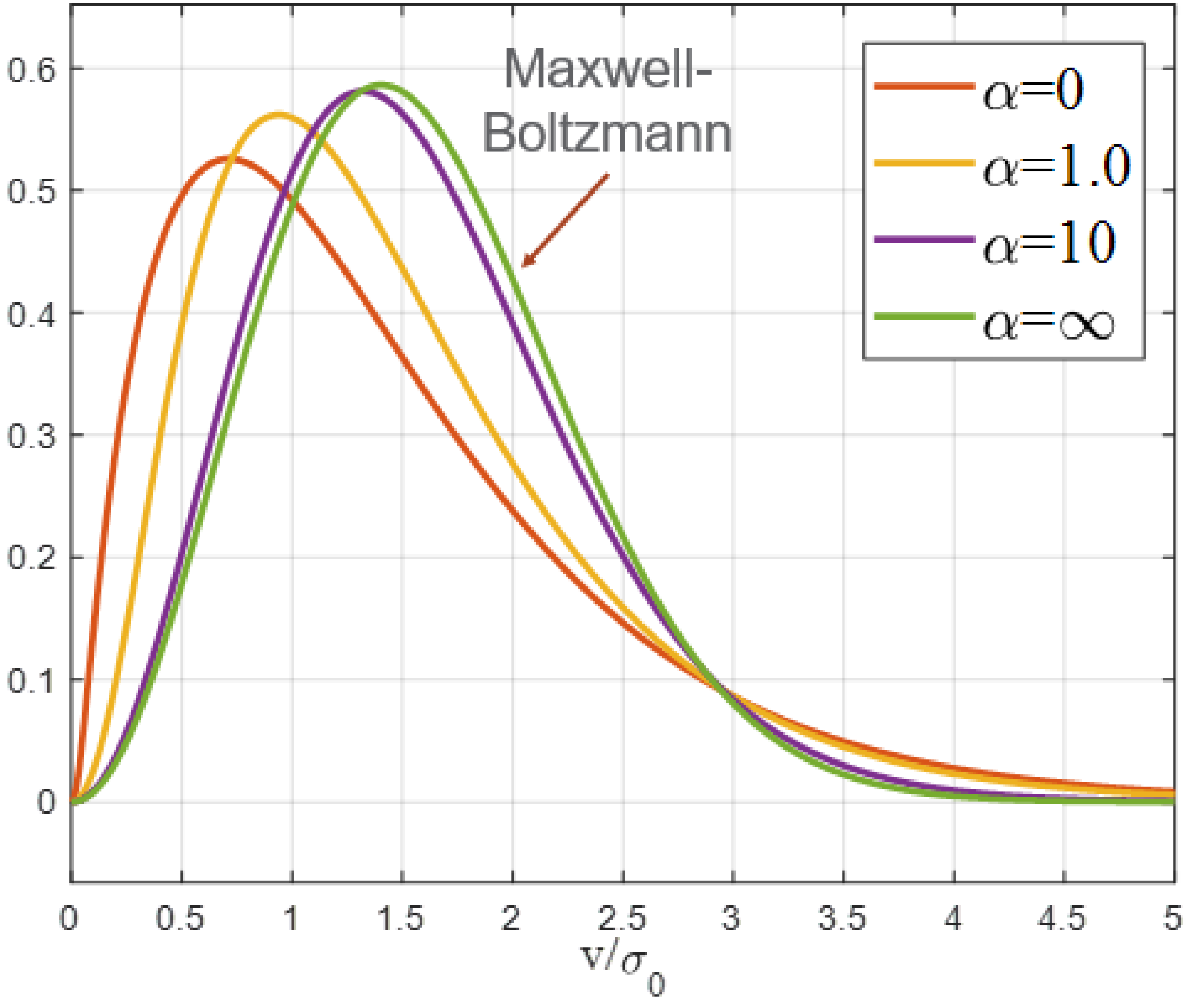
The X distribution with different shape parameter α

X is a two-parameter distribution with shape parameter α and velocity scale v_0

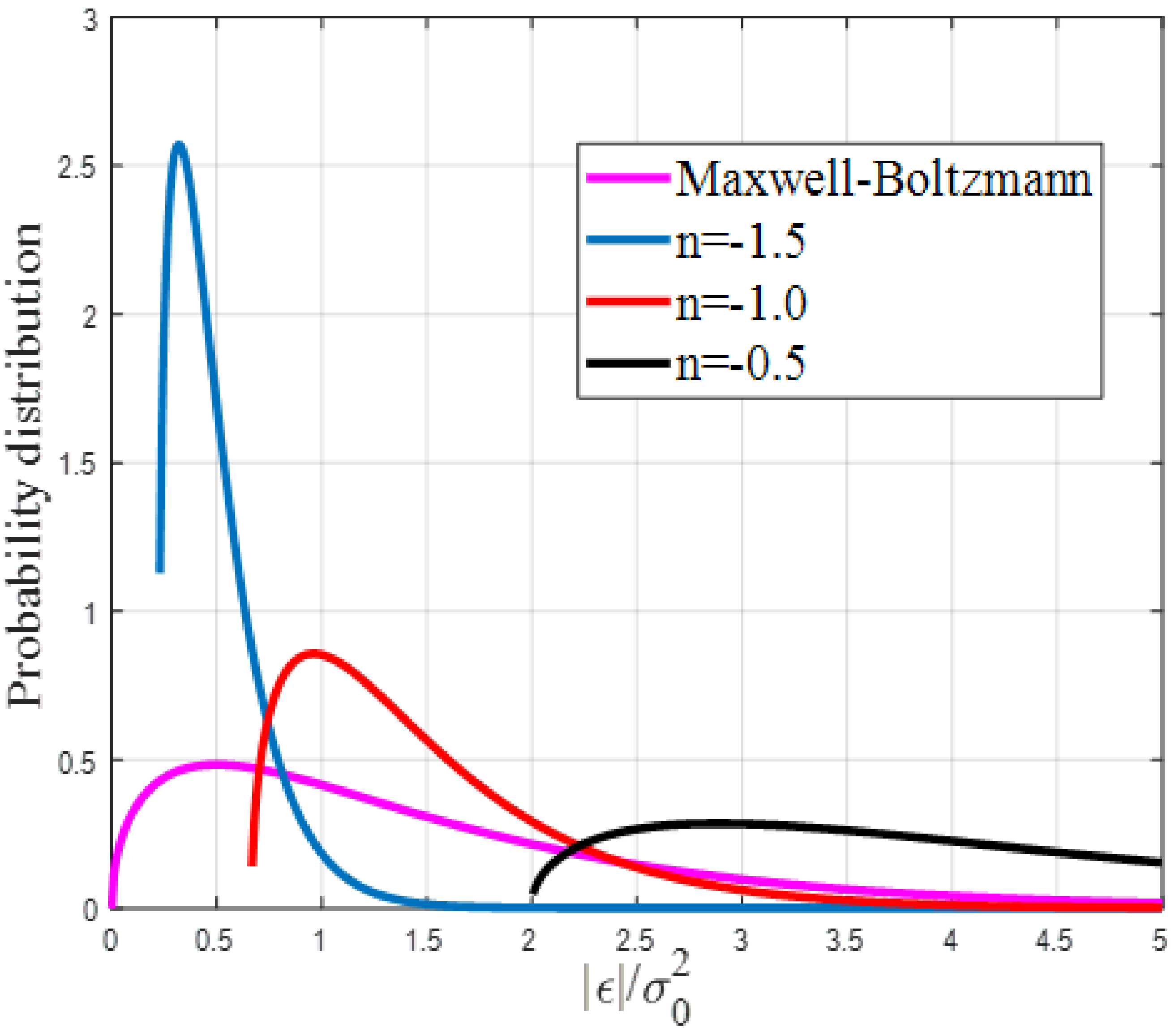


Comparison with N-body simulation

Maximum entropy distributions in dark matter flow



The Z distribution for particle speed with different shape parameter α



The E distribution for particle energy with different potential exponent n

Particle energy in dark matter flow

$$X(v)=\frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)} \qquad \varepsilon(v)=-\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

Particle energy: $\varepsilon(v)=\frac{3}{2}\left(1+\frac{2}{n}\right)v_0^2\sqrt{\alpha^2+\left(\frac{v}{v_0}\right)^2}$

Gaussian core for $|v|\ll v_0$

$$\varepsilon(v)\approx\frac{3}{2}\left(1+\frac{2}{n}\right)\left(\alpha v_0^2+\frac{v^2}{2\alpha}\right)\propto v^2$$

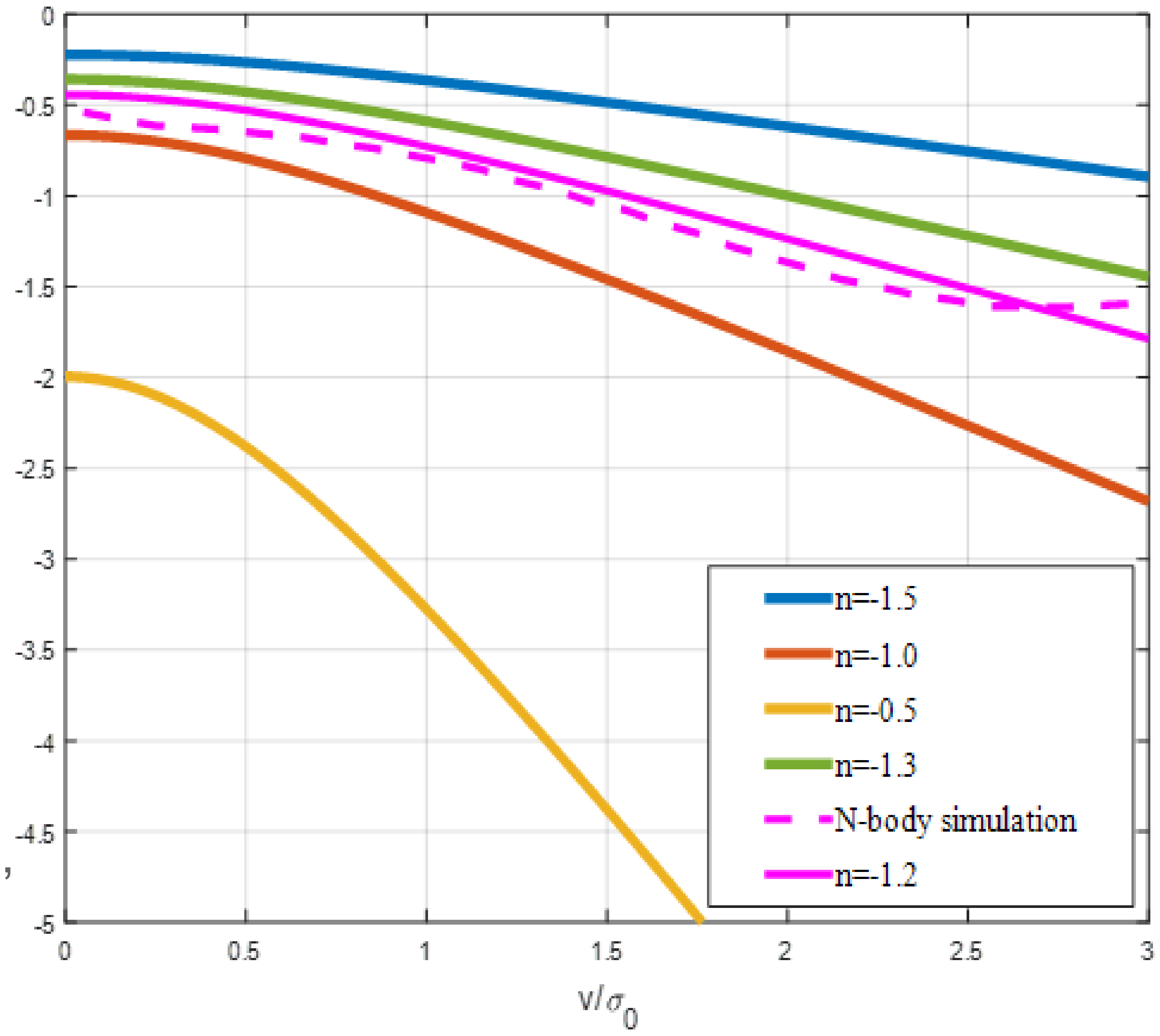
Inner halo, Newtonian behavior

Exponential wings for $|v|\gg v_0$

$$\varepsilon(v)\approx\frac{3}{2}\left(1+\frac{2}{n}\right)v_0 v\propto v$$

Outer region of halo, non-Newtonian behavior

[External field effects and MOND??]



Comparison with N-body simulation

Summary and key words

Maximum entropy	Velocity distribution	Entropy functional
Speed distribution	Energy distribution	Particle energy
Gaussian core & exponential wings	Shape parameter	Velocity scale

- Statistical theory for maximum entropy distributions of velocity, speed, and energy in dark matter flow
- Halo mass function can be a direct result to maximizing system entropy
- Maximum entropy velocity distribution (X distribution) naturally exhibits a Gaussian core at small velocity and exponential wings at large velocity (as observed from N-body simulations)
- Kinetic energy of dark matter particles follows a parabolic scaling for small speed ($\varepsilon \sim v^2$, Newtonian) and linear scaling ($\varepsilon \sim v$, non-Newtonian) for large speed. This might be relevant for "deep-MOND" behavior.

Halo mass functions from maximum entropy distributions in collisionless dark matter flow

arXiv:2110.09676 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2110.09676

Introduction

- Halo mass function, the most fundamental quantity
- Conventional Mass function from nonlinear collapse
  - Press-Schechter (PS) formalism
    - Threshold overdensity from spherical collapse
  - Extended PS using an excursion set approach
    - Overdensity as a random walk process
  - ST model
    - Ellipsoidal collapse model gives a mass-dependent overdensity threshold
- Mass function from mass cascade in dark matter flow
  - Double-λ mass function
    - Assume two different halo geometry parameter λ for different size of halos.
- The mass/energy cascade as an intermediate statistically steady state for non-equilibrium systems to continuously maximize system entropy.

Are there or what are the connections between halo mass function and maximum entropy??

$$\nu = \delta_c^2 / \sigma_\delta^2 (m_h) \qquad \delta_c = 1.6865$$

$$f_{PS}(\nu) = \frac{1}{\sqrt{2\pi}\sqrt{\nu}} e^{-\nu/2} \qquad \int_0^\infty f(\nu) d\nu = 1$$

$$f_{ST}(\nu) = A\sqrt{\frac{2q}{\pi}}\left(1 + \frac{1}{(q\nu)^p}\right)\frac{1}{2\sqrt{\nu}} e^{-q\nu/2}$$

$A = 0.32 \quad q = 0.75 \quad p = 0.3$

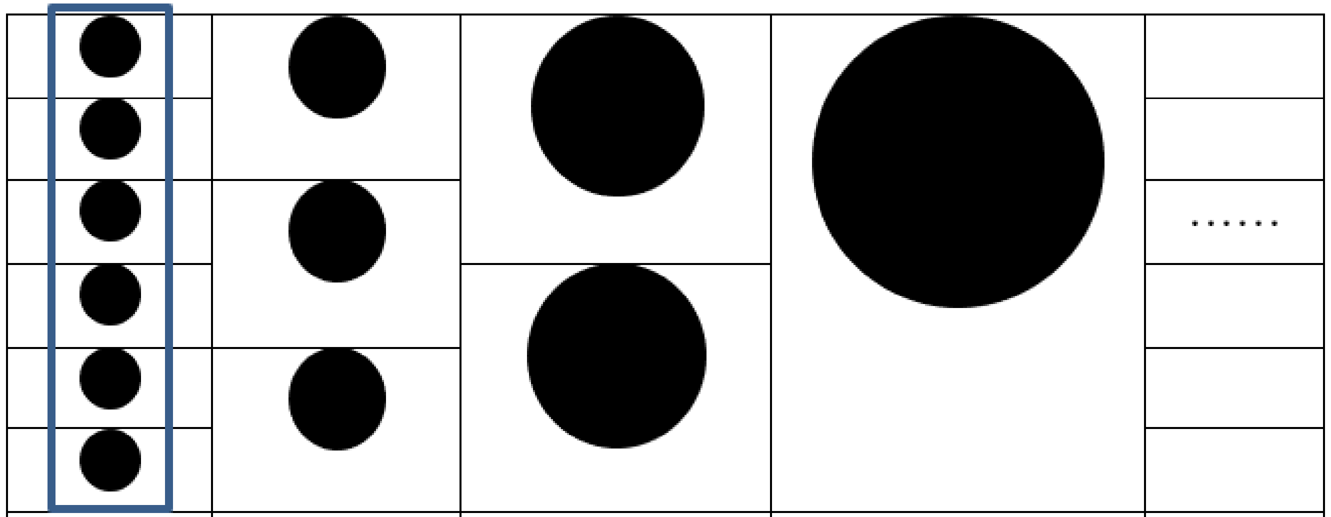
$A = 0.5 \quad q = 1.0 \quad p = 0 \rightarrow f_{PS}(\nu)$

$$f_{D\lambda}(\nu) = \frac{\left(2\sqrt{\eta_0}\right)^{-q}}{\Gamma(q/2)} \nu^{q/2-1} \exp\left(-\frac{\nu}{4\eta_0}\right)$$

$\eta_0 = 0.76 \quad q = 0.556$

$\eta_0 = 0.5 \quad q = 1 \rightarrow f_{PS}(\nu)$

Maximum entropy distributions



n_{p1}	n_{p2}	n_{p3}	n_{p4}	......
$\sigma_v^2(n_{p1})$	$\sigma_v^2(n_{p2})$	$\sigma_v^2(n_{p3})$	$\sigma_v^2(n_{p4})$	......
σ_{h0}^2	σ_{h0}^2	σ_{h0}^2	σ_{h0}^2	......

- Long-range and collisionless nature
- Identify all halos of different sizes at given *z*
- Group halos according to halo size n_p

Symbol	Physical meaning
X(v)	Distribution of one-dimensional particle velocity *v*
Z(v)	Distribution of particle speed v
E(ε)	Distribution of particle energy ε
$H(\sigma_v^2)$	Distribution of particle virial dispersion σ_v^2 (halo mass function)
$J(\sigma_v^2)$	Distribution of halos with virial dispersion σ_v^2
$P(v^2)$	Distribution of square of one-dimensional particle velocity v

$$n_p \equiv n_p\left(\sigma_v^2\right) \qquad \left\langle \sigma_v^2 \right\rangle = \int_0^\infty H\left(\sigma_v^2\right)\sigma_v^2 d\sigma_v^2$$

$$V(r) \propto r^n \qquad \text{n=-1 for standard gravity}$$

$$\left\langle \sigma_h^2 \right\rangle \equiv \bar{\sigma}_h^2 = \int_0^\infty H\left(\sigma_v^2\right)\sigma_h^2 d\sigma_v^2 \quad \left\langle \sigma^2 \right\rangle = \left\langle \sigma_v^2 \right\rangle + \left\langle \sigma_h^2 \right\rangle = \sigma_0^2$$

Relations between maximum entropy distributions

$$X(v)=\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma} e^{-v^2/2\sigma^2} H(\sigma_v^2)\, d\sigma_v^2$$

$$\int_{-\infty}^\infty X(v) e^{-vt} dv = \int_0^\infty H(\sigma_v^2) e^{\sigma^2 t^2/2} d\sigma_v^2$$

$$P(x=v^2)=\int_0^\infty \frac{1}{\sqrt{2\pi x}\sigma} e^{-x/2\sigma^2} H(\sigma_v^2)\, d\sigma_v^2$$

$$\int_0^\infty P(x) e^{-xt} dx = \int_0^\infty H(\sigma_v^2) \frac{1}{\sqrt{1+2\sigma^2 t}} d\sigma_v^2$$

$$H(\sigma_v^2) = J(\sigma_v^2)\, n_p(\sigma_v^2) / \bar{N}$$

$$\bar{N} = \int_0^\infty J(\sigma_v^2)\, n_p(\sigma_v^2)\, d\sigma_v^2$$

Average number of particles per halo

The X distribution for maximum entropy principle:

$$X(v)=\frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)} \Rightarrow P(x)=\frac{e^{-\sqrt{\alpha^2+x/v_0^2}}}{2\alpha v_0 K_1(\alpha)\sqrt{x}}$$

$$\Downarrow$$

$$\int_0^\infty H(\sigma_v^2) e^{-\sigma^2 t} d\sigma_v^2 = \frac{K_1\left(\alpha\sqrt{1+2v_0^2 t}\right)}{K_1(\alpha)\sqrt{1+2v_0^2 t}}$$

$$v=\left(\frac{m_h}{m_h^*}\right)^{2/3} = \frac{\sigma_v^2(m_h)}{\sigma_v^2(m_h^*)} = \frac{\sigma_v^2(m_h)}{\bar{\sigma}_h^2}$$

Introduce dimensionless variable

$$\Downarrow$$

$$f(v) = H(v\bar{\sigma}_h^2)\bar{\sigma}_h^2$$

Halo mass function is intrinsically related to H, and hence X, the maximum entropy distribution

Parameters and distributions for some typical potential exponents n

Laplacian or exponential

	n	β	α	v_0^2	$\langle \sigma_h^2 \rangle$	$\langle \sigma_v^2 \rangle$	$X(v)$	$H(x=\sigma_v^2)$	$P(x=v^2)$
	0	1	0	$\frac{\sigma_0^2}{2}$	0	σ_0^2	$\frac{e^{-\sqrt{2}v/\sigma_0}}{\sqrt{2}\sigma_0}$	$\frac{e^{-x/\sigma_0^2}}{\sigma_0^2}$	$\frac{e^{-\sqrt{2x}/\sigma_0}}{\sigma_0\sqrt{2x}}$
Long range interaction	-1	$\frac{3}{2}$	$\frac{K_1(\alpha)}{K_2(\alpha)} = \frac{\langle \sigma_h^2 \rangle}{\sigma_0^2}$	$\frac{\sigma_0^2 K_1(\alpha)}{\alpha K_2(\alpha)}$	$\sim \frac{\sigma_0^2}{2}$	$\sim \frac{\sigma_0^2}{2}$	$\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{2\alpha v_0 K_1(\alpha)}$	X distribution	$\frac{e^{-\sqrt{\alpha^2+x/v_0^2}}}{2\alpha v_0 K_1(\alpha)\sqrt{x}}$
Short range interaction	-2	3	∞	0	σ_0^2	0	$\frac{e^{-v^2/2\sigma_0^2}}{\sqrt{2\pi}\sigma_0}$	$\delta(x)$	$\frac{e^{-x/2\sigma_0^2}}{\sigma_0\sqrt{2\pi x}}$

Gaussian

Integral transformations between distributions:

$$\int_{-\infty}^{\infty} X(v) e^{-vt} dv = \int_0^{\infty} H(\sigma_v^2) e^{\sigma^2 t^2/2} d\sigma_v^2$$

$$\int_0^{\infty} P(x) e^{-xt} dx = \int_0^{\infty} H(\sigma_v^2) \frac{1}{\sqrt{1+2\sigma^2 t}} d\sigma_v^2$$

H and J Distributions for large halos

We first consider an extreme case, large halos with $\sigma_h^2 << \sigma_v^2$:

Halo group temperature $\rightarrow \sigma_h^2 \to 0$ and $\sigma^2 \approx \sigma_v^2 \leftarrow$ Halo temperature

[From integral transformations between distributions](#):

$$\int_0^\infty H\left(\sigma_v^2\right) e^{-\sigma^2 t} d\sigma_v^2 = \frac{K_1\left(\alpha\sqrt{1+2v_0^2 t}\right)}{K_1(\alpha)\sqrt{1+2v_0^2 t}}$$

With $\sigma^2 = \sigma_v^2$ H distribution for large halos:

$$H_\infty\left(\sigma_v^2\right) = \frac{1}{2\alpha v_0^2 K_1(\alpha)} \cdot \exp\left[-\frac{\alpha}{2}\left(\frac{\sigma_v^2}{\alpha v_0^2} + \frac{\alpha v_0^2}{\sigma_v^2}\right)\right]$$

Dimensionless H distribution for large halos:

$$f_{H_\infty}(\nu) = \frac{1}{2\gamma K_1(\alpha)} \cdot \exp\left[-\frac{\alpha}{2}\left(\frac{\nu}{\gamma} + \frac{\gamma}{\nu}\right)\right] \qquad \gamma = \frac{\alpha v_0^2}{\bar{\sigma}_h^2}$$

$$H\left(\sigma_v^2\right) = J\left(\sigma_v^2\right) n_p\left(\sigma_v^2\right) / \bar{N}$$

J distribution for large halos:

$$J_\infty\left(\sigma_v^2\right) = \frac{1}{2\alpha v_0^2 K_{\beta-1}(\alpha)} \left(\frac{\alpha v_0^2}{\sigma_v^2}\right)^\beta \exp\left[-\frac{\alpha}{2}\left(\frac{\sigma_v^2}{\alpha v_0^2} + \frac{\alpha v_0^2}{\sigma_v^2}\right)\right]$$

Halo size: $n_p\left(\sigma_v^2\right) = \bar{N} \dfrac{K_{\beta-1}(\alpha)}{K_1(\alpha)} \left(\dfrac{\sigma_v^2}{\alpha v_0^2}\right)^\beta$

$$\beta = 3/(3+n) \qquad \beta = 3/2 \text{ for } n = -1$$

Interestingly, H_∞ distribution can be obtained directly using the maximum entropy principle without resorting to X distribution (Next slides)

H_∞ and J_∞ Distributions from maximum entropy principle

Following the maximum entropy principle for velocity distrution:

H_∞ distribution is a maximum entropy distribution satisfying three constraints:

$$\int_0^\infty H_\infty\left(\sigma_v^2\right)d\sigma_v^2 = 1$$

$$\int_0^\infty H_\infty\left(\sigma_v^2\right)\sigma_v^2 d\sigma_v^2 = \left\langle \sigma_v^2 \right\rangle$$

$$\int_0^\infty J_\infty\left(\sigma_v^2\right)d\sigma_v^2 = \int_0^\infty \frac{H_\infty\left(\sigma_v^2\right)}{\mu\left(\sigma_v^2/v_0^2\right)^\beta}d\sigma_v^2 = 1$$

Write down the entropy functional with Lagrangian multiplier:

$$\begin{aligned} S\left[H_\infty\left(\sigma_v^2\right)\right] = &-\int_0^\infty H_\infty\left(\sigma_v^2\right)\ln H_\infty\left(\sigma_v^2\right)d\sigma_v^2 \\ &+\lambda_1\left(\int_0^\infty H_\infty\left(\sigma_v^2\right)d\sigma_v^2 - 1\right) \\ &+\lambda_2\left(\int_0^\infty H_\infty\left(\sigma_v^2\right)\sigma_v^2 d\sigma_v^2 - \left\langle \sigma_v^2 \right\rangle\right) \\ &+\lambda_3\left(\int_0^\infty \frac{H_\infty\left(\sigma_v^2\right)}{\mu\left(\sigma_v^2/v_0^2\right)^\beta}d\sigma_v^2 - 1\right) \end{aligned}$$

Taking the variation of the entropy functional with respect to distribution *H*:

$$H_\infty\left(\sigma_v^2\right) = e^{\lambda_1 - 1}\exp\left(\lambda_2\sigma_v^2 + \frac{\lambda_3}{\mu}\left(\frac{v_0^2}{\sigma_v^2}\right)^\beta\right)$$

$$H_\infty\left(\sigma_v^2\right) = \frac{1}{2\alpha v_0^2 K_1(\alpha)}\cdot\exp\left[-\frac{\alpha}{2}\left(\frac{\sigma_v^2}{\alpha v_0^2} + \frac{\alpha v_0^2}{\sigma_v^2}\right)\right]$$

Modeling halo virial dispersion and halo velocity dispersion

To solve H distribution using integral transformation:

$$\int_0^\infty H\left(\sigma_v^2\right)e^{-\sigma^2 t}d\sigma_v^2 = \frac{K_1\left(\alpha\sqrt{1+2v_0^2 t}\right)}{K_1(\alpha)\sqrt{1+2v_0^2 t}}$$

We need model for velocity dispersion σ^2:

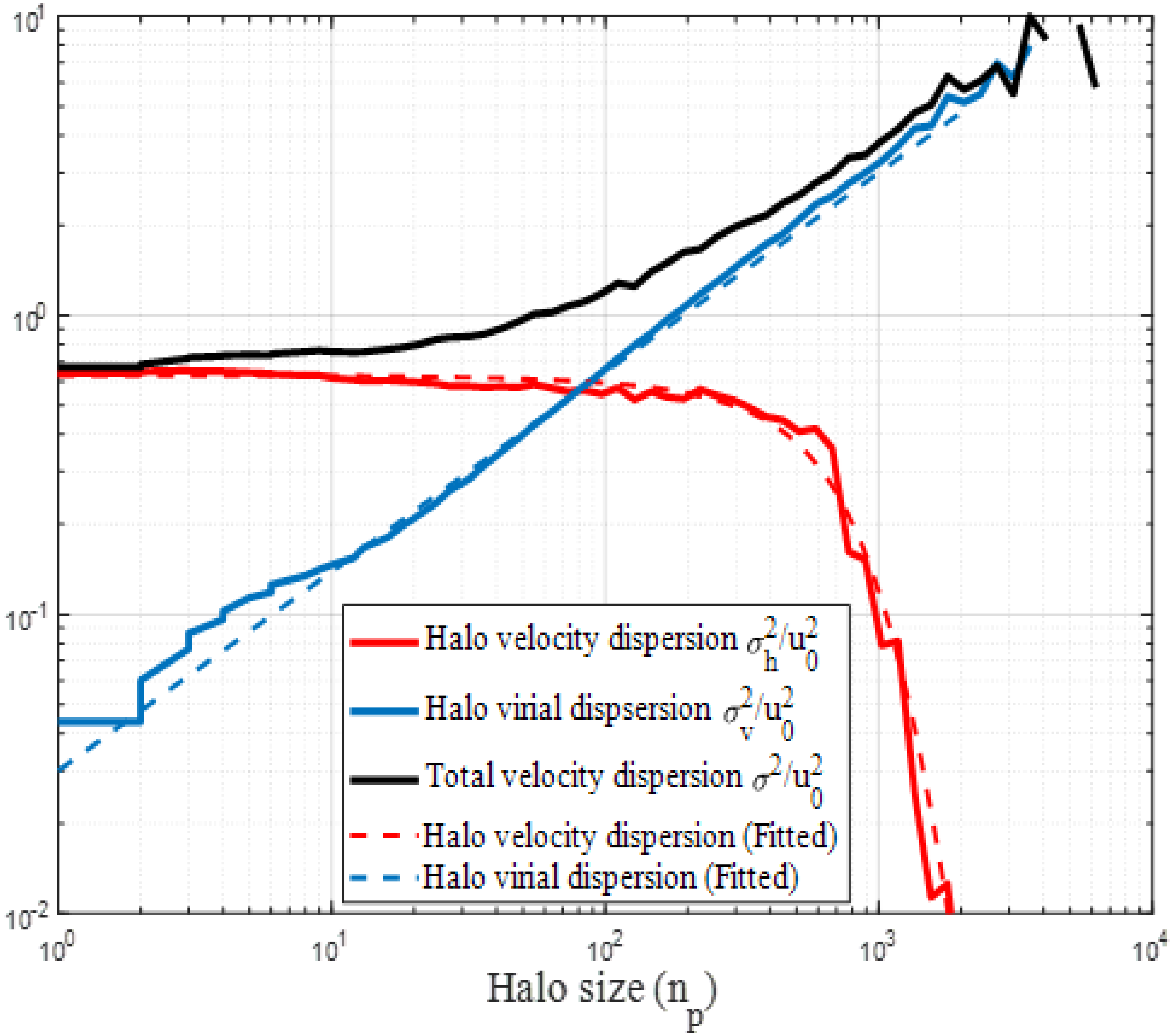
$$\sigma^2 = \sigma_v^2 + \sigma_h^2$$

Model for halo virial dispersion (halo temperature):

$$\sigma_v^2(m_h) = 0.03 n_p^{2/3} u_0^2 = 0.03\left(m_h/m_p\right)^{2/3} u_0^2$$

Model for halo velocity dispersion (halo group temperature):

$$\sigma_h^2(m_h) = 0.375\left[1-\tanh\left(\frac{m_h/m_p - 500}{600}\right)\right]u_0^2$$



H Distribution for small halos

We consider another extreme case, small halos with $\sigma_v^2 << \sigma_h^2$:

Halo group temperature → $\sigma_v^2 \to 0$ and $\sigma^2 \approx \sigma_h^2$ ← Halo temperature

If approximate the virial dispersion with v^2 $\quad v^2 \approx \sigma_v^2$

$$H\left(x=\sigma_v^2\right) \approx P\left(x=v^2\right) = \frac{e^{-\alpha}}{2\alpha v_0 K_1(\alpha)\sqrt{x}} \exp\left(-\frac{x}{2\alpha v_0^2}\right)$$

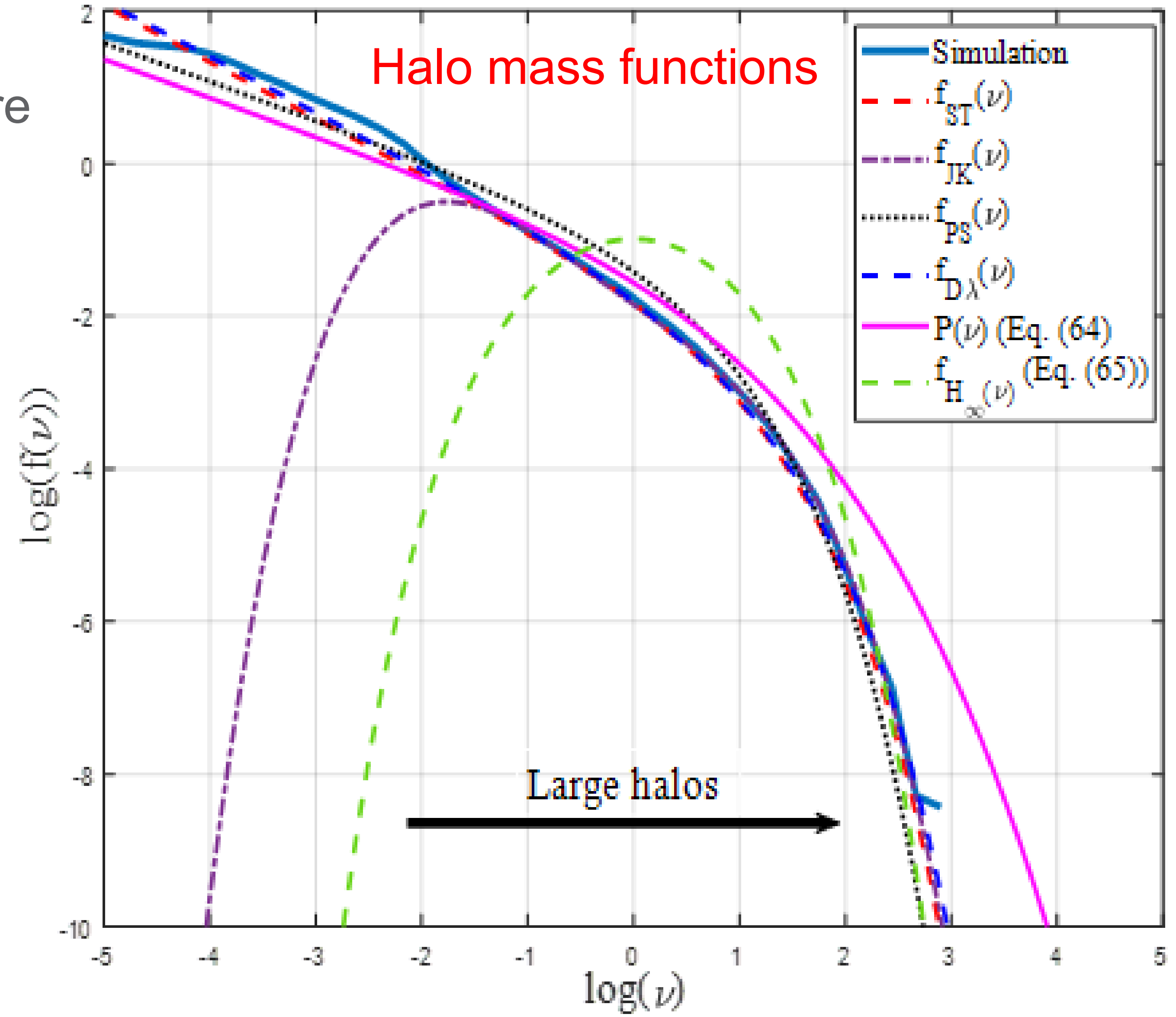
H distribution for small halos:

$$H_s\left(\sigma_v^2\right) = \frac{1}{\sqrt{2\pi\alpha v_0^2 \sigma_v^2}} \exp\left(-\frac{\sigma_v^2}{2\alpha v_0^2}\right)$$

PS mass function for γ=1

Dimensionless mass function for small halos:

$$f_{H_s}(\nu) = \frac{1}{\sqrt{2\pi\gamma\nu}} \exp\left(-\frac{\nu}{2\gamma}\right) \qquad \gamma = \frac{\alpha v_0^2}{\bar{\sigma}_h^2}$$



Halo mass function from maximum entropy distributions

From integral transformations between distributions:

H distribution from maximum entropy distribution should satisfy:

$$\int_0^\infty H\left(\sigma_v^2\right)e^{-\sigma^2 t}d\sigma_v^2 = \frac{K_1\left(\alpha\sqrt{1+2v_0^2 t}\right)}{K_1(\alpha)\sqrt{1+2v_0^2 t}}$$

Relation between dimensionless halo mass function and H distribution:

$$f(\nu) = H\left(\nu\bar{\sigma}_h^2\right)\bar{\sigma}_h^2$$

Dimensionless maximum entropy halo mass function:

$$\int_0^\infty f_{ME}(\nu)e^{-(\nu+\nu_h)t}d\nu = \frac{K_1\left(\alpha\sqrt{1+2\gamma t/\alpha}\right)}{K_1(\alpha)\sqrt{1+2\gamma t/\alpha}}$$

$\nu_h = \sigma_h^2/\bar{\sigma}_h^2$ and $\bar{\sigma}_h^2(a) = \sigma_v^2\left(m_h^*, a\right)$

Laplace transform of halo mass functions:

$$\int_0^\infty f_{PS}(\nu)e^{-\nu t}d\nu = \frac{1}{\sqrt{1+2t}}$$

$$\int_0^\infty f_{ST}(\nu)e^{-\nu t}d\nu = \frac{\sqrt{q}}{\sqrt{q+2t}}\frac{\sqrt{\pi}+\Gamma(1/2-p)(1/2+t/q)^p}{\sqrt{\pi}+2^{-p}\Gamma(1/2-p)}$$

$$\int_0^\infty f_{D\lambda}(\nu)e^{-\nu t}d\nu = \frac{1}{(1+4\eta_0 t)^{q/2}}$$

Moments of halo mass functions:

$$\int_0^\infty f_{PS}(\nu)\nu^n d\nu = 2^n\frac{\Gamma(1/2+n)}{\sqrt{\pi}}$$

$$\int_0^\infty f_{ST}(\nu)\nu^n d\nu = \left(\frac{2}{q}\right)^2\frac{\Gamma(1/2+n)+2^{-p}\Gamma(1/2+n-p)}{\Gamma(1/2)+2^{-p}\Gamma(1/2-p)}$$

$$\int_0^\infty f_{D\lambda}(\nu)\nu^n d\nu = \frac{(4\eta_0)^n\Gamma(q/2+n)}{\Gamma(q/2)}$$

Halo mass function from maximum entropy distributions

Equation for maximum entropy halo mass function:

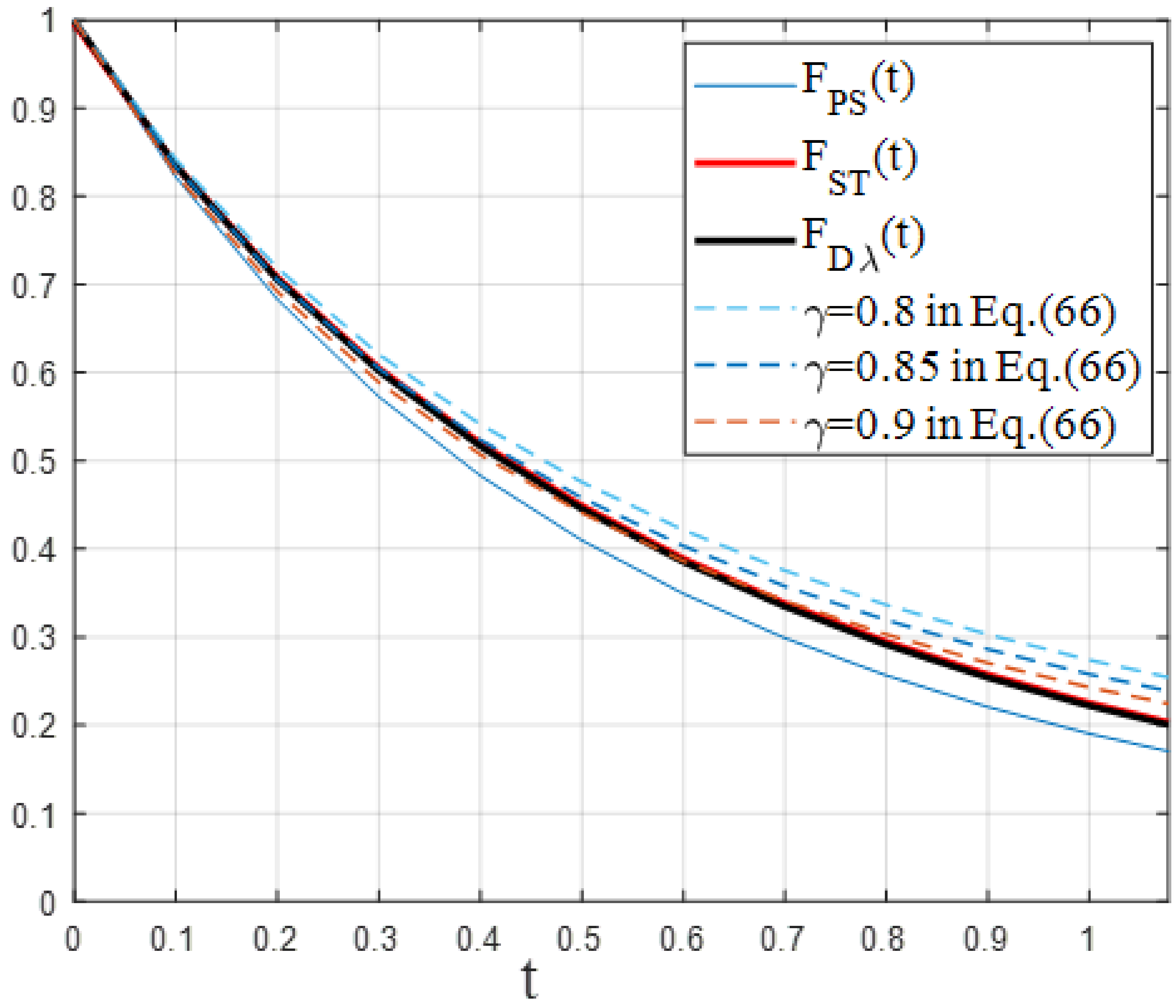
$$\int_0^\infty f_{ME}(\nu)e^{-(\nu+\nu_h)t}d\nu = \frac{K_1\left(\alpha\sqrt{1+2\gamma t/\alpha}\right)}{K_1(\alpha)\sqrt{1+2\gamma t/\alpha}}$$

No analytical solutions can be found. Instead Introduce a transformed function F_x to compare different halo mass functions:

$$F_X(t) = \int_0^\infty f_X(\nu)e^{-(\nu+\nu_h)t}d\nu$$

Subscript X is the abbreviation of the mass function model, PS, ST, Dλ and ME.

- ST and Dλ almost coincide with each other.
- Both agree better with the ME than the PS mass function.
- Halo mass function can be an intrinsic distribution to maximize system entropy.



Summary and keywords

Maximum entropy	Velocity distribution	Spherical collapse
Halo mass function	Energy distribution	H and P distributions

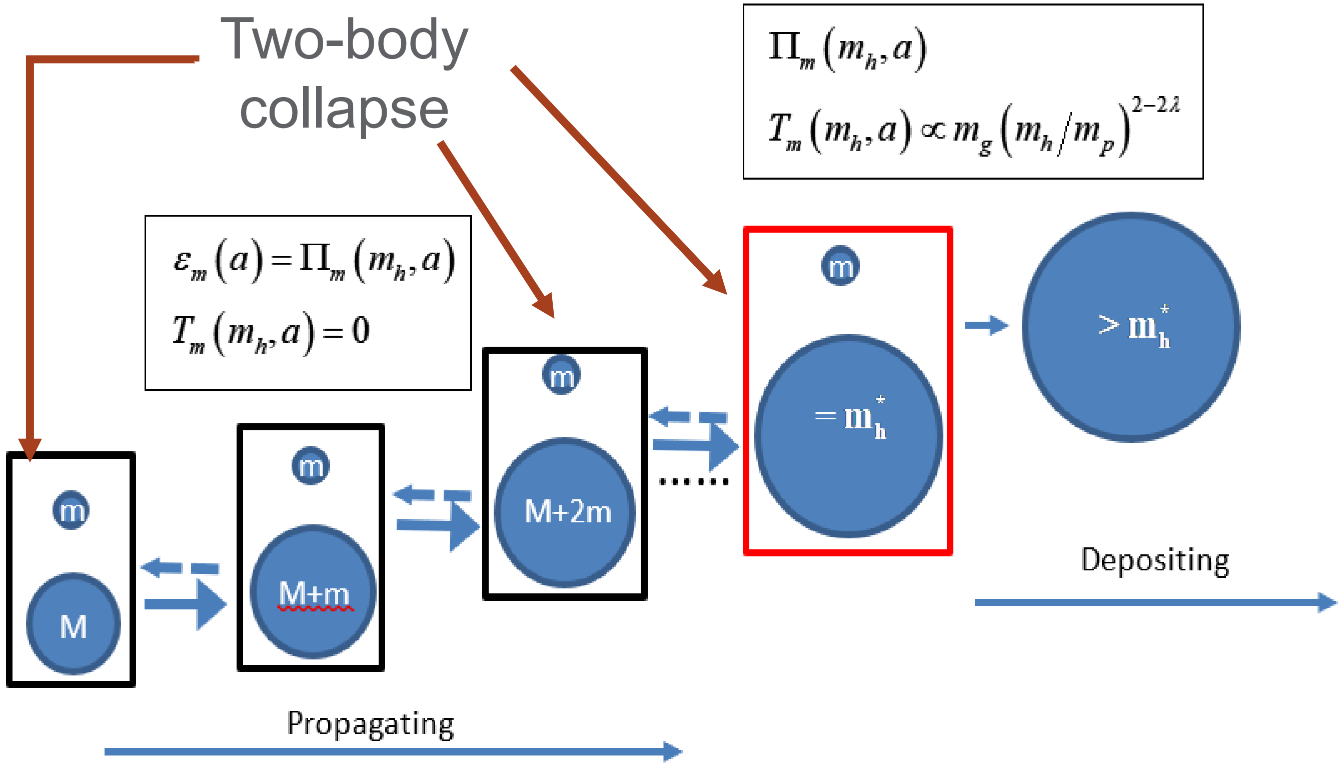
- Halo mass function is a fundamental quantity for structure formation and evolution.
- Conventional halo mass functions are based on simplified spherical/elliptical collapse models
- The H distribution for particle virial dispersion is essentially the halo mass function that can be related to X distribution that maximizes system entropy.
- The H distribution for large halos is also a maximum entropy distribution.
- For small halos, H approximates the distribution of square velocity (P) and recovers the Press-Schechter mass function.
- Halo mass function can be interpreted as an intrinsic distribution to maximize the system entropy

Two-body collapse model (TBCM): an elementary step of mass cascade and GSCH for pairwise velocity

Xu Z., 2021, arXiv:2110.05784v1 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2110.05784

Introduction: TBCM as an elementary step of inverse mass cascade

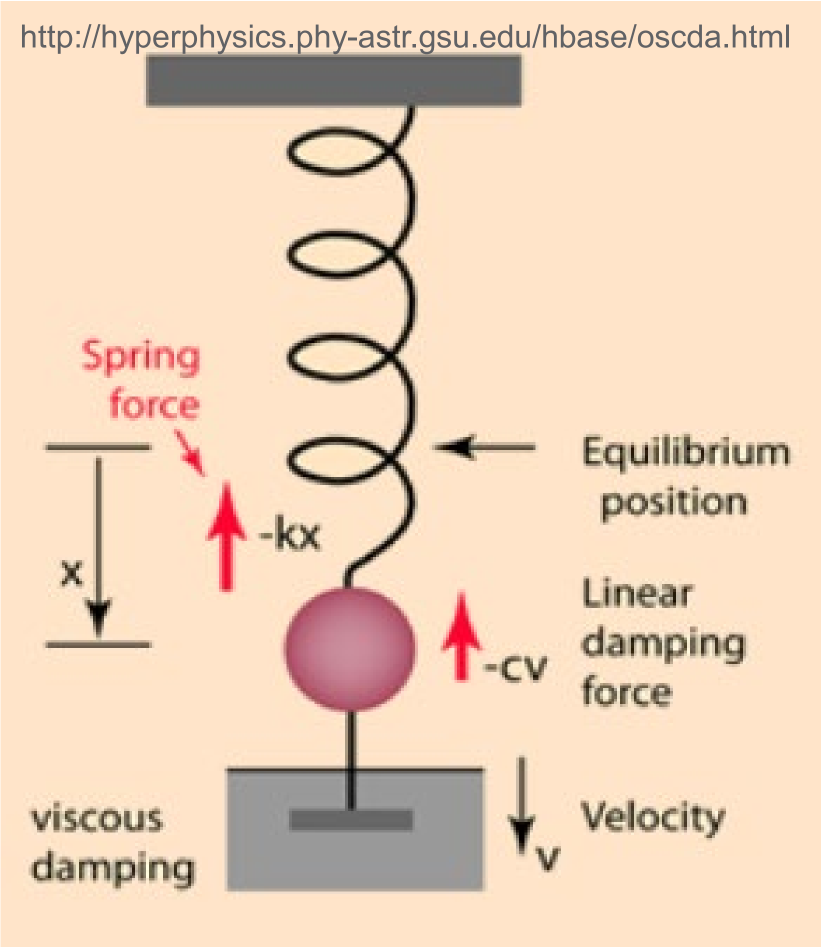
- Analytical tools are invaluable.
- Solutions are extremely difficult to find due to the highly non-linear nature of collapse.
- Two examples: the spherical collapse model (SCM) and stable clustering hypothesis (SCH).
- For an infinitesimal interval, mass cascade should involve the merging of two and only two substructures.
- Two-body problem in static background is known: Kepler's laws.
- Goal: solutions for two-body in expanding background and relations with SCM and SCH



Two-body collapse in expanding background is an elementary step of mass cascade.

- Goal: Prove SCH and Generalized SCH for moments of pairwise velocity.

Introduction: Damped harmonic oscillator as a fundamental model in dynamics



$$\ddot{\mathbf{r}} + (c/m)\dot{\mathbf{r}} + (k/m)\mathbf{r} = 0$$

damping spring force

Competition

Define a critical ratio to quantify competition: $\beta_s = \dfrac{(c/2m)^2}{(k/m)} = 1$ → Critical damping: $c_s = 2\sqrt{km}$

Energy evolution: $\dfrac{dE}{dt} + \left(\dfrac{2c}{m}\right)K = 0$

E: total energy (potential + kinetic)
K: kinetic energy

- Damped harmonic oscillator is a fundamental model in dynamics that is extremely insightful.
- There exist a critical damping c_s. For $c<c_s$, spring force is dominant (underdamped); For $c>c_s$, damping is dominant (overdamped).

- Does two-body collapse model play a similar role as harmonic oscillator?
- Overdamped and underdamped in gravitational collapse?
- Insights into the energy/momentum evolution?

Equations of motion in comoving and transformed systems

Equations of motion in a comoving system with expanding background

Equation of motion in a transformed system with fixed damping in static background

$$\frac{d^2\mathbf{x}_i}{dt^2}+2H\frac{d\mathbf{x}_i}{dt}=-\frac{Gm_p}{a^3}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^3}$$

Potential with an arbitrary exponent of ***n*** for particle-particle interacting

$$V_p(r)=-G_n m_p^2/r^{-n}$$

$$\frac{d^2\mathbf{x}_i}{dt^2}+2H\frac{d\mathbf{x}_i}{dt}=\frac{nG_n m_p}{a^3}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}$$

Introduce a new transformed time scale **s**

$$ds/dt=a^p$$

- If p=-2, *s* is the time variable for integration in N-body simulation.
- Transformed system: fixed damping and no scale factor *a*;

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{d\mathbf{x}_i}{ds}(p+2)a^{-p}H=\frac{nG_n m_p}{a^{3+2p}}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}$$

$$\dot{H}=-3H^2/2$$

$$H^2=8\pi G\bar{\rho}_y(a)/3$$

$$H_0^2=H^2a^3$$

$p=-3/2$ Matter dominant

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{H_0}{2}\frac{d\mathbf{x}_i}{ds}=nG_n m_p\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}=\frac{\mathbf{F}_i}{m_p}$$

Peculiar velocity in comoving**:**

$$\mathbf{u}_i=a\frac{d\mathbf{x}_i}{dt}=\frac{d\mathbf{r}_i}{dt}-H\mathbf{r}_i=a^{-1/2}\mathbf{v}_i$$

Velocity in time scale **s:**

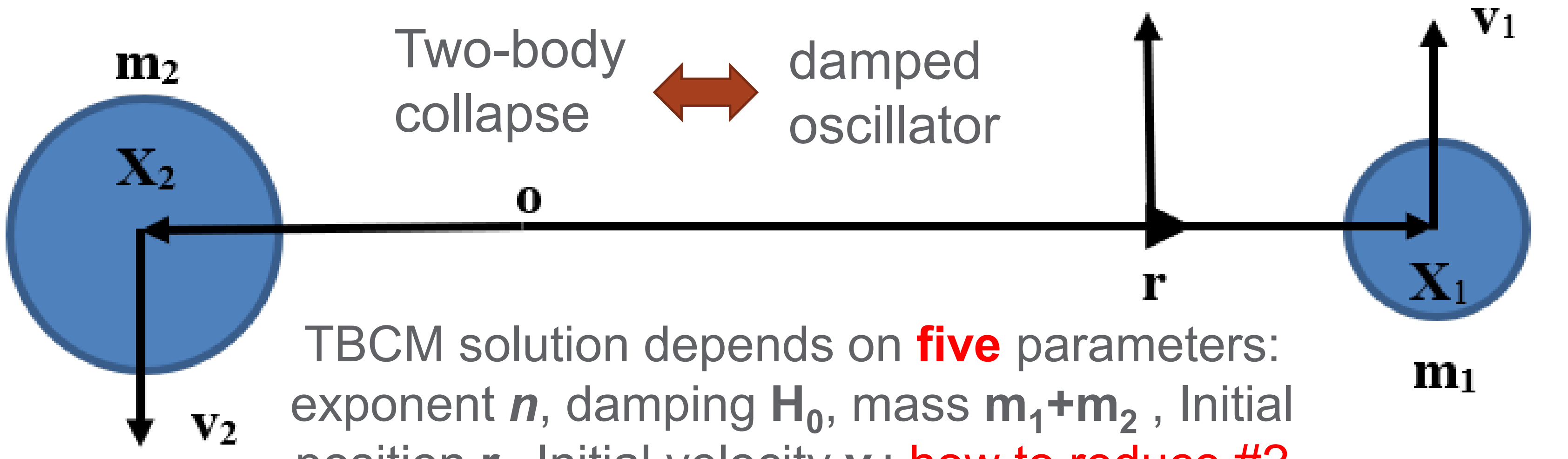
$$\mathbf{v}_i=\frac{d\mathbf{x}_i}{ds}=a^{3/2}\frac{d\mathbf{x}_i}{dt}=a^{1/2}\mathbf{u}_i$$

Formulation of a TBCM model in transformed system

Reduce to equations of motion for two-body:

$$\ddot{\mathbf{x}}_1 + \frac{H_0}{2}\dot{\mathbf{x}}_1 = \frac{nG_n m_2}{(2r)^{1-n}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$$

$$\ddot{\mathbf{x}}_2 + \frac{H_0}{2}\dot{\mathbf{x}}_2 = -\frac{nG_n m_1}{(2r)^{1-n}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$$



TBCM solution depends on **five** parameters: exponent ***n***, damping $\mathbf{H_0}$, mass $\mathbf{m_1+m_2}$, Initial position $\mathbf{r_i}$, Initial velocity $\mathbf{v_i}$; how to reduce #?

Displacement vector **r**:

$$\mathbf{r} = (\mathbf{x}_1 - \mathbf{x}_2)/2$$

Standard damped oscillator Eq.:

$$\boxed{\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} + (k/m)\mathbf{r} = 0}$$

Reduce to Eq. of motion for vector **r**:

$$\ddot{\mathbf{r}} + \frac{H_0}{2}\dot{\mathbf{r}} = \frac{nG_n(m_1+m_2)}{2(2r)^{1-n}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$$

Similar equation as damped oscillator in dynamics

Equation of motion for radius function r (magnitude of **r**): **(similar to spherical collapse model)**

$$\ddot{r} + \frac{H_0}{2}\dot{r} - \frac{nG_n(m_1+m_2)}{2(2r)^{1-n}} = \frac{(r_i v_i)^2}{r^3}\exp(-H_0 s)$$

Expanding background or damping

Gravitational interaction

Angular momentum

Compute particle position and velocity:

$$\mathbf{x}_1 = \mu\mathbf{r} \qquad \mathbf{v}_1 = \mu\dot{\mathbf{r}} \qquad \mu = \frac{2m_2}{m_1+m_2}$$

$$\mathbf{x}_2 = -(2-\mu)\mathbf{r} \qquad \mathbf{v}_2 = -(2-\mu)\dot{\mathbf{r}}$$

Competition between three terms determines the collapse regimes

Formulation of a TBCM model in transformed system

Introduce frequency function F(s):

$$r(s)=(r_i v_i)^{1/2} F(s)\exp\left(-\frac{1}{4}H_0 s\right)$$

Equation of motion for r:

$$\ddot{r}+\frac{H_0}{2}\dot{r}-\frac{nG_n(m_1+m_2)}{2(2r)^{1-n}}=\frac{(r_i v_i)^2}{r^3}\exp(-H_0 s)$$

Equation for frequency function:

$$\frac{\partial^2 F}{\partial s^2}=\underbrace{\frac{H_0^2}{16}F(s)}_{1}\underbrace{-\gamma_s\left(\frac{v_i}{r_i}\right)^{1+n/2}F^{n-1}(s)\exp\left(-\frac{n-2}{4}H_0 s\right)}_{2}+\underbrace{F^{-3}(s)}_{3}$$

$$\left.\frac{\partial F}{\partial s}\right|_{s=0}=\frac{H_0}{4}\left(\frac{r_i}{v_i}\right)^{1/2} \qquad F(s=0)=\left(\frac{r_i}{v_i}\right)^{1/2}$$

$$\gamma_s=\left(\frac{v_{ri}}{v_i}\right)^2$$

stable orbital speed (virial theorem): $v_{ri}=\sqrt{\frac{-nG_n r_i}{(2r_i)^{1-n}}\frac{m_1+m_2}{2}}$

Frequency ω: $\omega\equiv\omega(s)$

Frequency function F(s): $F(s)\equiv(\omega+s\dot{\omega})^{-1/2}=\left(\frac{\partial(\omega s)}{\partial s}\right)^{-1/2}$

Ratio γ_s reflects competition: gravity vs. angular momentum; System in initial virial equilibrium if $\gamma_s=1$;

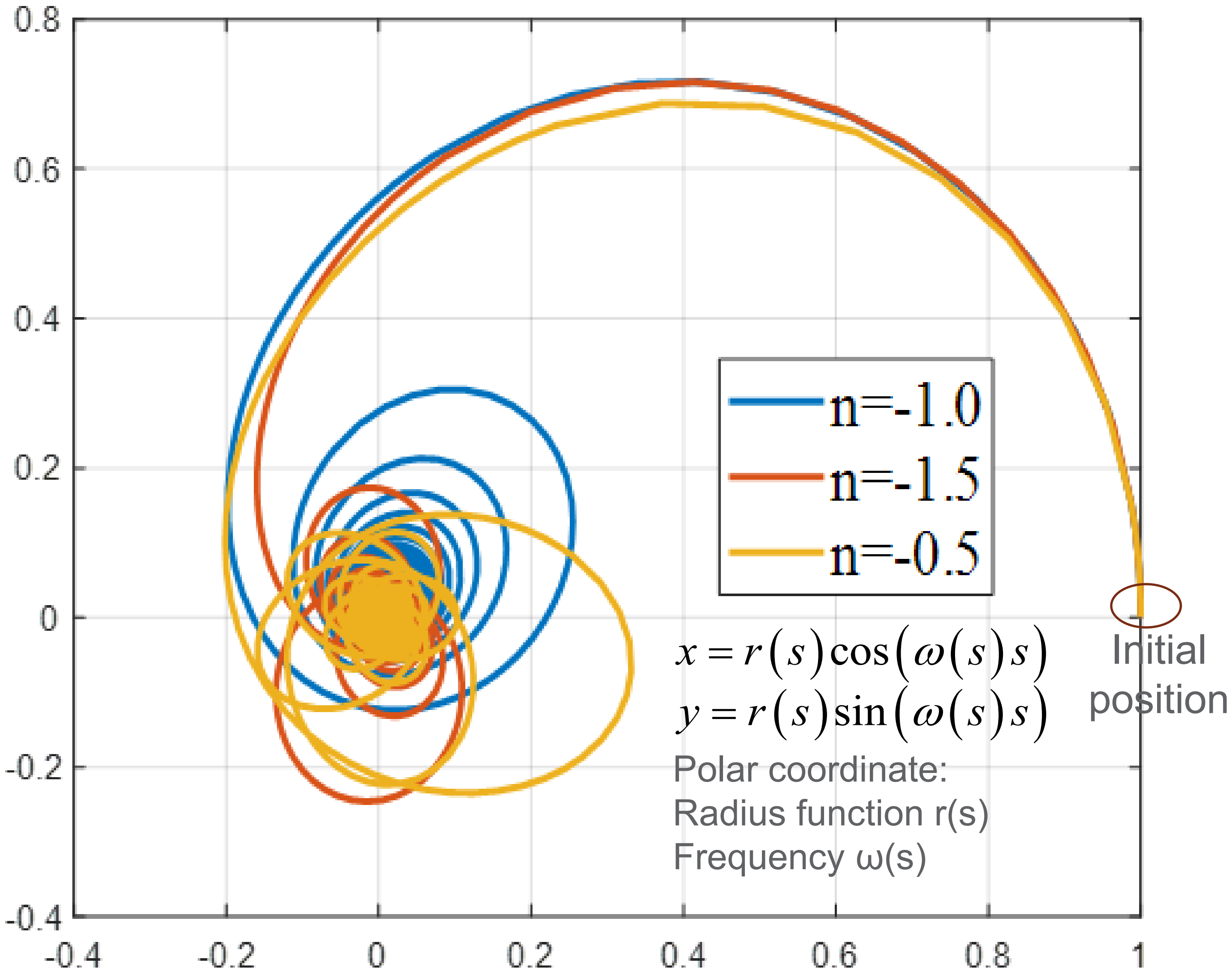
term 2 (gravitational force) = term 3 (angular momentum) leads to mean solutions:

$$F_m(s)=\gamma_s^{-1/(2+n)}\left(\frac{r_i}{v_i}\right)^{1/2}\exp\left(-\frac{2-n}{2+n}\cdot\frac{H_0 s}{4}\right)$$

$$\omega_m(s)=\frac{1}{2\lambda_s s}\frac{2+n}{2-n}\gamma_s^{2/(2+n)}\exp\left(\frac{2-n}{2+n}\cdot\frac{H_0 s}{2}\right)$$

$$r_m(s)=\gamma_s^{-1/(2+n)}r_i\exp\left(-\frac{H_0 s}{2+n}\right)$$

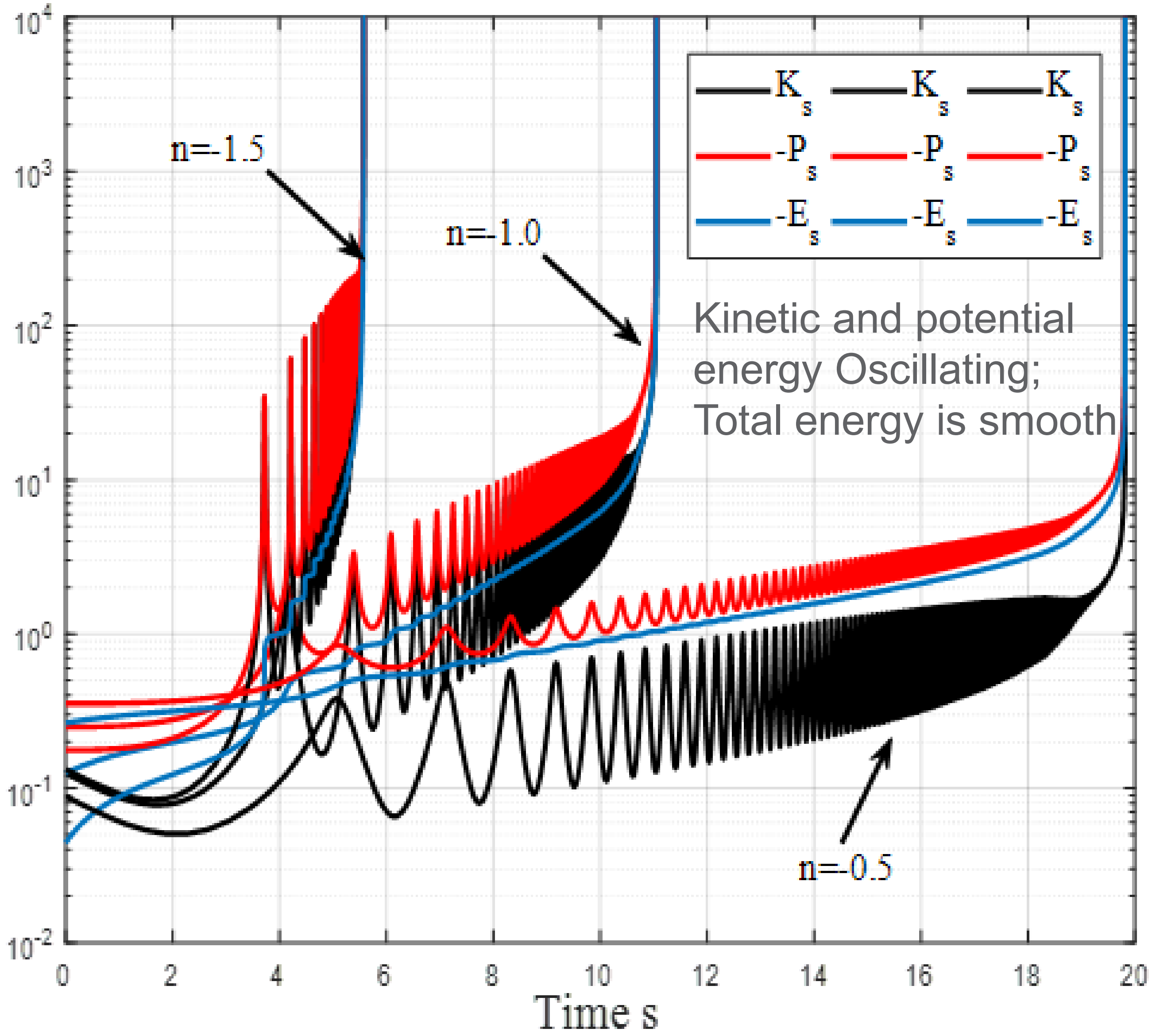
Examples of numerical solutions



$$x = r(s)\cos(\omega(s)s)$$
$$y = r(s)\sin(\omega(s)s)$$

Polar coordinate:
Radius function r(s)
Frequency ω(s)

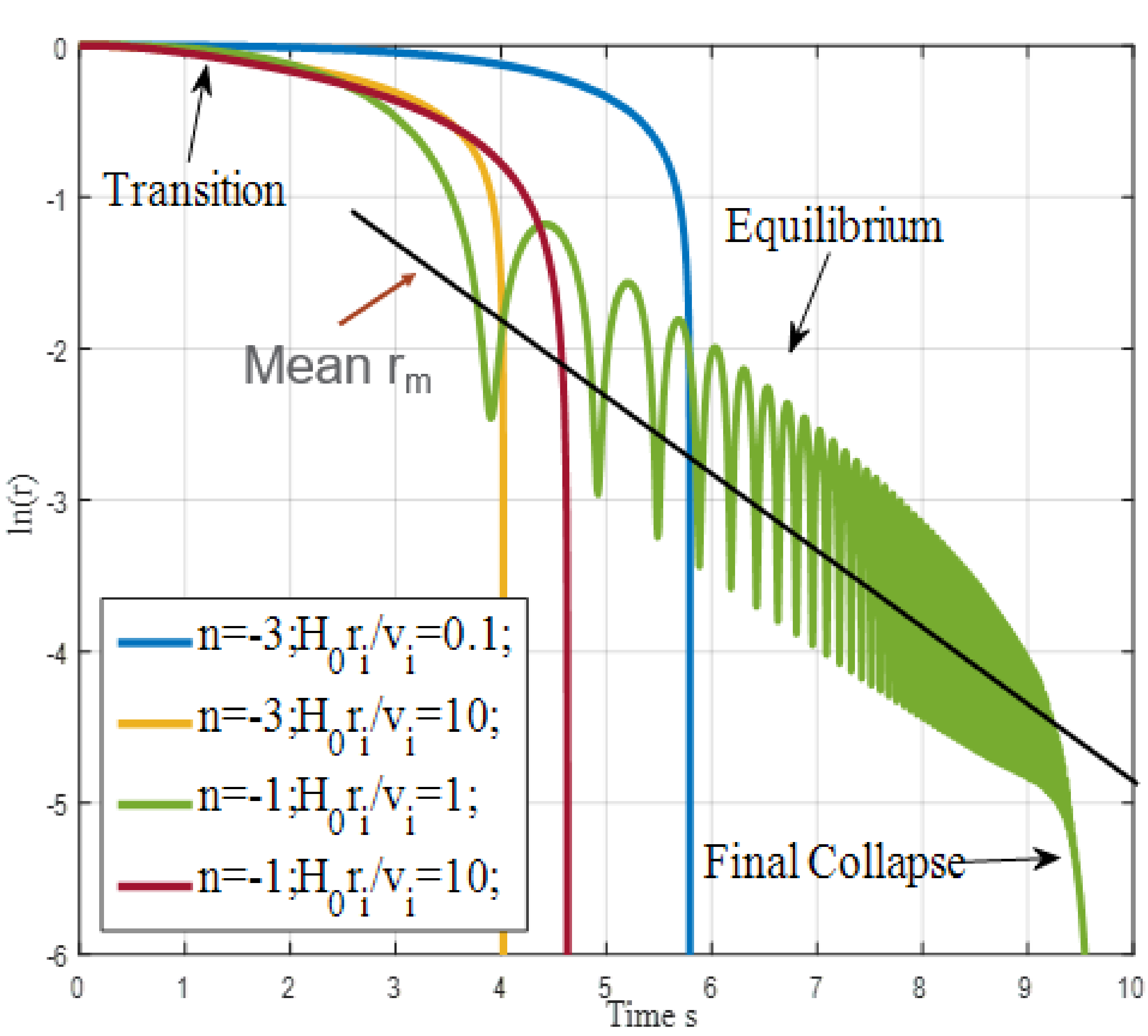
Trajectory of the motion of displacement vector **r**



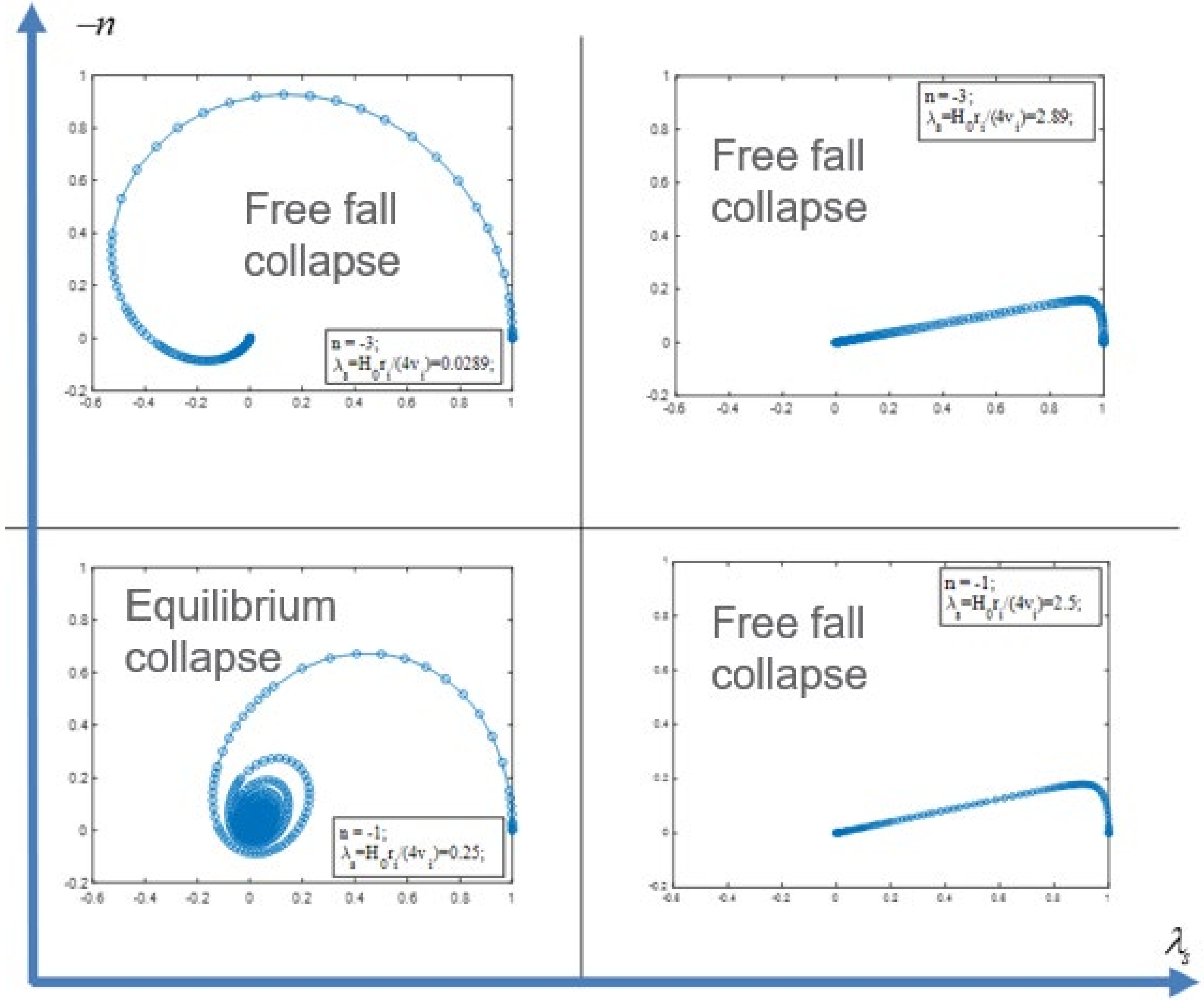
Kinetic and potential energy Oscillating; Total energy is smooth

Time evolution of system kinetic, potential and total energy

Two-body collapse: free fall or equilibrium?



Variation of radius **r** with time **s** exbibits two different collapse. Equilibrium collapse involves a mean and fluctuation.



Depending on the competition between three forces, two types of collapse can be identified.

$$\lambda_s = \frac{H_0 r_i}{4 v_i}$$

TBCM model in the simplest form and perturbative solutions for equilibrium collapse

Decompose frequency function F(s) into the mean and amplitude and substitute to equation for F(s):

$$F(s) = F_m(s) F_a(\omega_m s) = F_m(s) F_a(x)$$

mean amplitude

The simplest form of TBCM for amplitude function F_a:

$$\frac{\partial^2 F_a(x)}{\partial x^2} = \underbrace{\frac{2n}{(2-n)^2}\frac{F_a(x)}{x^2}}_{1} - \underbrace{F_a^{n-1}(x)}_{2} + \underbrace{F_a^{-3}(x)}_{3} \qquad x = \omega_m(s)s$$

$$F_a(x_0) = \gamma_s^{1/(2+n)} \qquad \left.\frac{\partial F_a}{\partial x}\right|_{x=x_0} = \frac{\beta_s \gamma_s^{-1/(2+n)}}{2+n} \qquad x_0 = \frac{2\gamma_s^{2/(2+n)}}{\beta_s}\frac{2+n}{2-n}$$

Mean solutions:

$$F_m(s) = \gamma_s^{-1/(2+n)}\left(\frac{r_i}{v_i}\right)^{1/2} \exp\left(-\frac{2-n}{2+n}\cdot\frac{H_0 s}{4}\right)$$

$$\omega_m(s) = \frac{2}{\beta_s s}\frac{2+n}{2-n}\gamma_s^{2/(2+n)} \exp\left(\frac{2-n}{2+n}\cdot\frac{H_0 s}{2}\right)$$

For long-range interaction n>-2, the competition between terms 2 and 3 leads to an oscillatory solution vibrating around the mean value F_a=1

Solution now only depends on **three** parameters:

- ratio γ_s reflects competition: gravity vs. angular momentum
- ratio β_s reflects competition: damping (or expanding background) vs. angular momentum
- exponent ***n***

$$\gamma_s = (v_{ri}/v_i)^2$$

$$\beta_s = H_0 r_i / v_i$$

Stable orbital speed:

$$v_{ri} = \sqrt{\frac{-nG_n r_i}{(2r_i)^{1-n}}\frac{m_1+m_2}{2}}$$

Classifying two-body collapse

$$\gamma_s = \left(v_{ri}/v_i\right)^2 \qquad\qquad \beta_s = H_0 r_i/v_i$$

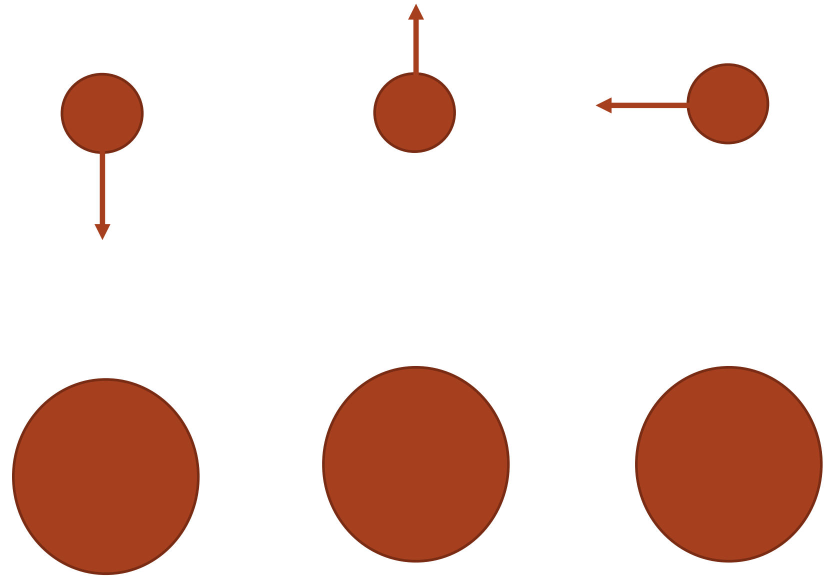
Freefall collapse :

- Short-range interaction with exponent *n*<-2
- γ_s >>1 : gravity is dominant over angular momentum
- β_s >>1 : damping is dominant
- γ_s <<1 : There is a turnaround before free fall

Equilibrium collapse :

- γ_s ≈1 and β_s <<1 : stable orbit (angular momentum comparable with gravity) with week damping
- β_s =0 : Standard two-body problem in static background

Equilibrium collapse has an oscillatory motion with a much longer time to fully collapse than free fall collapse!



γ_s>>1 or β_s>>1

Freefall collapse

γ_s<<1 or β_s>>1

Freefall collapse

γ_s ≈1 and β_s<<1

Equilibrium collapse

Solutions of free fall collapse and free fall time

Zero initial speed (no angular momentum):

$$v_i = 0 \quad \Rightarrow \quad \gamma_s = \left(v_{ri}/v_i\right)^2 \to \infty$$

Free fall time in static background:

$$s_{ce} = \frac{\pi r_i^{3/2}}{\sqrt{G\left(m_1 + m_2\right)}}$$

$$\lambda_{si} = \frac{H_0 r_i}{4 v_{ri}} \Leftarrow$$ Competition between damping and gravity

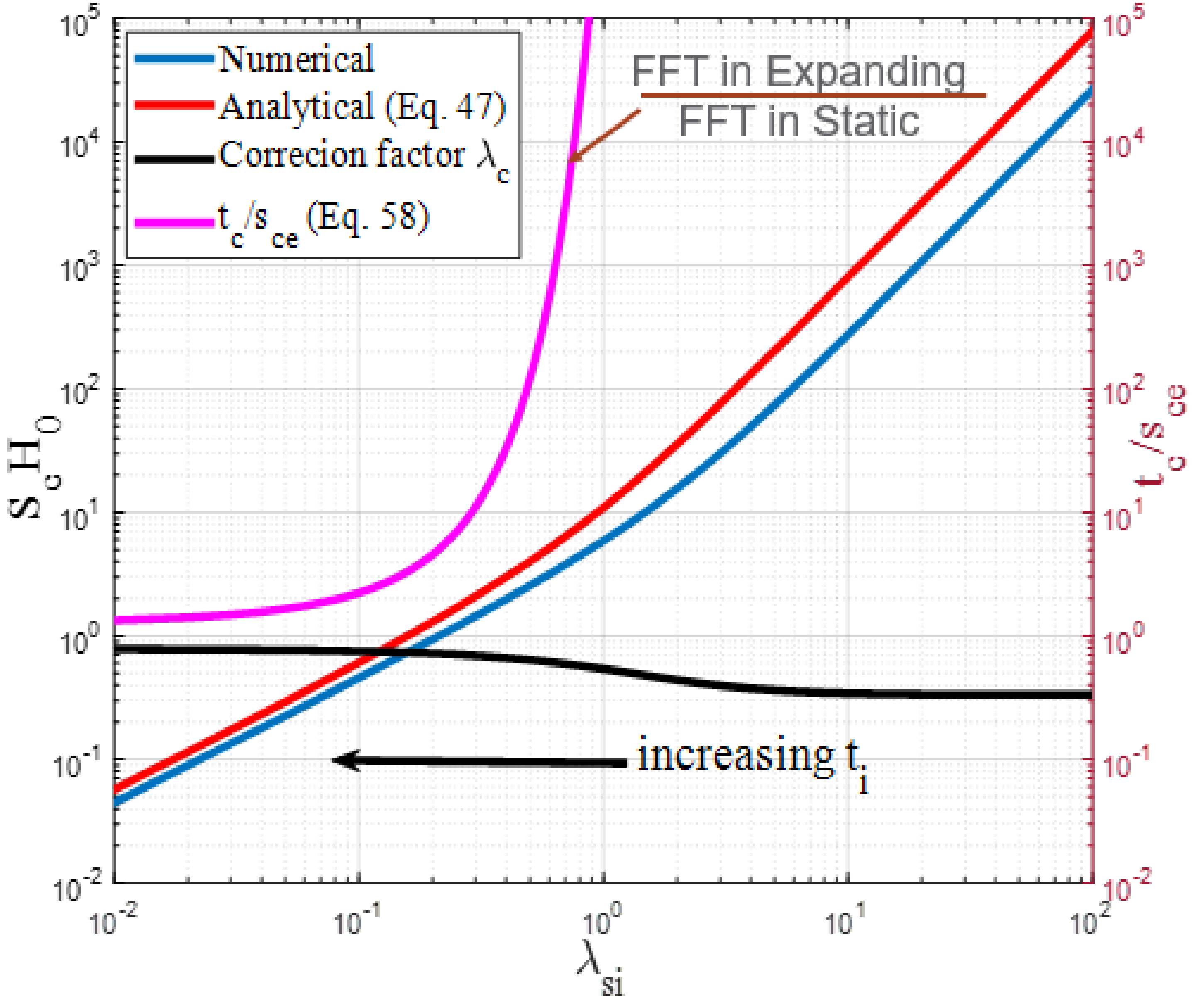
For small λ_{si} (weak damping):

$$s_c \approx s_{c1} = 4\sqrt{2}\frac{\lambda_{si}}{H_0} = \sqrt{\frac{2^{3-n} r_i^{2-n}}{-nG_n\left(m_1 + m_2\right)}}$$

For large λ_{si} (strong damping):

$$s_c \approx s_{c2} = 8\frac{\lambda_{si}^2}{H_0} = \frac{H_0 2^{1-n} r_i^{2-n}}{-nG_n\left(m_1 + m_2\right)}$$

- Due to damping, free fall time of two-body in expanding background is greater than the free fall time of same two-body in static background.



The earlier collapse starts (the smaller t_i), the greater the free fall time (H is decreasing)

Perturbative solutions for equilibrium collapse

Solve: $$\frac{\partial^2 F_a(x)}{\partial x^2} = \frac{2n}{(2-n)^2}\frac{F_a(x)}{x^2} - F_a^{n-1}(x) + F_a^{-3}(x)$$

Frequency function:

mean

fluctuation

$$F(s) = \left(\frac{r_i}{v_i}\right)^{1/2} \exp\left(-\frac{2-n}{2+n}\cdot\frac{H_0 s}{4}\right)\left\{1+\frac{\beta_s}{(2+n)^{3/2}}\sin(\theta_s)\right\}$$

Angle function:

$$\theta_s(s) = \frac{2\sqrt{2+n}}{\beta_s}\frac{2+n}{2-n}\left[\exp\left(\frac{2-n}{2+n}\cdot\frac{H_0 s}{2}\right)-1\right]$$

Radius function:

$$r(s) = r_i \exp\left(-\frac{H_0 s}{2+n}\right)\left\{1+\frac{\beta_s}{(2+n)^{3/2}}\sin(\theta_s)\right\}$$

Radial velocity:

$$\dot{r} = \frac{\partial r(s)}{\partial s} = \frac{H_0 r_i}{(2+n)}\exp\left(-\frac{nH_0 s}{2(2+n)}\right)\cos(\theta_s) - \frac{H_0 r}{2+n}$$

Specific kinetic energy:

$$K_s \approx \frac{2m_1 m_2 v_i^2}{(m_1+m_2)^2}\exp\left(\frac{-nH_0 s}{2+n}\right)\left[1-\frac{2\beta_s}{(2+n)^{3/2}}\sin\theta_s\right]$$

Specific potential energy:

$$P_s \approx \frac{2m_1 m_2 v_i^2}{(m_1+m_2)^2}\exp\left(\frac{-nH_0 s}{2+n}\right)\left[\frac{2}{n}+\frac{2\beta_s}{(2+n)^{3/2}}\sin\theta_s\right]$$

Specific total energy (fluctuation cancelled):

$$E_s = K_s + P_s = \frac{-(2+n)m_1 m_2}{(m_1+m_2)}\frac{G_n r_i}{(2r_i)^{1-n}}\exp\left(\frac{-nH_0 s}{2+n}\right)$$

Radial momentum: $$G_s = \frac{4m_1 m_2}{(m_1+m_2)^2}\mathbf{r}\cdot\mathbf{v} = \frac{4m_1 m_2}{(m_1+m_2)^2}\dot{r}r$$

Angular momentum: $$\mathbf{H}_s = \frac{4m_1 m_2}{(m_1+m_2)^2}r^2 F^{-2}(s)\hat{\mathbf{z}} = \frac{4m_1 m_2 v_i r_i}{(m_1+m_2)^2}\exp\left(-\frac{1}{2}H_0 s\right)\hat{\mathbf{z}}$$

Mean energy satisfying virial theorem: $2\langle K_s\rangle - n\langle P_s\rangle = 0$

All have exponential evolution in time scale s!

Critical values of β_s (analogue of critical damping) and critical halo density

Equilibrium collapse :

- $\gamma_s \approx 1$ and $\beta_s \ll 1$: stable orbit with week damping
- $\beta_s = 0$: Standard two-body problem in static background

$$\gamma_s = \left(\frac{v_{ri}}{v_i}\right)^2 \approx 1 \quad \Rightarrow \quad \beta_s = \frac{H_0 r_i}{v_i} \approx \frac{H_0 r_i}{v_{ri}} = \frac{Radial}{Circular} \ll 1$$

angular momentum comparable with gravity

Weak damping

Also see angle of incidence

First critical value for existence of equilibrium collapse with oscillator solution:

$$\frac{\beta_s}{(2+n)^{3/2}} \le 1 \quad \Rightarrow \quad \beta_{s1} = (2+n)^{3/2}$$

Second critical value for equilibrium collapse with oscillator solution:

$$\sin\left[\theta_s\left(t = kt_i\right)\right] = 0 \quad \Rightarrow \quad \beta_{s2} = \frac{(2+n)^{3/2}}{(2-n)\pi}$$

$$n = \frac{2-6m}{1+3m}$$

$$m = 1, 2, \ldots \infty$$

$$n = -1, -10/7, -8/5 \ldots -2$$

Critical halo density:

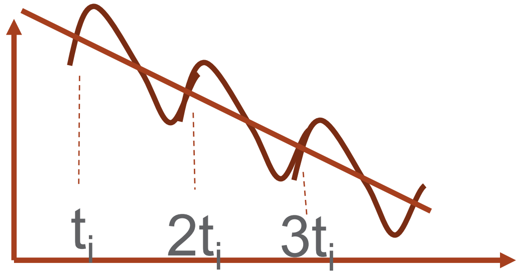
$$\Delta_c = 2/\beta_{s2}^2 = 18\pi^2$$

Radius function:

$$r(s) = r_i \exp\left(-\frac{H_0 s}{2+n}\right)\left\{1 + \frac{\beta_s}{(2+n)^{3/2}} \sin\left(\theta_s\right)\right\}$$

Angle function:

$$\theta_s(s) = \frac{2(2+n)^{3/2}}{\beta_s(2-n)}\left[\left(\frac{t}{t_i}\right)^{\frac{2-n}{3(2+n)}} - 1\right]$$



Evolution in comoving system for two-body angular velocity, spin parameter and angle of incidence

Evolution in transformed system with time scale *s* can be equivalently transformed back to original comoving system:

$$ds/dt = a^{-3/2} \quad \Rightarrow \quad s = t_0 \ln\left(t/t_i\right)$$

$$\exp\left(\tau H_0 s\right) \to \left(a/a_i\right)^{\tau}$$

Exponential evolution in time scale s ⇒ Power-law evolution in time t

Two-body kinetic energy:

$$K_s \approx \frac{2m_1 m_2 v_i^2}{\left(m_1+m_2\right)^2} \exp\left(\frac{-nH_0 s}{2+n}\right)\left[1-\frac{2\beta_s}{\left(2+n\right)^{3/2}}\sin\theta_s\right]$$

Kinetic energy in terms of angular velocity:

$$\frac{1}{2}\left(m_1\mu^2 + m_2\left(2-\mu\right)^2\right)\omega_s^2 r^2 = \left(m_1+m_2\right)K_s$$

$$\omega_s \approx \frac{v_i}{r_i}\exp\left[\frac{2-n}{2\left(2+n\right)}H_0 s\right] \quad \Rightarrow \quad \omega_t = \frac{r_i^{3/2}}{\beta_s} H r_m^{-3/2}$$

Two-body spin parameter:

$$\lambda_p = \frac{\left|\mathbf{H}_s\right|\left|E_s\right|^{1/2}}{G\left(m_1+m_2\right)} = \frac{\sqrt{2}}{2}\frac{\left(m_1 m_2\right)^{3/2}}{\left(m_1+m_2\right)^3} = \frac{\sqrt{2}}{16} \approx 0.0884$$

Evolution of two-body angle of incidence:

$$\cot\left(\theta_{\mathbf{vr}}\right) = \frac{\mathbf{r}\cdot\mathbf{v}}{\left|\mathbf{r}\times\mathbf{v}\right|} = \frac{G_s}{\left|\mathbf{H}_s\right|} = -\frac{\beta_s}{\left(2+n\right)}\left(\frac{a}{a_i}\right)^{-\frac{3}{2}}$$

Kinetic energy for large halos with an infinitesimal lifetime:

$$dK_h = \frac{K_s\left(s=0\right)}{a} = \frac{2M}{M^2} dM \cdot \alpha_s \frac{GM}{a r_i} \quad \Rightarrow \quad \sigma_v^2 = \frac{2}{3}K_h = \frac{GM}{2r_h}$$

Angular velocity in co-moving system dependent on halo size r_m , larger halo has smaller angular velocity

Prove stable clustering hypothesis (SCH) and derive generalized SCH

Peculiar pairwise velocity:

$$\Delta u_L(2r) = (u_L' - u_L)$$

$$a^{1/2}\Delta u_L = 2\frac{\mathbf{r}\cdot\mathbf{v}_1}{r} = 2\frac{G_s(s)}{r} = 2\dot{r}$$

See two-body virial quantity for radial flow

$$\Delta u_L = -2Har + 2\beta_s u_i \cos\theta_s$$

$$\langle \Delta u_L \rangle = -2Har + 2\langle \cancel{\beta_s u_i \cos\theta_s} \rangle$$

$$\langle \Delta u_L \rangle = -2Har = -2a^{-1/2}H_0 r$$

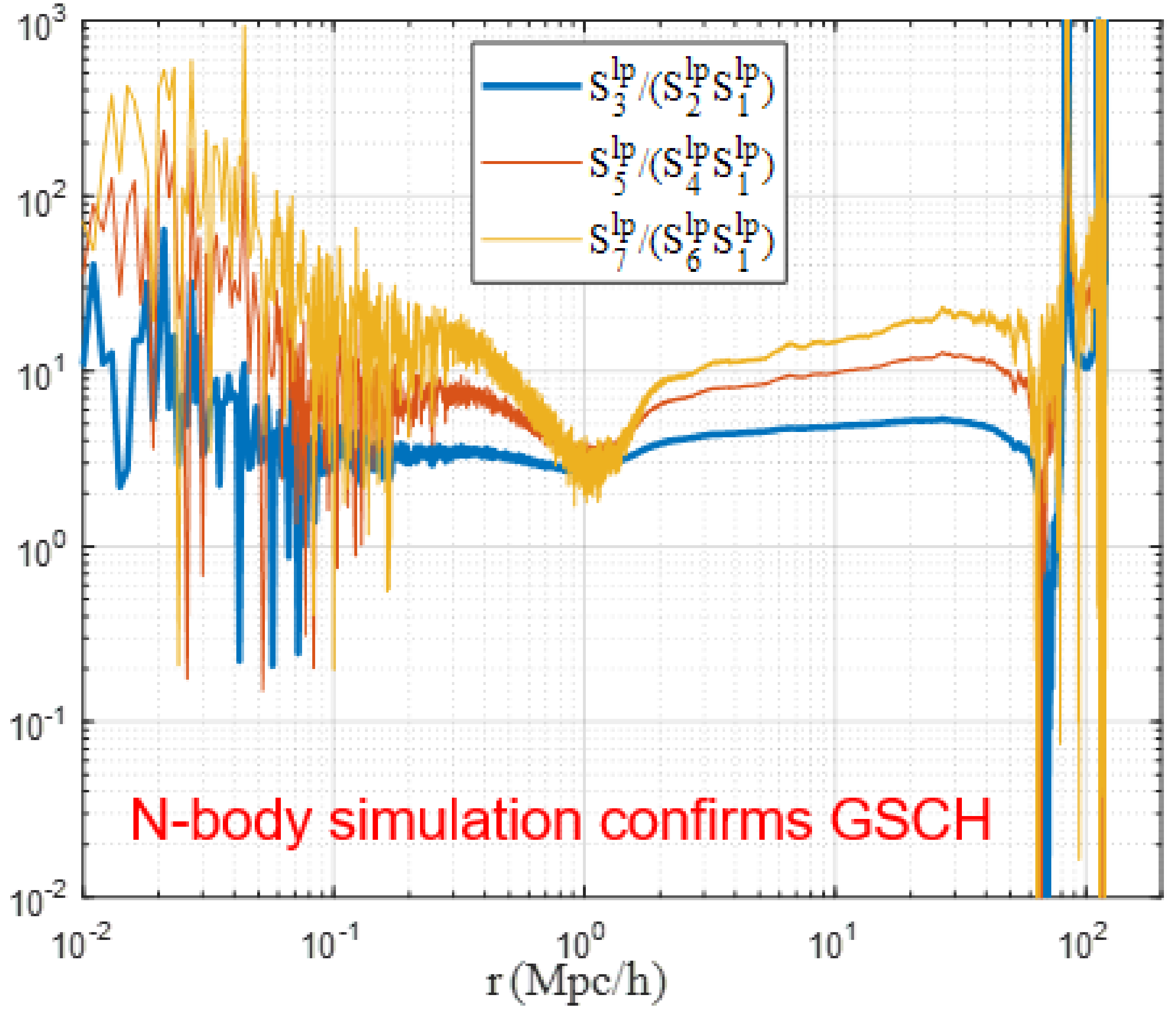
Stable clustering hypothesis (SCH) proved

$$\langle \Delta u_L^2 \rangle (r \to 0) = 4\langle \beta_s^2 u_i^2 \cos^2\theta_s \rangle > 0$$

Non-zero pairwise dispersion, a feature of collisionless flow

$$\langle \Delta u_L^{2m+1} \rangle = (2m+1)\langle \Delta u_L^{2m} \rangle \langle \Delta u_L \rangle = (2m+1)\langle \Delta u_L^{2m} \rangle (-2Har)$$

Generalized stable clustering hypothesis (GSCH)



Connections with spherical collapse model (SCM)

- Spherical collapse model (SCM) solves the motion of spherical shells. Many important insights can be obtained from SCM.
- There are fundamental connections between two-body collapse model (TBCM) and SCM.
- The original SCM describe exactly a two-body collapse with one-dimensional radial motion only and zero angular momentum.
- TBCM model describes a spherical collapse model with a non-zero angular momentum and non-radial orbits
- Both models predict a critical halo density ratio $\Delta=18\pi^2$, while TBCM can predict freefall and equilibrium collapse and SCH and GSCH.

$$\frac{d^2R}{dt^2}=-\frac{GM}{R^2}$$

Equation of motion for SCM in physical coordinate

$$\frac{\partial^2 r}{\partial s^2}+\frac{H_0}{2}\frac{\partial r}{\partial s}+\frac{GM}{2(2r)^2}=\underbrace{\frac{H_0^2}{2}r}_{1}$$

Equation of motion for SCM in comoving system

Term 1: due to the absence of a uniform background density

$$\frac{\partial^2 r}{\partial s^2}+\frac{H_0}{2}\frac{\partial r}{\partial s}+\frac{GM}{2(2r)^2}=\underbrace{\frac{(r_i v_i)^2}{r^3}\exp(-H_0 s)}_{2}$$

Equation of motion for two-body collapse model (TBCM)

Term 2: angular momentum

Summary and keywords

Harmonic oscillator	Transformed system	Free fall time
Critical damping	Two-body collapse	Expanding background
Stable clustering	Generalized SCH	Spherical collapse model
Equilibrium collapse	Freefall collapse	Critical halo density

- Formulate two-body collapse model (TBCM) that plays the same role as harmonic oscillator for fundamental understanding of gravitational collapse
- Propose the competition between gravity, expanding background, and angular momentum and classify collapse into: 1) freefall collapse for weak angular momentum; and 2) equilibrium collapse for weak damping
- Identify two critical values, $\beta_{s1}=1$ for free fall collapse and $\beta_{s2}=1/(3\pi)$ for equilibrium collapse, that quantifies the competition between damping and gravity
- Predict a critical halo density ratio of $18\pi^2$, same as the spherical collapse model.
- Prove the stable clustering hypothesis (SCH), i.e. mean pairwise velocity proportional to the separation r.
- Develop a generalized stable clustering hypothesis (GSCH) for higher order moments of pairwise velocity.

Pacific Northwest NATIONAL LABORATORY

Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion

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Introduction

Review: In freely decaying turbulence, there is no energy injection on large scale and total energy is continuously decaying with time.

- Both integral scale l (energy-contained scale) and energy dissipation rate ε vary with time.
- What is the large-scale dynamics of freely decaying turbulence? How does energy evolve with time?

$$\varepsilon \equiv A\frac{u^2}{(l/u)} = A\frac{u^3}{l}$$

Loitsyansky integral invariant (integral of velocity correlation):

$$\int \langle \mathbf{u}\cdot\mathbf{u}' \rangle \mathbf{r}^2 d\mathbf{r} \approx u^2 l^5 = const$$

$$u^2 \sim t^{-10/7}$$

$$l \sim t^{2/7}$$

$$\varepsilon \sim t^{-17/7}$$

Due to the formation and virilization of halos, the kinetic energy in dark matter flow continuously increases with time. In this regard, dark matter flow is a freely growing turbulence.

What is the large-scale dynamics of dark matter flow? How do energy/momentum evolve with time?

- Goal 1: Formulate large scale dynamics in dark matter flow (how energy and momentum evolves?)
- Goal 2: Energy, momentum and spin parameter in halos
- Goal 3: Formulate integral "constants" on large and halo scales (are they still constants?)

Equations of motion in comoving and transformed systems

Equation of motion in a comoving system with expanding background

Equation of motion in a transformed system with fixed damping in static background

$$\frac{d^2\mathbf{x}_i}{dt^2}+2H\frac{d\mathbf{x}_i}{dt}=-\frac{Gm_p}{a^3}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^3}$$

Potential with an arbitrary exponent of ***n*** for particle-particle interacting

$$V_p(r)=-G_n m_p^2/r^{-n}$$

$$\frac{d^2\mathbf{x}_i}{dt^2}+2H\frac{d\mathbf{x}_i}{dt}=\frac{nG_n m_p}{a^3}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}$$

Introduce a new transformed time scale **s**

$$ds/dt=a^p$$

- If p=-2, *s* is the time variable for integration in N-body simulation.
- Transformed system: fixed damping and no scale factor *a*;

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{d\mathbf{x}_i}{ds}(p+2)a^{-p}H=\frac{nG_n m_p}{a^{3+2p}}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}$$

$$\dot{H}=-3H^2/2$$

$$H^2=8\pi G\bar{\rho}_y(a)/3$$

$$H_0^2=H^2a^3$$

$$p=-3/2$$

Matter dominant

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{H_0}{2}\frac{d\mathbf{x}_i}{ds}=nG_n m_p\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}=\frac{\mathbf{F}_i}{m_p}$$

Peculiar velocity in comoving:

$$\mathbf{u}_i=a\frac{d\mathbf{x}_i}{dt}=\frac{d\mathbf{r}_i}{dt}-H\mathbf{r}_i=a^{-1/2}\mathbf{v}_i$$

Velocity in time scale **s**:

$$\mathbf{v}_i=\frac{d\mathbf{x}_i}{ds}=a^{3/2}\frac{d\mathbf{x}_i}{dt}=a^{1/2}\mathbf{u}_i$$

Energy evolution in transformed system

Starting from equation of motion in transformed system:

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{H_0}{2}\frac{d\mathbf{x}_i}{ds}=nG_n m_p\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}=\frac{\mathbf{F}_i}{m_p}$$

Express force as potential gradient:

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{1}{2}H_0\frac{d\mathbf{x}_i}{ds}+N\frac{\partial P_s}{\partial \mathbf{x}_i}=0$$

$$\bullet\left(\frac{d\mathbf{x}_i}{ds}\right)$$

Dot product on both sides:

Time evolution of energy in ***s***:

$$\frac{\partial\left(P_s+K_s\right)}{\partial s}+H_0K_s=0$$

Exactly same as damped oscillator. Need additional relation to close.

Time evolution of energy in ***t***:

$$\frac{\partial}{\partial t}\left(K_p+a^{-n-1}P_y\right)+H\left(2K_p+a^{-n-1}P_y\right)=0$$

With n=-1

$$\frac{\partial}{\partial t}\left(K_p+P_y\right)+H\left(2K_p+P_y\right)=0$$

Standard cosmic energy equation

Specific potential energy (radial moment):

$$P_s=\frac{1}{N}\sum_{i}^{N}\phi\left(\mathbf{x}_i\right)=a^{-n}P_y$$

Potential energy in physical coordinate

Specific kinetic energy of entire system:

$$K_s=\frac{1}{2N}\sum_{i=1}^{N}\mathbf{v}_i^2=\frac{a}{2N}\sum_{i=1}^{N}\mathbf{u}_i^2=aK_p$$

Peculiar kinetic energy

Recall solution from Two-body collapse model (TBCM): exponential evolution of energy

$$\boxed{K_s=\alpha\exp\left(-\frac{H_0 s}{1+\beta/\alpha}\right)\quad P_s=\beta\exp\left(-\frac{H_0 s}{1+\beta/\alpha}\right)}$$

Energy evolution in comoving system and ε_u

Transformation back to comoving system:

$$ds/dt = a^{-3/2} \Rightarrow s = t_0 \ln(t/t_i)$$

Exponential in ***s*** corresponds to power-law in ***t***:

$$\exp(\tau H_0 s) \to (a/a_i)^\tau$$

Power-law time evolution for energy in terms of rate of energy cascade ε_u:

$$K_p = -\varepsilon_{\mathbf{u}} t^{-\frac{2}{3}\frac{(2+\beta/\alpha)}{(1+\beta/\alpha)}} = -\varepsilon_{\mathbf{u}} a^{-\frac{(2+\beta/\alpha)}{(1+\beta/\alpha)}}$$

Power-law for Peculiar kinetic energy

$$P_y = a^n P_s = \frac{\beta}{\alpha}\varepsilon_{\mathbf{u}} a^{n-\frac{1}{(1+\beta/\alpha)}}$$

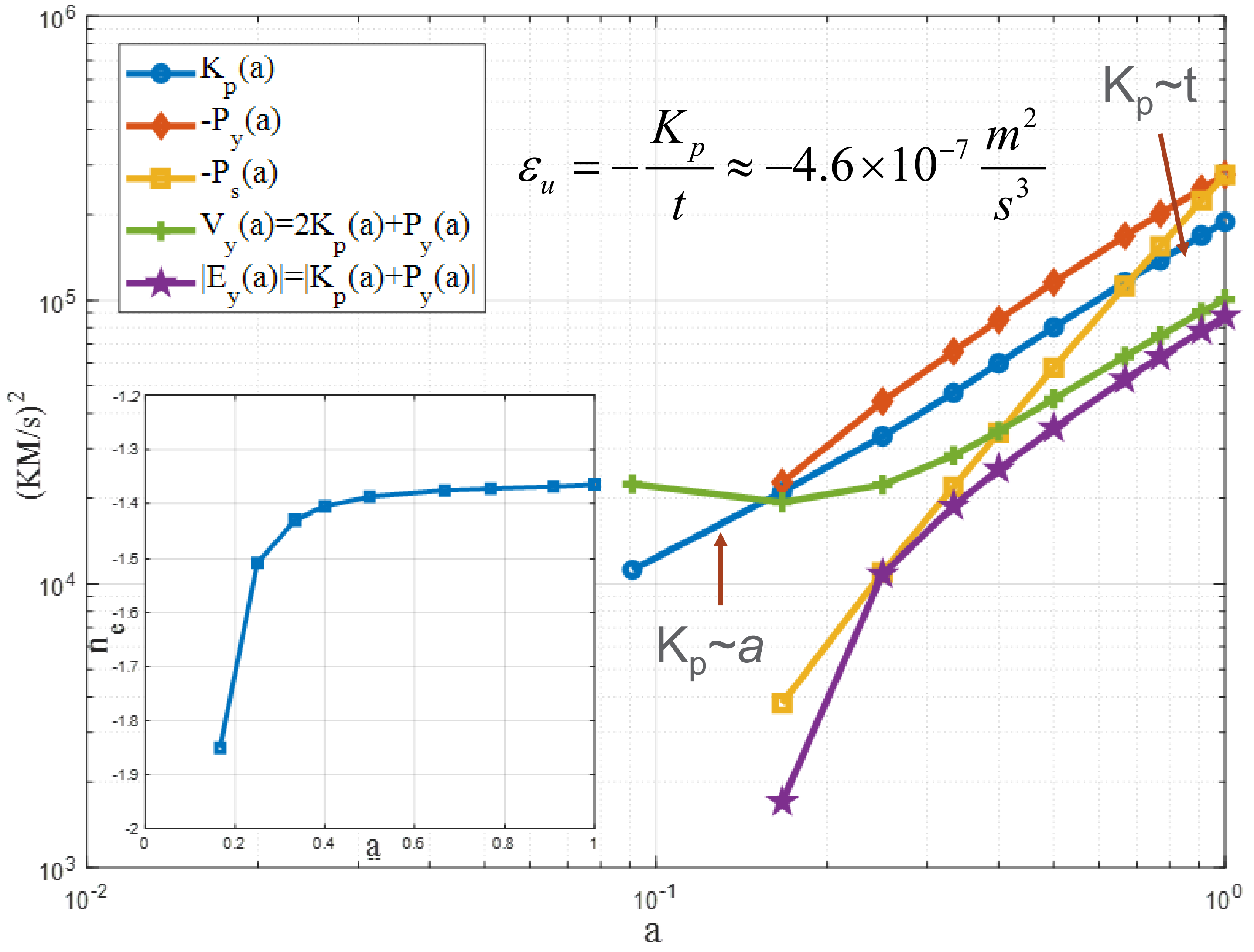
Power-law for potential energy

N-body simulation Early time: $K_p \propto a$; Early time: $K_p \propto t$;

Effective exponent for virial theorem:

$$n_e = \frac{2K_p}{P_y} = -\frac{10}{7} \neq -1$$

[Mostly from Halo surface energy]



The variation of kinetic and potential energies with scale factor *a* from a N-body simulation. Both energies exhibit a power-law scaling.

Momentum evolution in transformed system

Starting from equation of motion in transformed system:

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{H_0}{2}\frac{d\mathbf{x}_i}{ds}=nG_n m_p\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}=\frac{\mathbf{F}_i}{m_p}$$

Express force as potential (P_s) gradient:

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{1}{2}H_0\frac{d\mathbf{x}_i}{ds}+N\frac{\partial P_s}{\partial \mathbf{x}_i}=0$$

Dot product on both sides

$\bullet\mathbf{x}_i$

Time evolution of virial quantity in ***s***:

$$\frac{dG_s}{ds}+\frac{1}{2}H_0G_s=2K_s-nP_s$$

Time evolution of virial quantity in ***t***:

$$\frac{dG_p}{dt}+HG_p=\frac{2aK_p-na^{-n}P_y}{a^2}$$

Specific virial quantity (radial moment):

$$G_s=\frac{1}{N}\sum_i^N\mathbf{v}_i\cdot\mathbf{x}_i=a^{1/2}\frac{1}{N}\sum_i^N\mathbf{u}_i\cdot\mathbf{x}_i=a^{1/2}G_p$$

Peculiar virial quantity

Specific angular momentum:

$$\mathbf{H}_s=\frac{1}{N}\sum_i^N\mathbf{x}_i\times\mathbf{v}_i=\frac{1}{N}\sum_i^N\mathbf{x}_i\times\mathbf{u}_i a^{1/2}=a^{1/2}\mathbf{H}_p$$

Peculiar angular momentum

Taking cross product on both sides

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{1}{2}H_0\frac{d\mathbf{x}_i}{ds}+N\frac{\partial P_s}{\partial \mathbf{x}_i}=0 \qquad \times\mathbf{x}_i$$

Time evolution of angular momentum in ***s***:

$$\frac{d\mathbf{H}_s}{ds}+\frac{H_0}{2}\mathbf{H}_s=-\sum_i^N\mathbf{x}_i\times\frac{\partial P_s}{\partial \mathbf{x}_i}=\frac{1}{m_p}\sum_i^N\mathbf{x}_i\times\mathbf{F}_i=0$$

- This is for open system without boundary.
- Extra care is needed for N-body systems with periodic boundaries

The evolution of momentum on halo and large scale

Virial quantity in entire N-body system:

$$G_p = \frac{1}{N}\sum_i^N \mathbf{u}_i \cdot \mathbf{x}_i = \frac{1}{a} G_{py}$$

Subscript
- "p" for Comoving
- "py" for physical coordinate

Angular momentum in entire N-body system:

$$\mathbf{H}_p = \frac{1}{N}\sum_{i=1}^N \mathbf{x}_i \times \mathbf{u}_i = \frac{1}{a}\mathbf{H}_{py}$$

Decompose both position x and velocity u:

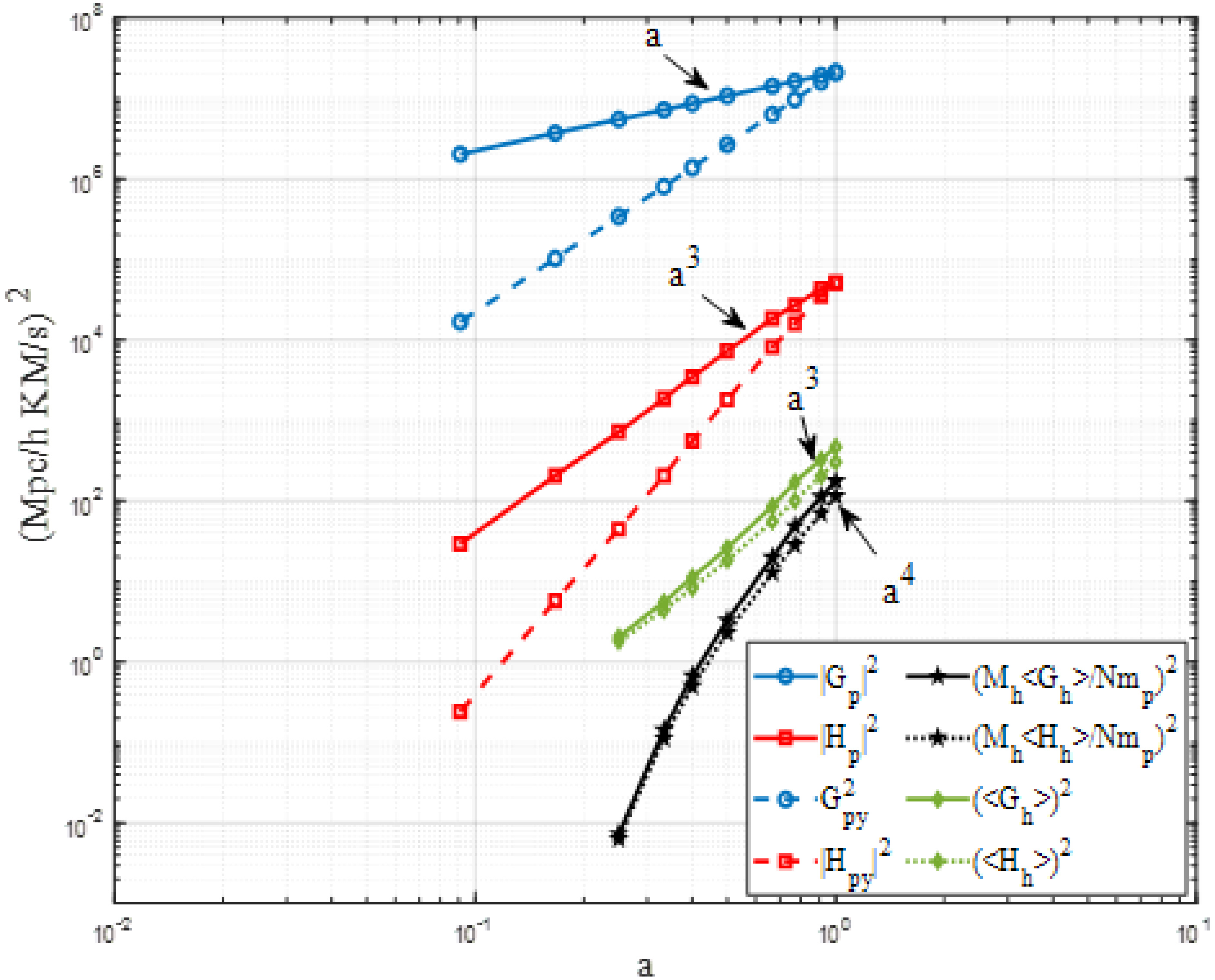
$$\mathbf{x}_i = \mathbf{x}_h + \mathbf{x}_i' \qquad \mathbf{u}_i = \mathbf{u}_h + \mathbf{u}_i'$$

Halo virial quantity (radial momentum):

$$G_{hc} = \frac{1}{n_p}\sum_{i=1}^{n_p}\left(\mathbf{x}_p' \cdot \mathbf{u}_p'\right) = \frac{1}{a} G_h$$

Halo angular momentum:

$$\mathbf{H}_{hc} = \frac{1}{n_p}\sum_{i=1}^{n_p}\left(\mathbf{x}_p' \times \mathbf{u}_p'\right) = \frac{1}{a}\mathbf{H}_h$$



On large scale (see here for proof)

$$G_{py} \propto a^{3/2}$$
$$\left|\mathbf{H}_{py}\right| \propto a^{5/2}$$

On halo scale (Consistent with previous results)

$$\left\langle G_h \right\rangle \propto a^{3/2} \propto t$$
$$\left\langle \left|\mathbf{H}_h\right| \right\rangle \propto a^{3/2} \propto t$$

The variation of energy in halos

- Identify all halos of different sizes
- Group halos according to halo size n_p or m_h
- Compute mean square radius r_g for each halo
- Compute halo virial kinetic energy σ_v^2 for each halo
- Compute intra-halo potential energy Φ_h
- Compute the group average and std.

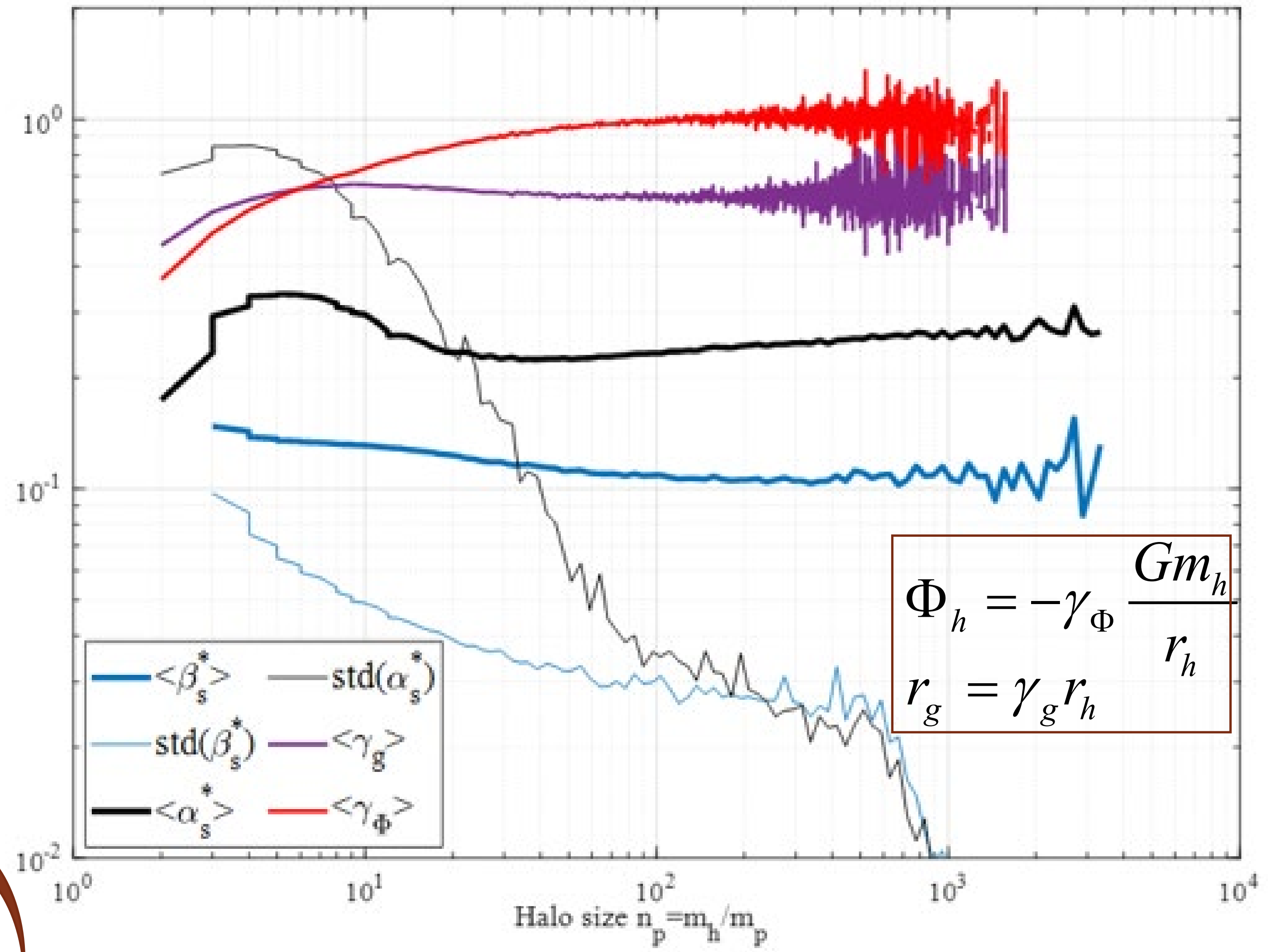
Halo virial kinetic energy: $\sigma_v^2 = \frac{1}{3n_p}\sum_{k=1}^{n_p}\left|\mathbf{u}_k - \mathbf{u}_h\right|^2$

Halo velocity: $\mathbf{u}_h = \frac{1}{n_p}\sum_{k=1}^{n_p}\mathbf{u}_k$

$$\beta_s^* = \frac{Hr_g}{\sigma_v} \qquad \alpha_s^* = \frac{\sigma_v^2 r_g}{Gm_h} \qquad \Delta_c = 18\pi^2 \qquad \gamma_v = -\frac{3\sigma_v^2}{\Phi_h}$$

Angle of incidence | Ratio of kinetic to potential energy | Critical density ratio | Halo virial ratio

$$\gamma_g = \left(\frac{1}{2}\alpha_s^*\beta_s^{*2}\Delta_c\right)^{1/3} \qquad \gamma_\Phi = \frac{6\left(\alpha_s^*\beta_s^{*2}\Delta_c/2\right)^{2/3}}{\gamma_v\beta_s^{*2}\Delta_c}$$



$$\Phi_h = -\gamma_\Phi\frac{Gm_h}{r_h}$$
$$r_g = \gamma_g r_h$$

Small halos of same size are generated at different time (large std). Large halos are synchronized and generated at the same time (small std).

The variation of momentum in halos

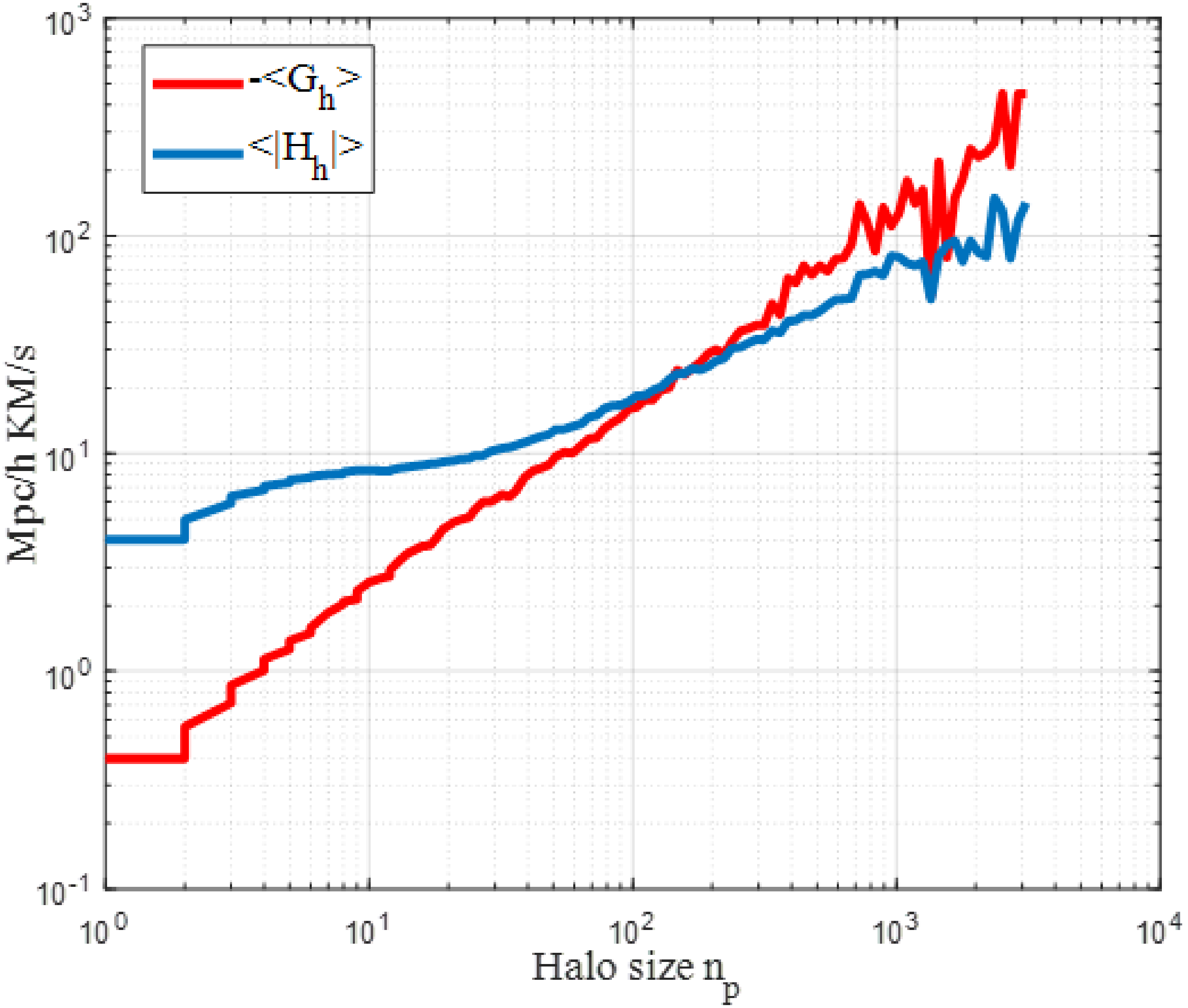
- Compute mean square radius r_g for each halo
- Compute halo virial kinetic energy σ_v^2 for each halo
- Compute radial momentum G_h for each halo
- Compute angular momentum H_h for each halo
- Compute the group average and std.

$$G_h = -\tau_s^* \sigma_v r_g a^{-1} \qquad |\mathbf{H}_h| = \eta_s^* \sigma_v r_g a^{1/2}$$

$$n_s^* = \frac{2K_h}{\Phi_h} = \frac{3\sigma_v^2}{\Phi_h} = -\gamma_v \qquad z_s^* = \frac{E_h}{\sigma_v^2} = \frac{K_h + \Phi_h}{\sigma_v^2}$$

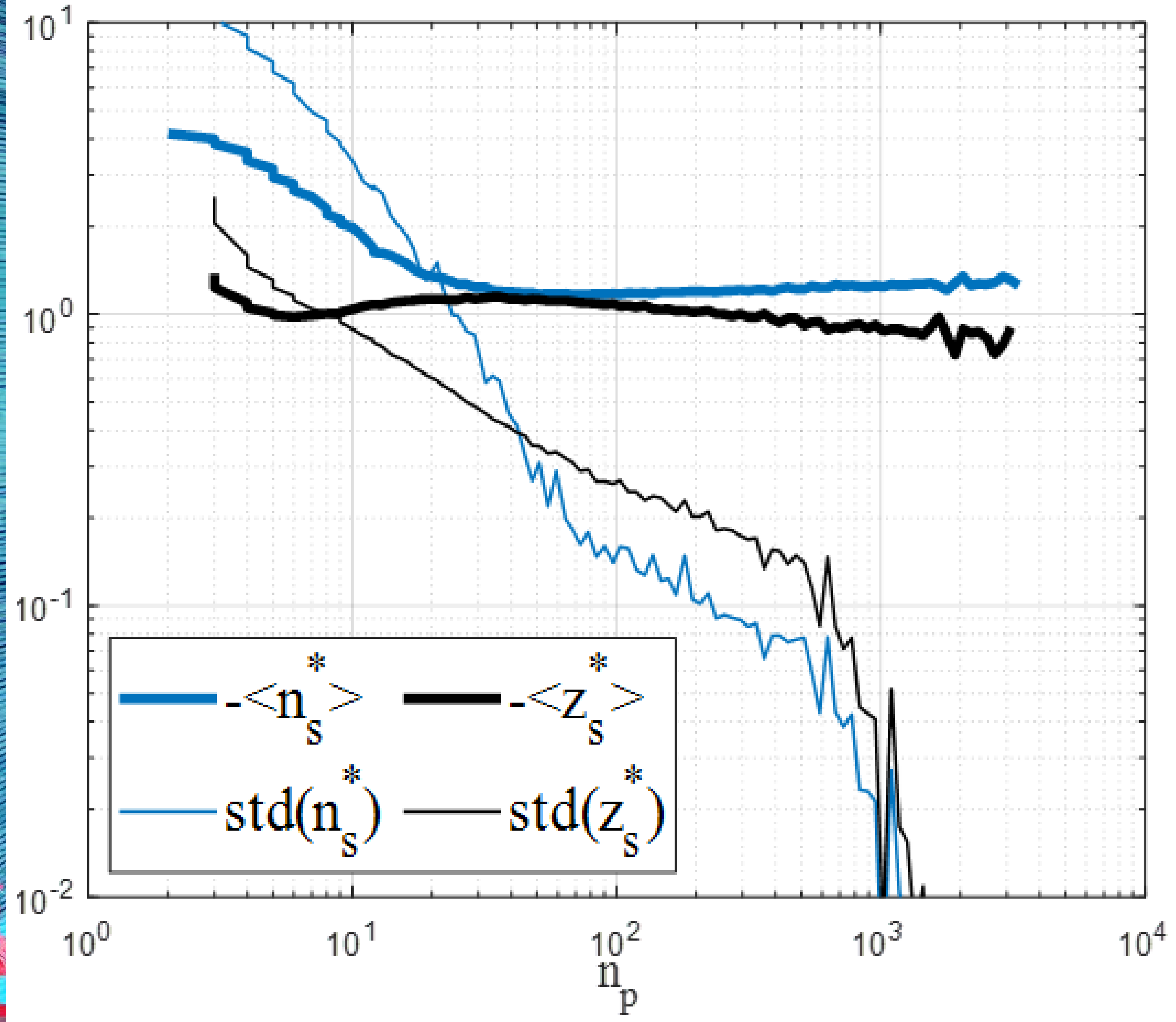
$$\lambda_p = a^{1/2} \alpha_s^* \eta_s^* \sqrt{|z_s^*|}$$

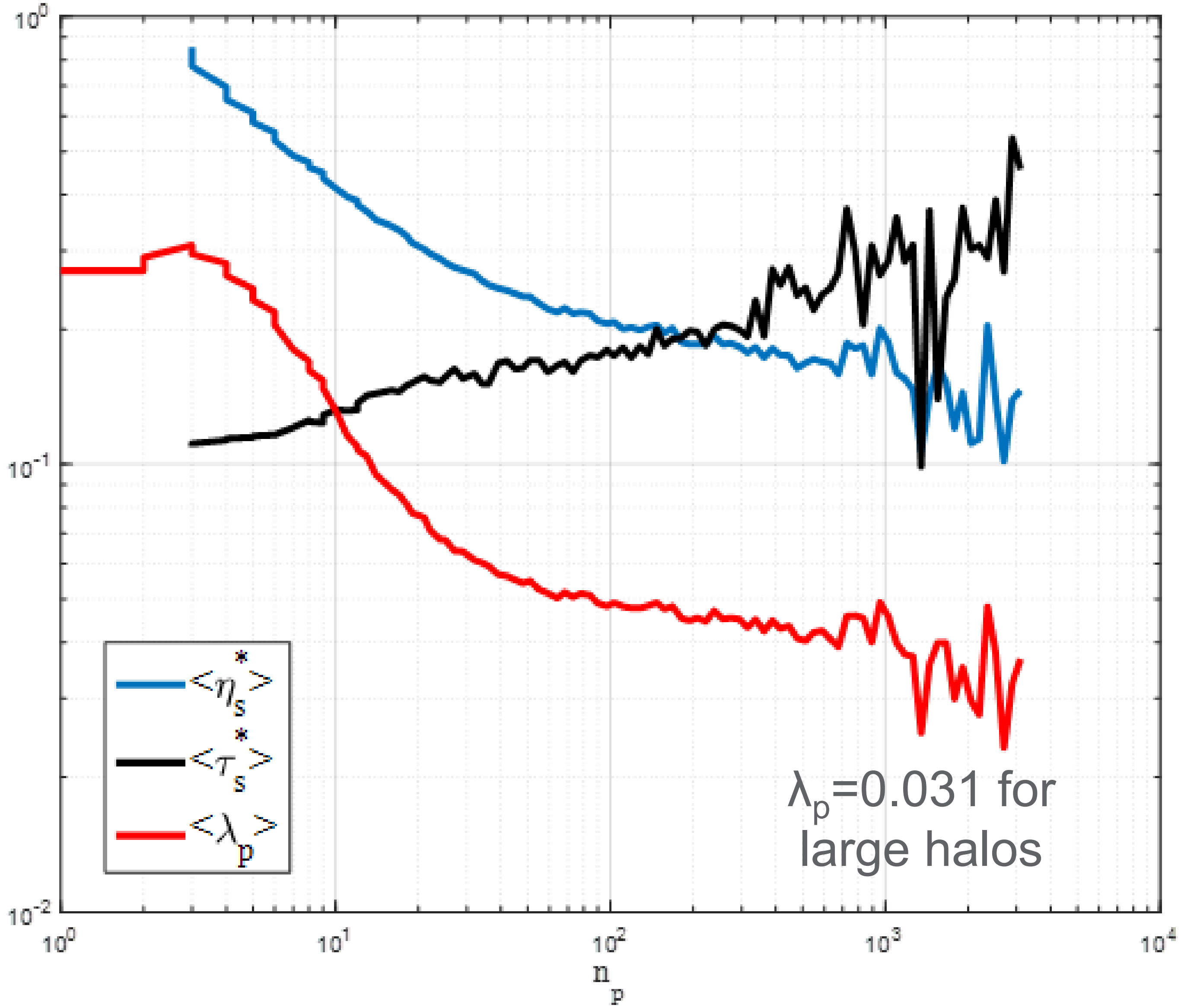
$$\lambda_p = \frac{\sqrt{2}}{2} \frac{(m_1 m_2)^{3/2}}{(m_1 + m_2)^3} = \frac{\sqrt{2}}{16} \approx 0.0884$$



The variation of momentum with halo size from a N-body simulation.

The variation of momentum in halos





Relevant parameters for halo energy and momentum

Table 2. Relevant parameters for halo energy, momentum and spin from theory and simulations

Type of halos	γ_Φ	γ_g	γ_v	Δ_c	α_s^* Eq.(54)	β_s^* Eq.(54)
Two-body halos (theory)	1/4	1/2	1.0	$18\pi^2$	1/24	$\sqrt{3}/(3\pi)$
Large halos (NFW)	0.936 Eq.(49)	0.567 Eq.(48)	1.3	$18\pi^2$	0.230	0.095
Large halos (isothermal)	1 Eq.(45)	$\sqrt{3}/3$ Eq.(43)	1.5	$18\pi^2$	$\sqrt{3}/6$ Eq.(54)	$\sqrt{2/3}/(3\pi)$ Eq.(54)

	n_s^* Eq.(68)	z_s^* Eq.(68)	η_s^* Eq.(63)	τ_s^* Eq.(63)	$f_H(m_h)$ Eq.(63)	$f_G(m_h)$ Eq.(63)	λ_p Eq.(70)
Two-body halos (theory)	-1.0	-1.5	$\sqrt{3}$	$\sqrt{3}/(3\pi)$	3π	1	$\sqrt{2}/16$
Large halos (NFW)	-1.3	-0.81	0.151	0.103	1.59	1.08	0.031
Large halos (isothermal)	-1.5	-0.5	$\sqrt{2/3}/(3\pi)$	$\sqrt{2/3}/(3\pi)$	1	1	$1/(18\pi)$

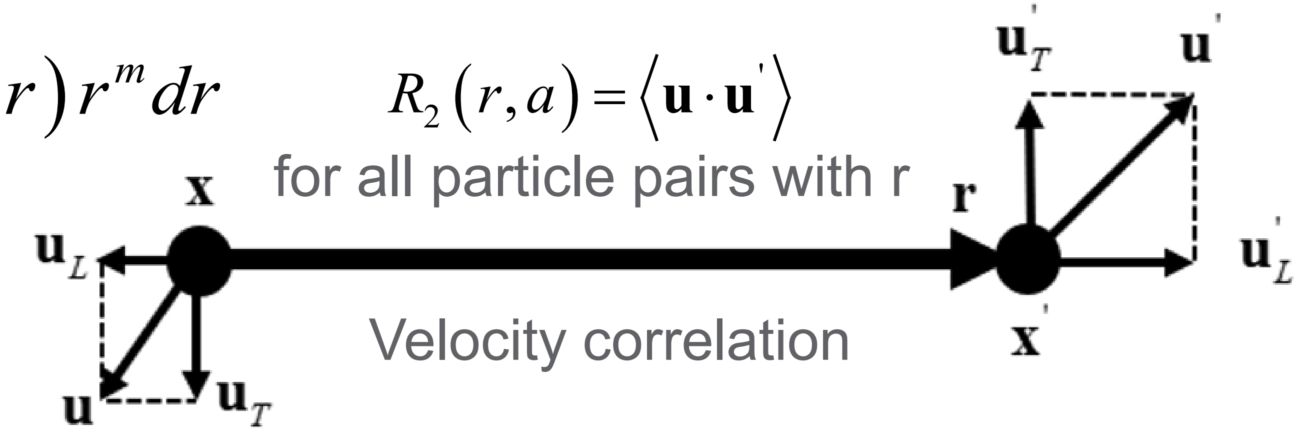
Integral constants I_m and physical meaning of I_2

The virial quantity (radial momentum) and angular momentum are intimately related to integral constants for dynamics of dark matter flow. Starting from the velocity correlation function R_2, defining

$$I_m = \int \langle \mathbf{u} \cdot \mathbf{u}' \rangle r^{m-2} d\mathbf{r}^3 = \int R_2(r) r^{m-2} d\mathbf{r}^3 = \int_0^\infty 4\pi R_2(r) r^m dr$$

$$R_2(r,a) = \langle \mathbf{u} \cdot \mathbf{u}' \rangle$$

for all particle pairs with r



Energy spectrum is Fourier transform of R_2:

$$E_u(k) = \frac{1}{\pi} \int_0^\infty R_2(r) kr \sin(kr) dr$$

Integral constant I_m is the derivative of spectrum at long wave-length limit (large scale):

$$I_m = 4\pi^2 \frac{(-1)^{1+m/2}}{m} \left. \frac{\partial^m E_u}{\partial k^m} \right|_{k=0} \propto a \quad \text{with} \quad E_u(k \to 0) \propto a$$

I_2 is related to the linear momentum. This leads to a k^4 velocity spectrum on large scale.

Assume linear momentum vanishes: $$\int_V \mathbf{u} d\mathbf{x}^3 = 0$$

$$I_2 = \int \langle \mathbf{u} \cdot \mathbf{u}' \rangle d\mathbf{r}^3 = \lim_{V \to \infty} \frac{1}{V} \int_V \int_V \langle \mathbf{u} \cdot \mathbf{u}' \rangle d\mathbf{x}^3 d\mathbf{x}'^3 = \lim_{V \to \infty} V \left\langle \left(\frac{1}{V} \int_V \mathbf{u} d\mathbf{x}^3 \right)^2 \right\rangle = 0 \Rightarrow \left. \frac{\partial^2 E_u}{\partial k^2} \right|_{k=0} = 0$$

Physical meaning of integral constants I_4

Defined in comoving coordinates:

$$G=\frac{1}{V}\int_V \mathbf{x}\cdot\mathbf{u}dx^3$$ Virial quantity

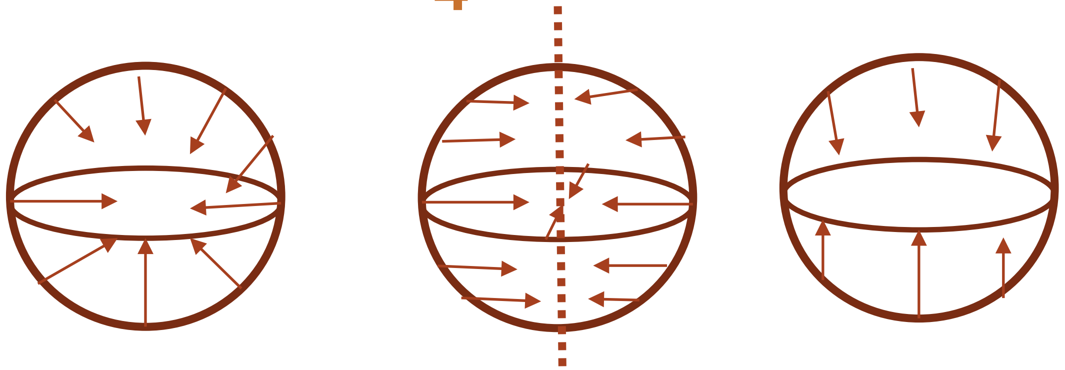
$$\mathbf{H}=\frac{1}{V}\int_V \mathbf{x}\times\mathbf{u}dx^3$$ Angular momentum

$$\mathbf{M}=\frac{1}{V}\int_V \mathbf{x}\otimes\mathbf{u}dx^3$$ Momentum tensor

$$\mathbf{I}=\frac{1}{V}\int_V \mathbf{x}\otimes\mathbf{x}dx^3$$ Inertial tensor

$$T=\mathbf{M}:\mathbf{M}$$ Contraction of momentum tensor

$\bar{u}_{k,k}$ is mean divergence



$$\mathbf{M}=\begin{bmatrix} G/3 & 0 & 0 \\ 0 & G/3 & 0 \\ 0 & 0 & G/3 \end{bmatrix}\quad \mathbf{M}=\begin{bmatrix} G/2 & 0 & 0 \\ 0 & G/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\quad \mathbf{M}=\begin{bmatrix} G & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\alpha_T=3$ $\alpha_T=2$ $\alpha_T=1$

$$I_4=-2\lim_{V\to\infty}\left(\langle T\rangle V\right)=\lim_{V\to\infty}\left[-|\mathbf{H}|^2+(\mathbf{M}:\mathbf{I})\bar{u}_{k,k}\right]V$$

$\alpha_T=3$ if structure collapsing into a point,
$\alpha_T=2$ if collapsing into a filament (N-body)
$\alpha_T=1$ if collapsing into a plane.

For incompressible flow with vanishing divergence ($u_{k,k}=0$), I_4 is related to the angular momentum of entire system

$$I_4=-\lim_{V\to\infty}\left(\left\langle|\mathbf{H}|^2\right\rangle V\right)$$

For dark matter flow with vanishing **H** on large scale, Both **M** and **I** are diagonal. I_4 is related to the virial quantity (radial momentum) of entire system.

$$\alpha_T T=\alpha_T(\mathbf{M}:\mathbf{M})\approx G^2\gg|\mathbf{H}|^2$$

$$I_4=-\frac{2}{\alpha_T}\lim_{V\to\infty}\left(\left\langle G^2\right\rangle V\right)$$

Evolution of momentum on large scale

Integral constant on large scale:

$$I_4 = -\frac{2}{\alpha_T}\lim_{V\to\infty}\left(\left\langle G^2\right\rangle V\right) = \lim_{V\to\infty}\left(\left\langle(\mathbf{M}:\mathbf{I})\bar{u}_{k,k}\right\rangle V\right)$$

$$\left\langle G^2\right\rangle = -\frac{\alpha_T}{2}\left\langle(\mathbf{M}:\mathbf{I})\bar{u}_{k,k}\right\rangle$$

$$\mathbf{I} = \frac{1}{V}\int_V x_i x_j d\mathbf{x}^3 = \frac{1}{3}\left(\frac{L}{2}\right)^2 \delta_{ij}$$

Inertial tensor on large scale

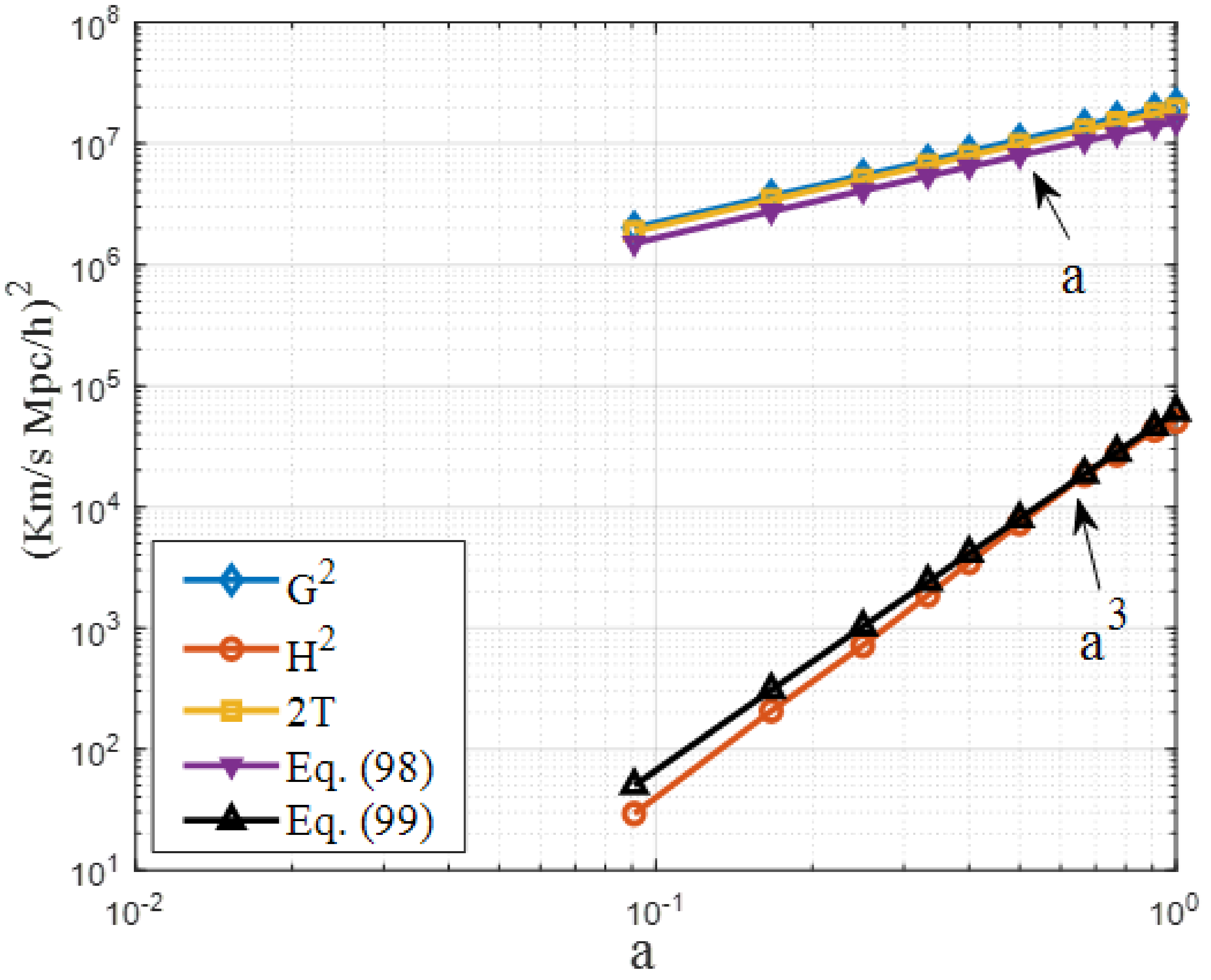
$$G = \frac{1}{V}\int_V x_j u_i \, d\mathbf{x}^3 \delta_{ij} = \mathbf{M}:\boldsymbol{\delta}$$

Virial quantity on large scale

$$G = \frac{\alpha_T}{24} L^2 \bar{u}_{k,k}$$

Virial quantity is related to divergence or density contrast

$$\boxed{2T \approx \left\langle G^2\right\rangle = \frac{\alpha_T^2}{8} a_0 u^2 r_2^2 \propto a \quad \left\langle |\mathbf{H}|^2\right\rangle = 0.002 a_0 u^2 r_2^2 a^2 \propto a^3}$$



Time variation of momentum with scale factor *a* from N-body simulation

Momentum and integration constants on halo scale

On small (halo) scale, velocity field is of constant divergence and matter density is non-uniform.

$$I_4 = -2\lim_{V\to\infty}\left(\langle T\rangle V\right) = \lim_{V\to\infty}\left[-|\mathbf{H}|^2 + (\mathbf{M}:\mathbf{I})\overline{u}_{k,k}\right]V$$

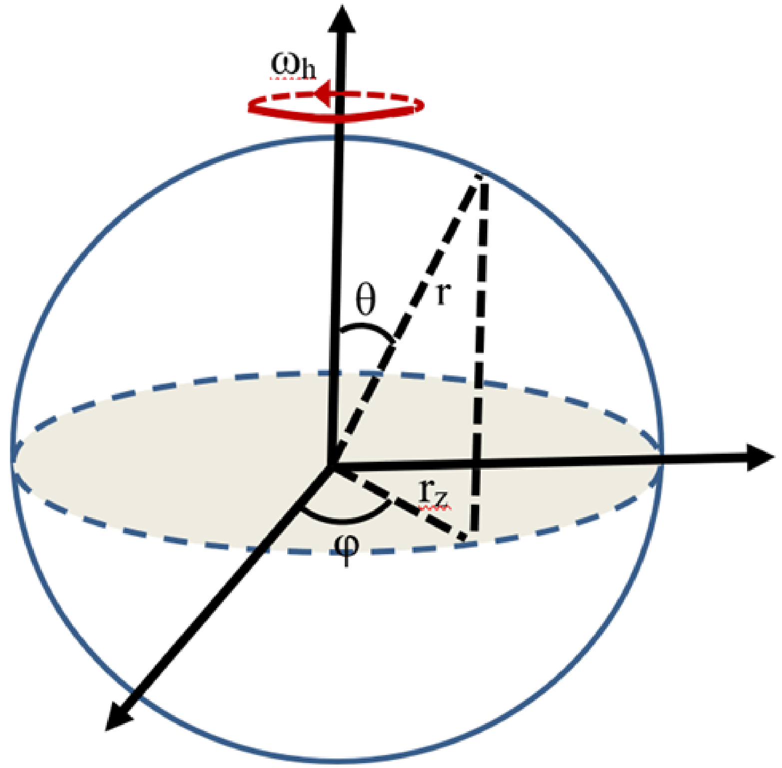
Momentum tensor:

$$\mathbf{M} = \frac{1}{m_h}\int_V \mathbf{x}\otimes\mathbf{u}\rho_h d\mathbf{x}^3 = \begin{bmatrix} G/3 & -|\mathbf{H}|/2 & 0 \\ |\mathbf{H}|/2 & G/3 & 0 \\ 0 & 0 & G/3 \end{bmatrix}$$

Inertial tensor:

$$\mathbf{I} = \frac{1}{m_h}\int_V x_i x_j \rho_h d\mathbf{x}^3 = \frac{1}{3}r_g^2\delta_{ij}$$

$$T = \mathbf{M}:\mathbf{M} = \left(\frac{1}{3}G^2 + \frac{1}{2}|\mathbf{H}|^2\right)$$



Halo radial and angular momentum are equal

$$G = -|\mathbf{H}| = -Hr_g^2$$

$$\alpha_T = \frac{G^2}{T} = \frac{6}{2+3|\mathbf{H}|^2/G^2} = \frac{6}{5}$$

Summary and keywords

Large scale dynamics	Comoving/transformed system	Rate of energy cascade
Integration constants	Radial/angular momentum	Spin parameter
Velocity correlation function	Velocity spectrum function	Effective potential exponent

- The energy and momentum evolution of N-body system is analytically derived. This is made possible by introducing a new time scale *s*.
- The kinetic and potential energy of N-body system increase linearly with time with a constant rate of energy production ε_u.
- For entire N-body system, the radial momentum scales as $G_{py} \sim a^{3/2}$, while angular momentum $H_{py} \sim a^{5/2}$.
- The specific momentum (radial and angular) in halos scale as $\sim a^{3/2}$
- At same redshift, the analytically derived halo spin parameter decreases with halo mass, i.e. $\lambda_p = 0.09$ for typical two-particle halos and $\lambda_p = 0.031$ for large halos.
- The spin parameter of a given halo is a constant of time for early-stage halos with faster mass accretion and increases with time for late-stage halos with slower mass accretion.
- The radial/angular momentum are closely related to integral "constants" I_m that is defined as integral of velocity correlation or the *m*th derivative of energy spectrum at small *k*.

Statistical (correlation-based) approach for dark matter flow

The statistical theory of dark matter flow (second order)

Xu Z., 2022, arXiv:2202.00910 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2202.00910

Introduction

Review:
Statistical theory in hydrodynamic turbulence

- Kinematic relations between statistical measures
  - Correlation functions
  - Structure functions
  - Power spectrum functions
- Incompressible on all scales
  - Divergence-free
  - Constant density

- N-body simulations are invaluable to understand dark matter flow (DMF).
- Fundamental problems when projecting N-body velocity field onto structured grids:
  - Velocity field is only sampled by N-body simulations at discrete locations of particles.
  - The sampling has a poor quality at locations with low particle density
  - Velocity field can be multi-valued and discontinuous due to the collisionless nature.

Goal 1: what are the kinematic relations in dark matter flow?

Goal 2: what is the nature of dark matter flow on different scales?

Approach:

- Use pairwise average for real-space two-point statistics to avoid projecting

- Take advantage of symmetry implied by the assumptions of homogeneity and isotropy.

- Develop kinematic relations between different statistical measures

- Identify the nature of DM flow, i.e. incompressible, constant divergence, or irrotational flow.

Two-point first order velocity correlation tensor

General correlation tensor between velocity field and a scalar field p(x):

$$Q_i(\mathbf{x},\mathbf{r}) = \langle u_i(\mathbf{x})\, p(\mathbf{x}') \rangle \qquad \mathbf{x}' = \mathbf{x} + \mathbf{r}$$

Pairwise average: Averaging over all particle pairs with the same separation r.

Reduced to function of r due to homogeneity and isotropy:

$$Q_i(\mathbf{x},\mathbf{r}) \equiv Q_i(\mathbf{r}) \equiv Q_i(r) = A_1(r)\, r_i$$

Divergence of first order tensor:

$$\frac{\partial Q_i(r)}{\partial r_i} = -\langle (\nabla\cdot\mathbf{u}(\mathbf{x}))\, p(\mathbf{x}') \rangle = 3A_1 + \frac{\partial A_1}{\partial r} r$$

Curl of first order tensor (always zero):

$$\nabla\times\mathbf{Q}(\mathbf{x},\mathbf{r}) = \langle (\nabla\times\mathbf{u}(\mathbf{x}))\, p(\mathbf{x}') \rangle = -\varepsilon_{ijk}\left(A_1\delta_{ik} + \frac{r_i r_k}{r}\frac{\partial A_1}{\partial r} \right) = 0$$

the Levi-Civita symbol satisfies the identity $\varepsilon_{ijk}\delta_{jk} = 0 \quad \varepsilon_{ijk} r_j r_k = \mathbf{r}\times\mathbf{r} = 0$

Incompressible flow	Constant divergence
$A_1(r) = 0$	$A_1(r) = -\theta\langle p(\mathbf{x})\rangle / 3$
$Q_i(r) = 0$	$Q_i(r) = -\left(\frac{\theta}{3}\langle p(\mathbf{x})\rangle\right) r_i$

- The first order correlation tensor must vanish for incompressible flow
- The curl of the first order correlation tensor is always zero for any flow

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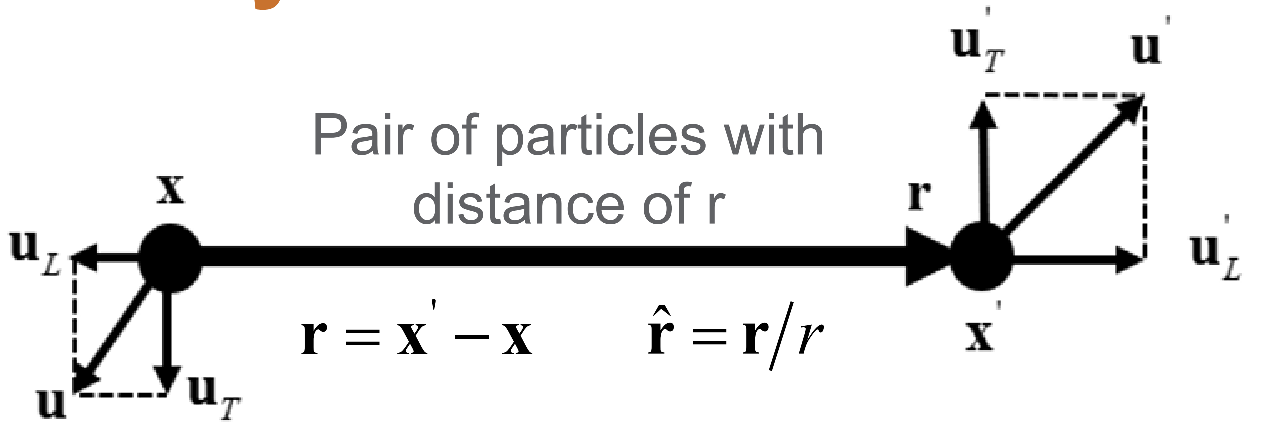
Two-point second order velocity correlation tensors

Second order velocity correlation tensor:

$$Q_{ij}(\mathbf{r}) = Q_{ij}(r) = \left\langle u_i(\mathbf{x})\, u_j(\mathbf{x}')\right\rangle$$



General form of isotropic second order tensor:

$$Q_{ij}(\mathbf{r}) = Q_{ij}(r) = A_2(r)\, r_i r_j + B_2(r)\,\delta_{ij}$$

Divergence of second order tensor:

$$Q_{ij,i} = \left(4A_2 + \frac{\partial A_2}{\partial r} r + \frac{1}{r}\frac{\partial B_2}{\partial r}\right) r_j$$

Used to derive
Kinematic relations

Longitudinal velocity:	Transverse velocity:
$u_L = \mathbf{u}\cdot\hat{\mathbf{r}} = u_i \hat{r}_i$	$\mathbf{u}_T = -(\mathbf{u}\times\hat{\mathbf{r}}\times\hat{\mathbf{r}})$
$u_L' = \mathbf{u}'\cdot\hat{\mathbf{r}} = u_i' \hat{r}_i$	$\mathbf{u}_T' = -(\mathbf{u}'\times\hat{\mathbf{r}}\times\hat{\mathbf{r}})$

$$Q_{ij,i} = -\left\langle (\nabla\cdot u_i(\mathbf{x}))\, u_j(\mathbf{x}')\right\rangle = 0$$ ⬅ Incompressible flow

$$Q_{ij,i} = -\left\langle (\nabla\cdot u_i(\mathbf{x}))\, u_j(\mathbf{x}')\right\rangle = -\theta\left\langle u_j(\mathbf{x}')\right\rangle = 0$$ ⬅ Constant divergence flow

Velocity difference or Pairwise velocity: $\Delta u_L = u_L' - u_L$

Velocity sum: $\sum u_L = u_L' + u_L$

Curl of second order tensor:

$$\nabla\times Q_{ij}(r) = \varepsilon_{imj} r_m \left(A_2 - \frac{1}{r}\frac{\partial B_2}{\partial r}\right) = 0$$ ⬅ Irrotational flow

Same even order kinematic relations for incompressible flow and constant divergence flow

Two-point second order velocity correlation functions

Using index contraction of second order tensor to define three scalar correlation functions

Total correlation function:

$$R_2(r) = Q_{ij}\delta_{ij} = \langle \mathbf{u}\cdot\mathbf{u}' \rangle = \langle u_i u_i' \rangle = A_2 r^2 + 3B_2$$

Longitudinal correlation function

$$L_2(r) = Q_{ij}\, r_i r_j / r^2 = \langle u_L u_L' \rangle = A_2 r^2 + B_2$$

Transverse correlation function

$$T_2(r) = Q_{ij} n_i n_j = \langle \mathbf{u}_T \cdot \mathbf{u}_T' \rangle / 2 = B_2(r)$$

$$R_2(r) = 2R(r) = L_2(r) + 2T_2(r)$$

Two correlation coefficients can be defined for longitudinal and transverse velocity:

$$\rho_L(r) = \frac{\langle u_L u_L' \rangle}{\langle u_L^2 \rangle} \quad \text{and} \quad \rho_T(r) = \frac{\langle \mathbf{u}_T \cdot \mathbf{u}_T' \rangle}{\langle |\mathbf{u}_T|^2 \rangle}$$

The velocity power spectrum and correlation function form Fourier transform pair

$$R(r) = \int_0^\infty E_u(k) \frac{\sin(kr)}{kr} dk$$

$$E_u(k) = \frac{2}{\pi} \int_0^\infty R(r)\, kr \sin(kr)\, dr$$

Integral scale: the length scale within which velocities are appreciably correlated

$$l_{u0} = \frac{1}{u^2} \int_0^\infty R(r)\, dr = \frac{\pi}{2u^2} \int_0^\infty E_u(k) k^{-1} dk$$

One-dimensional RMS (root-mean-square) velocity:

$$u(a) = \left(\frac{1}{3} \langle \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \rangle \right)^{1/2}$$

Kinematic relations for correlation functions

For incompressible flow or constant divergence flow:

$$T_2 = \frac{1}{2r}\left(r^2 L_2\right)_{,r} \qquad R_2 = \frac{1}{r^2}\left(r^3 L_2\right)_{,r}$$

$$Q_{ij}(r) = -\frac{1}{2r}\left[\left(L_2\right)_{,r} r_i r_j - \left(r^2 L_2\right)_{,r}\delta_{ij}\right]$$

$$L_2(r) = \int_0^\infty E_u(k)\frac{2 j_1(kr)}{kr}dk$$

$$T_2(r) = \int_0^\infty E_u(k)\left(j_0(kr) - \frac{j_1(kr)}{kr}\right)dk$$

$$l_{u0} = \frac{1}{u^2}\int_0^\infty R(r)\,dr = \frac{1}{u^2}\int_0^\infty L_2(r)\,dr$$

Relations between correlation functions

Correlation tensor in terms of correlations

Relations to power spectrum function

Integral length scale

For irrotational flow:

$$R_2 = \frac{1}{r^2}\left(r^3 T_2\right)_{,r} \qquad L_2 = \left(r T_2\right)_{,r}$$

$$Q_{ij}(r) = \left(T_2\right)_{,r}\frac{r_i r_j}{r} + T_2\delta_{ij}$$

$$L_2(r) = 2\int_0^\infty E_u(k)\left(j_0(kr) - 2\frac{j_1(kr)}{kr}\right)dk$$

$$T_2(r) = \int_0^\infty E_u(k)\frac{2 j_1(kr)}{kr}dk$$

$$l_{u0} = \frac{1}{u^2}\int_0^\infty R(r)\,dr = \frac{1}{u^2}\int_0^\infty T_2(r)\,dr$$

nth order spherical Bessel function of the first kind: $j_n(kr)$

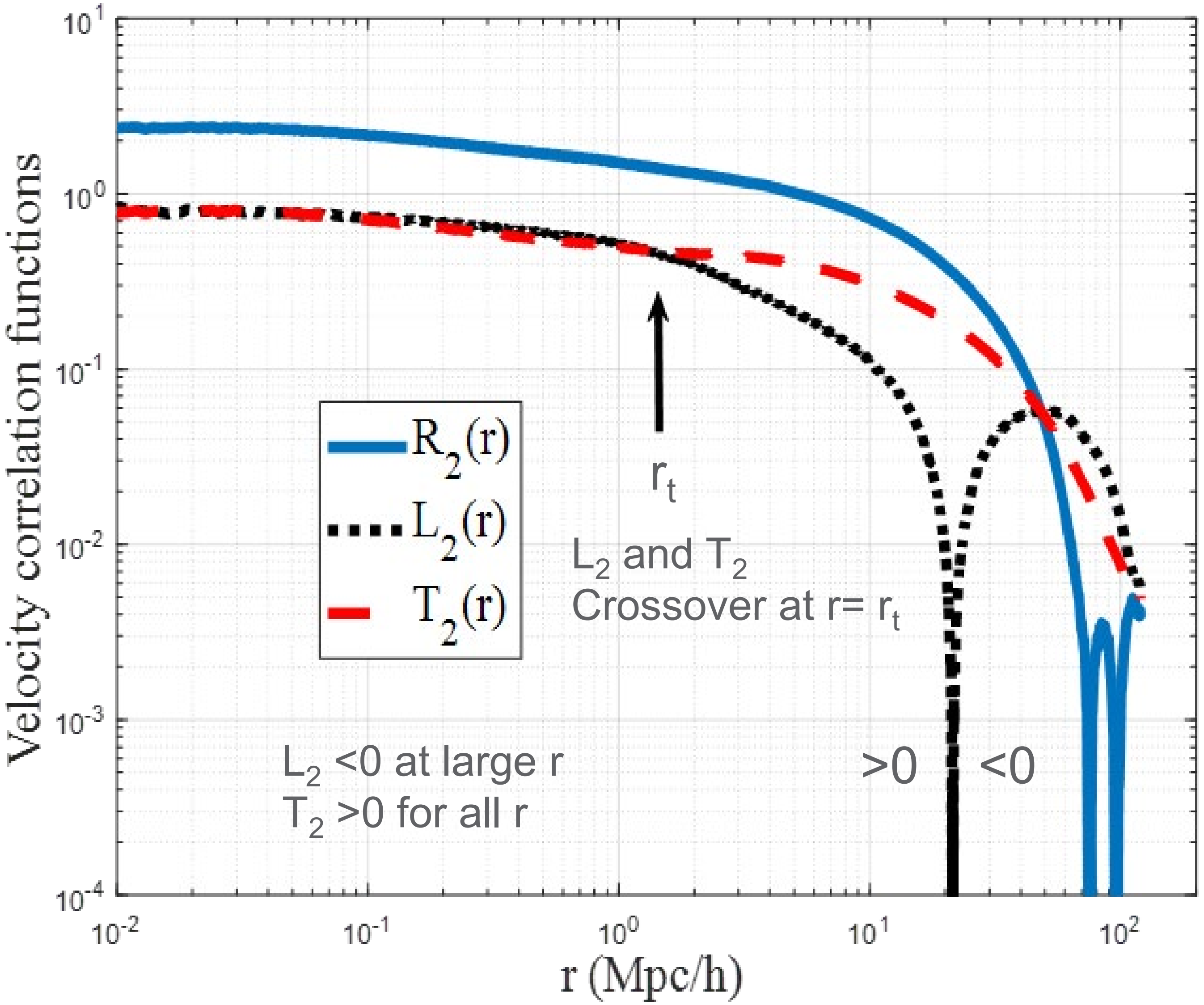
Characterizing the type of flow

$$R_{2i} = \frac{1}{r^3}\int_0^r R_2(y)\,y^2\,dy$$

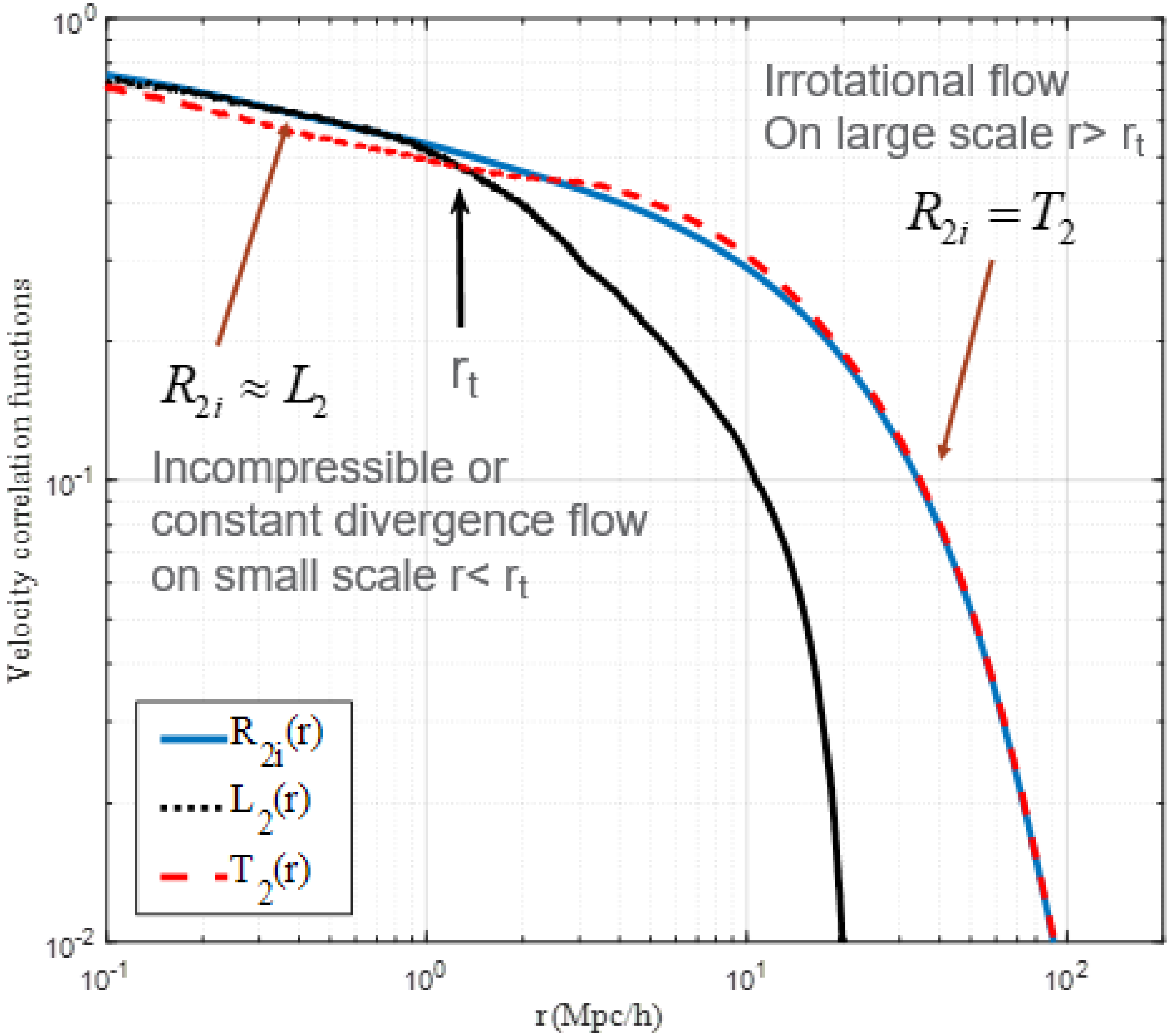
For incompressible or constant divergence flow: $R_{2i} = L_2$

For irrotational flow: $R_{2i} = T_2$

Correlation functions from N-body simulation and nature of dark matter flow

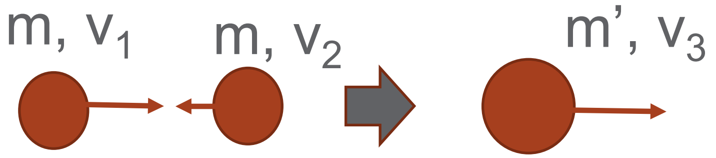


The variation of two-point second order velocity correlation functions (normalized by u2) with scale r at z=0



Using correlation functions to characterize different types of flow.

Velocity correlation and collisionless particle "annihilation"



annihilation

Momentum conservation:

$$mv_1 + mv_2 = m'v_3 \;\Rightarrow\; v_3 = \frac{m}{m'}\left(v_1 + v_2\right)$$

$$\left\langle v_3^2 \right\rangle = \left(\frac{m}{m'}\right)^2 \left\langle \left(v_1 + v_2\right)^2 \right\rangle = 2\left(\frac{m}{m'}\right)^2 \left\langle u_L^2 \right\rangle \left(1 + \boxed{\rho_{L0}}\right)$$

Mass-energy conservation:

$$mc^2 + \frac{1}{2}mv_1^2 + mc^2 + \frac{1}{2}mv_2^2 = m'c^2 + \frac{1}{2}m'v_3^2$$

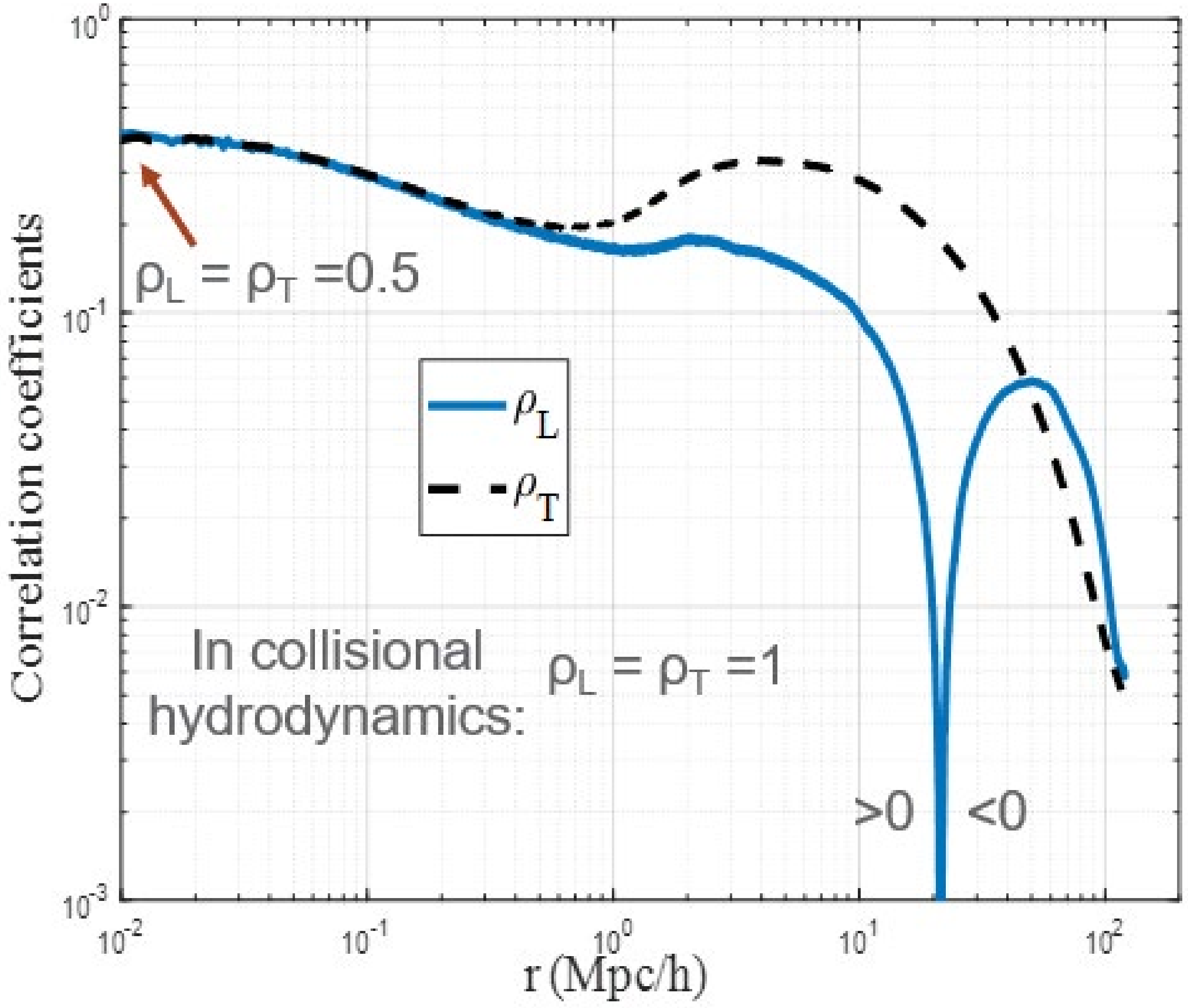
$$\Downarrow$$

$$m' = m\left[1 + \frac{\left\langle u_L^2 \right\rangle}{2c^2} + \sqrt{1 - \rho_{L0}\frac{\left\langle u_L^2 \right\rangle}{c^2} + \frac{\left\langle u_L^2 \right\rangle^2}{4c^4}}\right] \approx m\left[2 + \left(1 - \rho_{L0}\right)\frac{\left\langle u_L^2 \right\rangle}{2c^2}\right]$$

Particle "annihilation" (r=0) leads to extra mass converted from kinetic energy if gravity is the only interaction and no radiation is produced from that "annihilation".

Equipartition: halo T and halo group T

$$\sigma^2\left(m_h\right) = \sigma_v^2\left(m_h\right) + \sigma_h^2\left(m_h\right) \quad \rho_{cor} = \left\langle \sigma_h^2 \right\rangle / \left\langle \sigma^2 \right\rangle \approx 1/2$$



The correlation coefficients for longitudinal velocity and for transverse velocity

Modeling velocity correlation functions on large scale

On large scale, transverse velocity correlation can be well modelled by exponential function:

$$T_2(r,a) = a_0 u^2 \exp(-r/r_2) \propto a \qquad a_0 (u/u_0)^2 = 0.45a$$

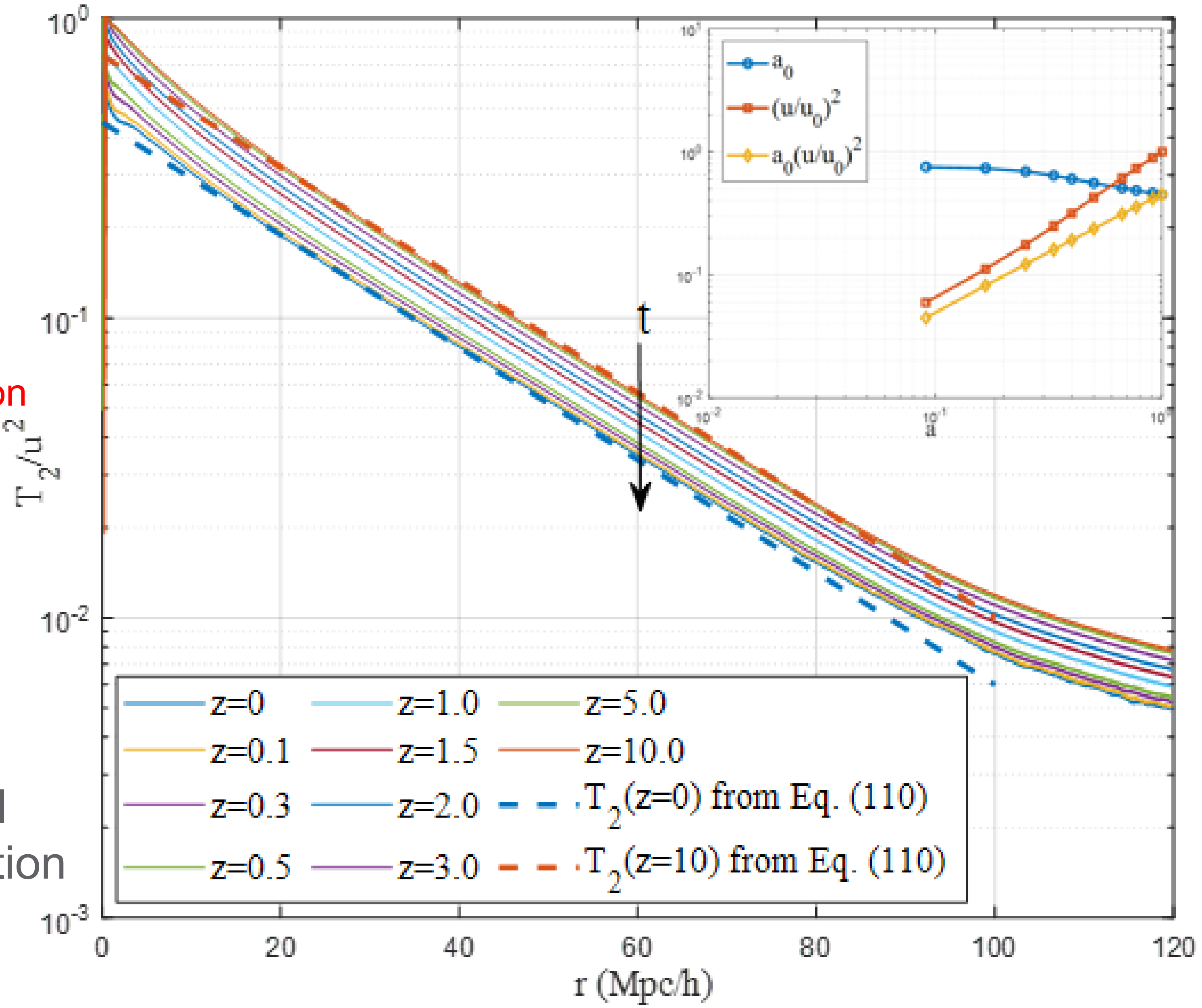
$$r_2 \approx 21.4\, Mpc/h$$ Redshift-independent length scale, might be related to the size of sound horizon

Using kinematic relations for irrotational flow on large scale

$$L_2(r,a) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(1-\frac{r}{r_2}\right)$$ Longitudinal correlation

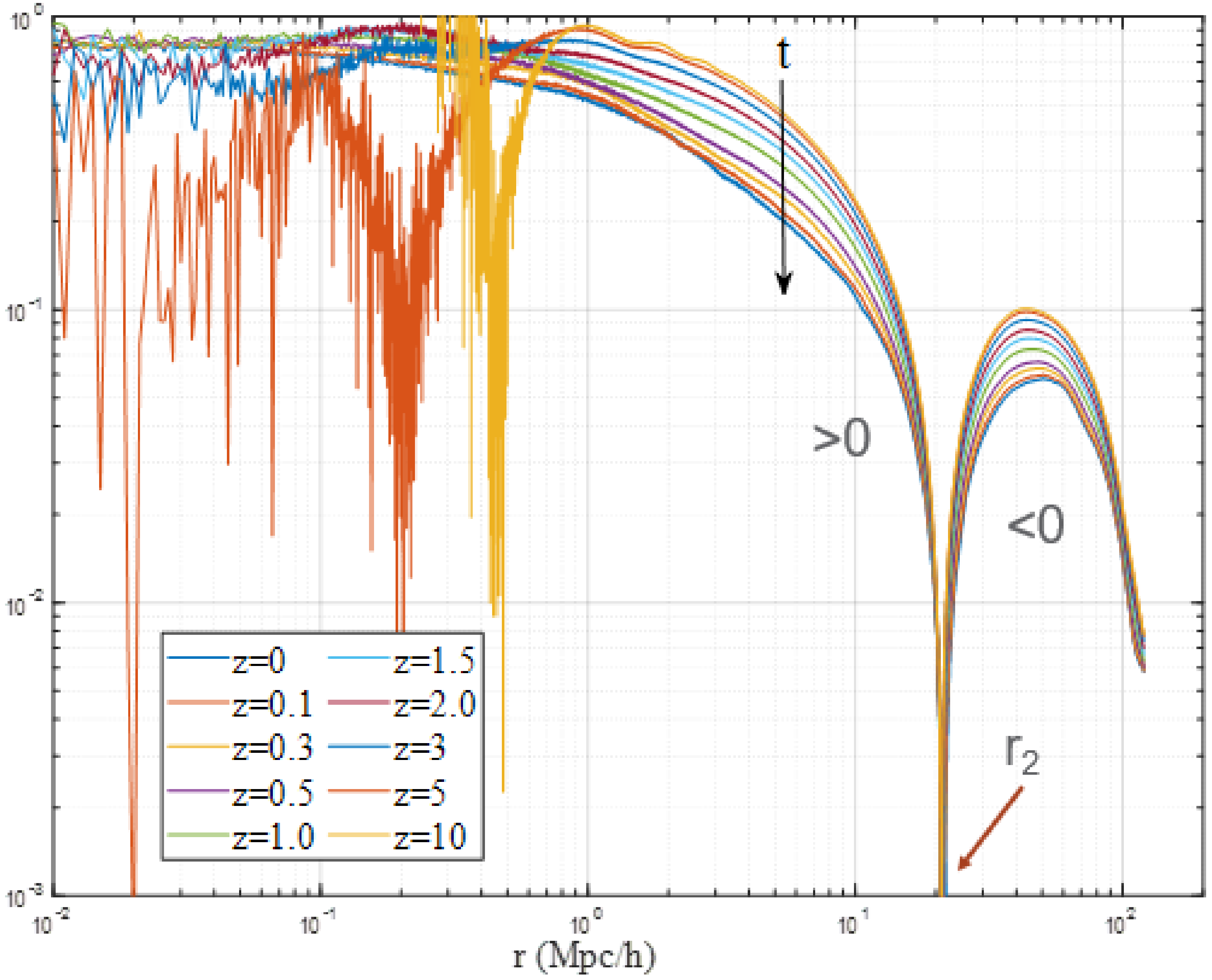
$$R_2(r,a) = \langle \mathbf{u}\cdot\mathbf{u}' \rangle = 2R(r) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(3-\frac{r}{r_2}\right)$$ Total correlation

$$l_{u0} = \frac{1}{u^2}\int_0^\infty R(r)\,dr = \frac{1}{2u^2}\int_0^\infty R_2(r)\,dr = 2a_0 r_2$$ Correlation length

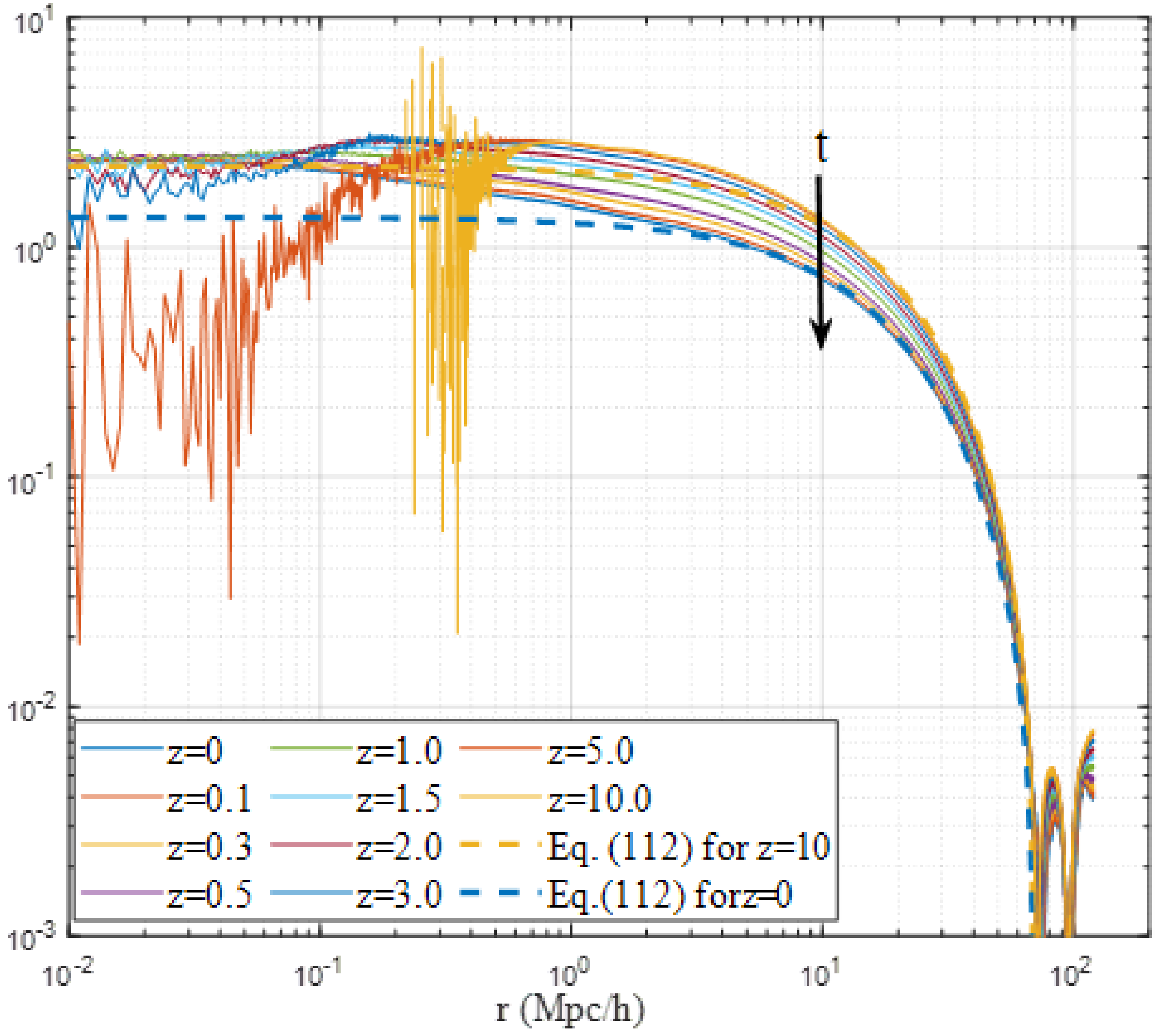


Transverse velocity correlation function T_2 varying with r at different redshifts z

Longitudinal and total velocity correlation



The variation of longitudinal velocity correlation function L_2 with scale r and redshift z



The variation of total velocity correlation function R_2 with scale r and redshift z

Density and potential correlations on large scale

Using kinematic relations and exponential transverse velocity correlation, we can analytically derive all correlations for velocity, density and potential on large scale.

Linear perturbation theory and Zeldovich approximation on large scale:

$$\delta \approx \eta = -\frac{\nabla \cdot \mathbf{u}}{aHf(\Omega_m)} \qquad \mathbf{u} = -\frac{Hf(\Omega_m)\nabla\phi}{4\pi G\rho a} \qquad \delta \approx \eta = \frac{\nabla^2\phi}{4\pi G\rho a^2}$$

Log-density field: $\eta(\mathbf{x}) = \log(1+\delta) \approx \delta$

Density correlation:

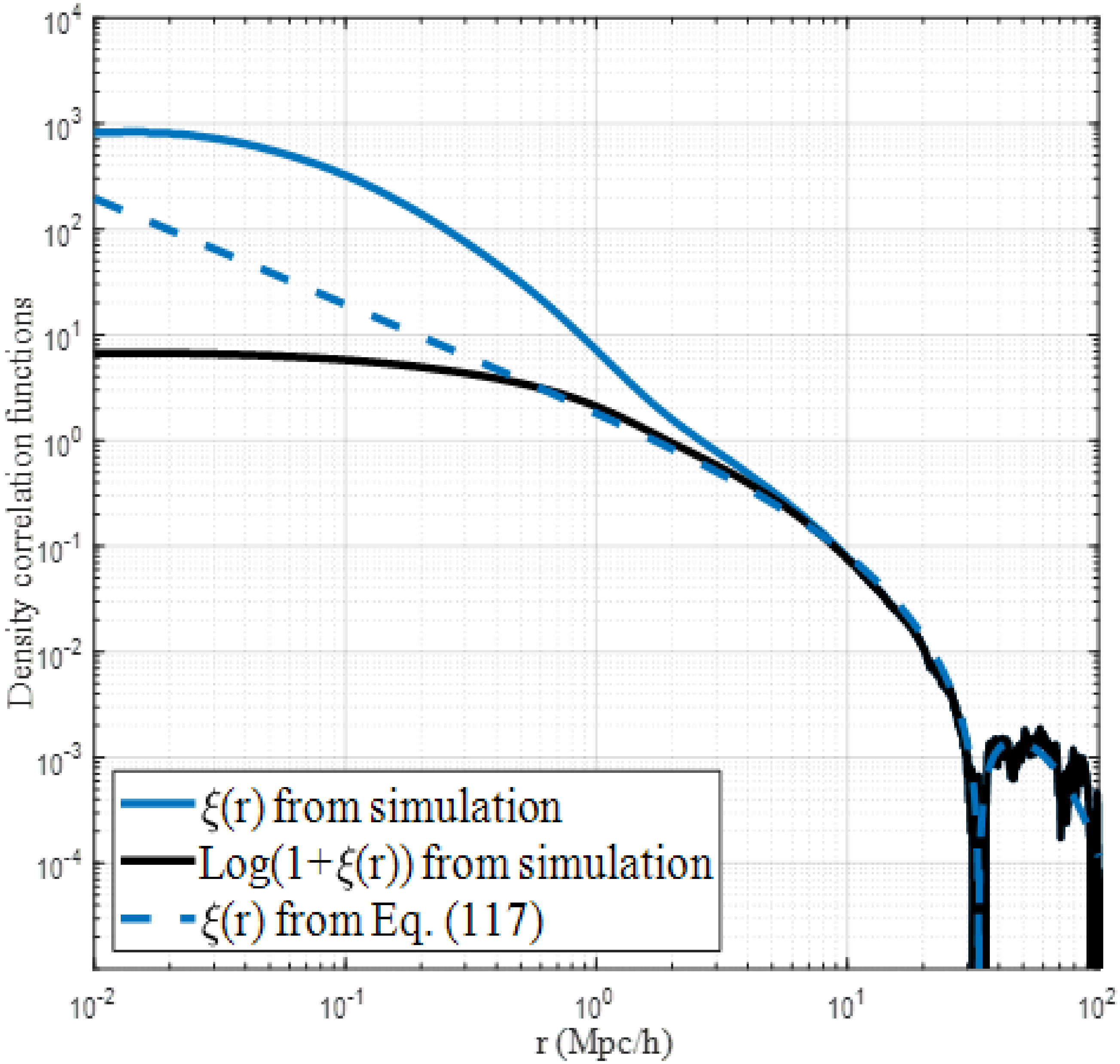
$$\xi(r,a) = \frac{1}{\left(aHf(\Omega_m)\right)^2} \cdot \frac{a_0 u^2}{r r_2} \exp\left(-\frac{r}{r_2}\right)\left[\left(\frac{r}{r_2}\right)^2 - 7\left(\frac{r}{r_2}\right) + 8\right]$$

Averaged density correlation:

$$\bar{\xi}(r,a) = \frac{3}{r^3}\int_0^r \xi(y,a) y^2 dy == \frac{a_0 u^2}{\left(aHf(\Omega_m)\right)^2} \frac{3}{r r_2} \exp\left(-\frac{r}{r_2}\right)\left(4 - \frac{r}{r_2}\right)$$

Potential correlation:

$$R_\phi = \frac{1}{2}\left\langle \phi(\mathbf{x}) \cdot \phi(\mathbf{x}') \right\rangle = \frac{9}{8}\left(\frac{aH}{f(\Omega_m)}\right)^2 a_0 u^2 r_2^2 \exp\left(-\frac{r}{r_2}\right)\left[\left(\frac{r}{r_2}\right) + 1\right] \propto a^0$$



Density correlation at z=0 and comparison with model

Velocity/density/potential spectrum functions on large scale

Velocity spectrum function:

$$E_u(k) = a_0 u^2 \frac{8}{\pi r_2} \frac{k^{-2}}{\left(1 + 1/(kr_2)^2\right)^3}$$

$$k_{max} r_2 = \sqrt{2}$$

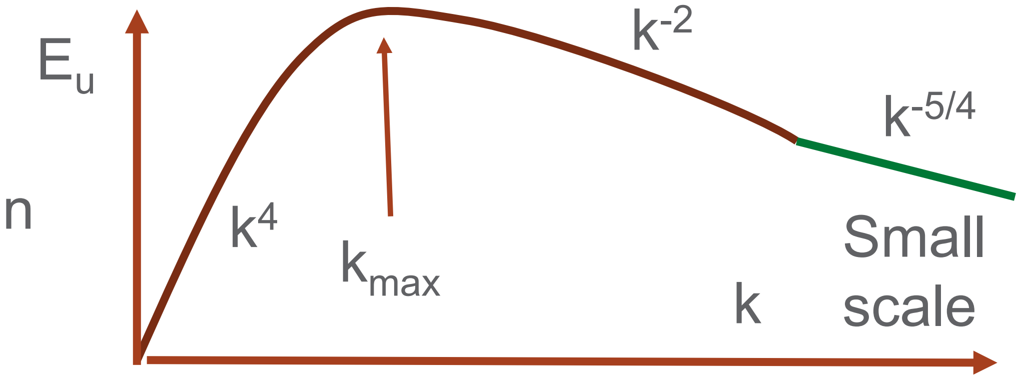
$$E_u(k_{max}) = \frac{256}{125\pi} r_2 a_0 u^2$$

$$E_u(k) \propto k^4 \quad \text{for} \quad kr_2 \ll 1$$

$$E_u(k) \propto k^{-2} \quad \text{for} \quad kr_2 \gg 1$$

[k^4 spectrum due to vanishing linear momentum]()

Signature of Burger's equation in weakly nonlinear regime



Density spectrum function:

$$E_\delta(k) = \frac{16 a_0 u^2}{\left(aHf(\Omega_m)\right)^2 \pi r_2} \frac{1}{\left(1 + 1/(kr_2)^2\right)^3}$$

Potential spectrum function:

$$E_\phi(k) = \frac{18}{\pi r_2} \left(\frac{aH}{f(\Omega_m)}\right)^2 \frac{a_0 u^2 k^{-4}}{\left(1 + 1/(kr_2)^2\right)^3}$$

Matter power spectrum:

$$P_\delta(k,a) = 2\pi^2 E_\delta(k,a)/k^2 = \frac{32\pi a_0 u^2 r_2}{\left(aHf(\Omega_m)\right)^2} \frac{1}{(kr_2)^2 \left(1 + 1/(kr_2)^2\right)^3}$$

$$P_\delta(k_{max}, a) = \frac{128\pi a_0 u^2 r_2}{27\left(aHf(\Omega_m)\right)^2}$$

Second order velocity dispersion functions and energy distribution in real space

Dispersion function for smoothed velocity (energy contained in scales above r):

$$\sigma_u^2(r) = \frac{1}{3}\int_{-\infty}^{\infty} E_u(k)\, W(kr)^2\, dk = \int_r^{\infty} E_{ur}(r')\, dr'$$

Window function for tophat spherical filter:

$$W(x) = \frac{3}{x^3}\left[\sin(x) - x\cos(x)\right] = 3\frac{j_1(x)}{x}$$

$$E_{ur}(r) = -\frac{\partial \sigma_u^2(r)}{\partial r}$$

Energy contained in scales between [r, r+dr]

Energy contained in scales below r:

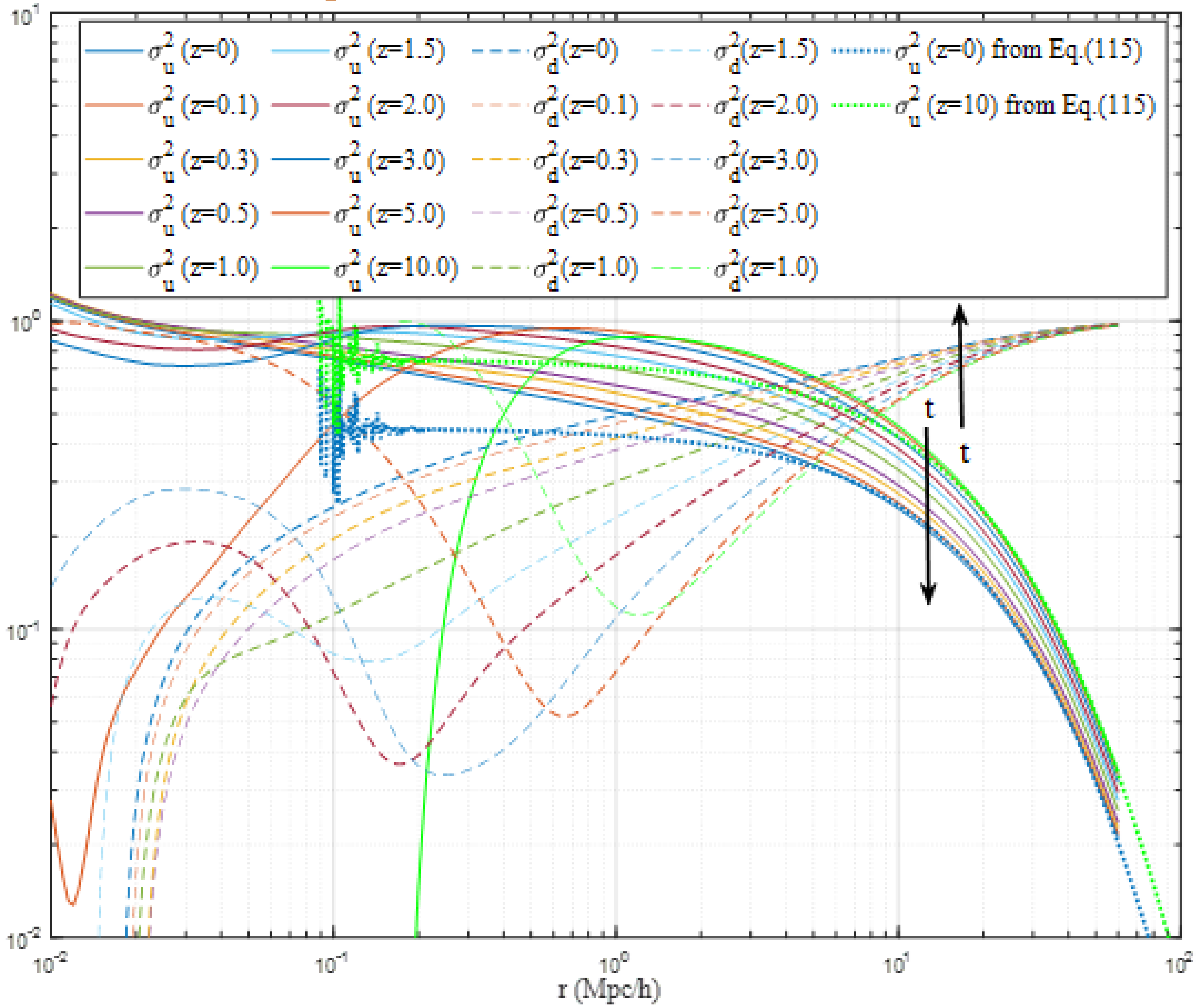
$$\sigma_d^2(r) = \frac{1}{3}\int_{-\infty}^{\infty} E_u(k)\left[1 - W(kr)^2\right] dk$$

$$\sigma_u^2(r) + \sigma_d^2(r) = u^2$$

Energy decomposed into scales below and above r:

Relations to velocity correlation function:

$$R_2(2r) = \frac{1}{24r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\sigma_u^2(r)\, r^4\right)\right)\right)$$



Variation of two dispersion functions with scale r (simulation). Fraction of energy contained in large scale decreases with time.

Second order velocity structure functions

Longitudinal Structure functions are moments of pairwise velocity:

$$S_m^{lp}(r)=\left\langle\left(\Delta u_L\right)^2\right\rangle=\left\langle\left(u_L^{'}-u_L\right)^m\right\rangle$$

$$S_1^{lp}(r)=\left\langle\Delta u_L\right\rangle=\left\langle u_L^{'}-u_L\right\rangle$$

Second order longitudinal structure function (pairwise velocity dispersion):

$$S_2^{lp}(r)=\left\langle\left(\Delta u_L\right)^2\right\rangle=\left\langle\left(u_L^{'}-u_L\right)^2\right\rangle=2\left(\left\langle u_L^2\right\rangle-L_2(r)\right)$$

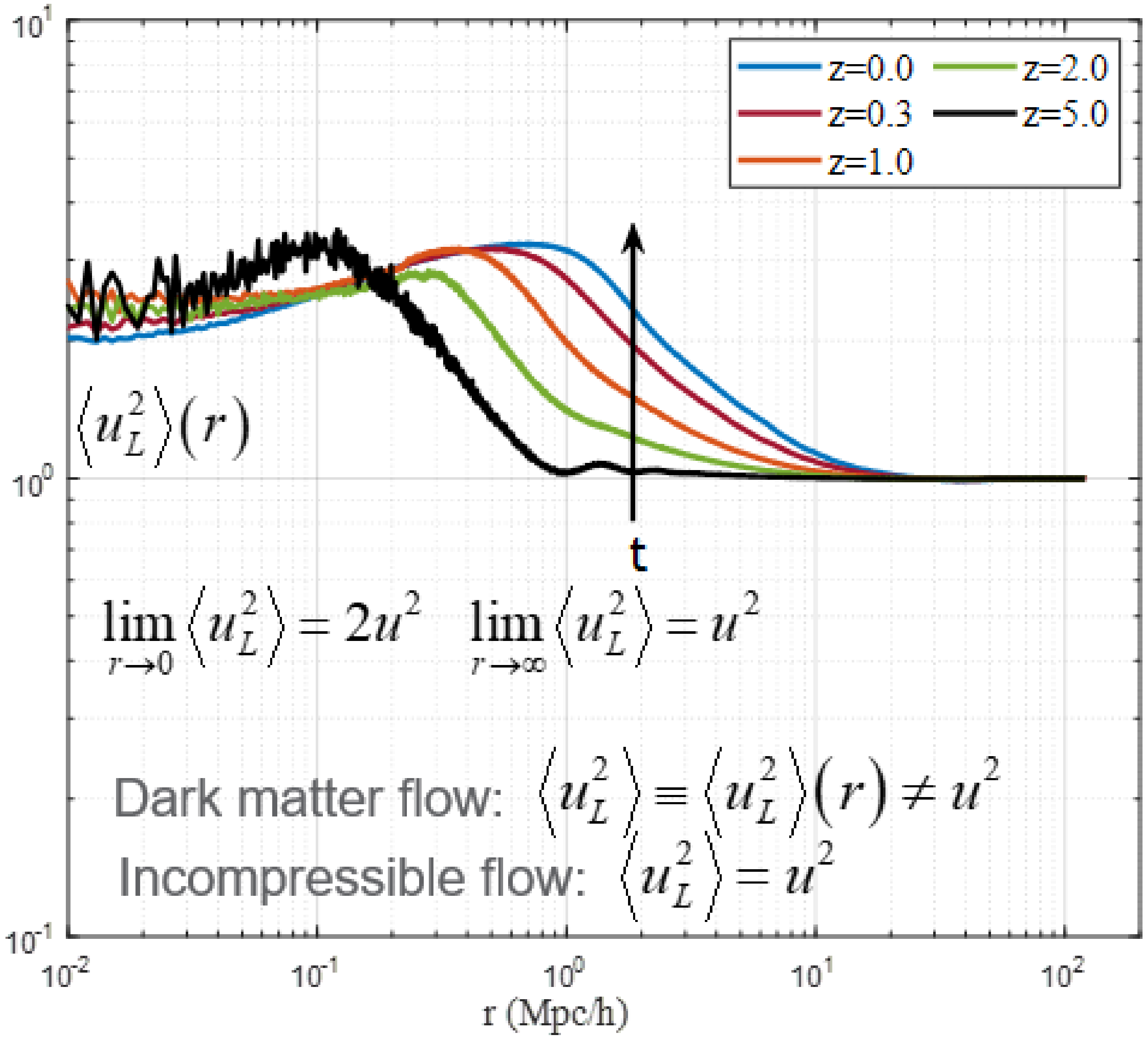
Second order longitudinal structure function (modified):

$$S_2^{l}(r)=2\left(\lim_{r\to 0}\left\langle u_L u_L^{'}\right\rangle-L_2(r)\right)=2\left(u^2-L_2(r)\right)$$

$S_2^{lp}(r)\neq S_2^{l}(r)$ because of $\left\langle u_L^2\right\rangle\neq u^2$

$$\lim_{r\to 0}\left\langle u_L^2\right\rangle=2u^2 \quad \lim_{r\to\infty}\left\langle u_L^2\right\rangle=u^2 \quad \lim_{r\to 0}S_2^{lp}=\lim_{r\to\infty}S_2^{lp}=2u^2$$

$$\lim_{r\to 0}L_2(r)=\lim_{r\to 0}T_2(r)=u^2 \quad \lim_{r\to\infty}L_2(r)=\lim_{r\to\infty}T_2(r)=0$$



The variation of longitudinal velocity dispersion $\langle\Delta u_L^2\rangle$ with scale r at different redshifts z

Second order velocity structure functions

Total velocity structure function:

$$S_2^{ip}(r) = \langle \Delta \mathbf{u}^2 \rangle = \left\langle (\mathbf{u}' - \mathbf{u})^2 \right\rangle = 6\langle u_L^2 \rangle - 2R_2(r)$$

Total velocity structure function (modified):

$$S_2^i(r) = 6u^2 - 2R_2(r)$$

$$S_2^{ip}(r) \neq S_2^i(r) \quad \text{because of} \quad \langle u_L^2 \rangle \neq u^2$$

Relation to velocity spectrum function:

$$S_2^i(r) = 4\int_0^\infty E_u(k)(1 - j_0(kr))\,dk$$

Relation to velocity dispersion function:

$$S_2^i(2r) = \frac{1}{12r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\sigma_d^2(r)r^4\right)\right)\right)$$

Structure function for enstrophy and real space enstrophy distribution:

Enstrophy: $$E_n = \int_0^\infty E_u(k)k^2\,dk$$

Enstrophy of smoothed velocity by a filter of size r:

$$\frac{S_2^x(r)}{2r^2} = \frac{1}{3}\int_0^\infty E_u(k)k^2W^2(kr)\,dk = \int_r^\infty E_{nr}(r')\,dr'$$

Real space distribution of enstrophy between [r r+dr]:

$$E_{nr}(r) = -\frac{\partial}{\partial r}\left[S_2^x(r)/(2r^2)\right]$$

Relation to total structure function:

$$\frac{1}{3r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(S_2^x(r)r^4\right)\right) = \frac{\partial S_2^i(2r)}{\partial r}$$

Kinematic relations for structure functions

For incompressible flow or constant divergence flow:

$$S_2^l(r) = \frac{4}{3}\int_0^\infty E_u(k)\left(1 - 3\frac{j_1(kr)}{kr}\right)dk$$

Relation between different structure functions:

$$S_2^i(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 S_2^l(r)\right]$$

Relation to velocity dispersion functions:

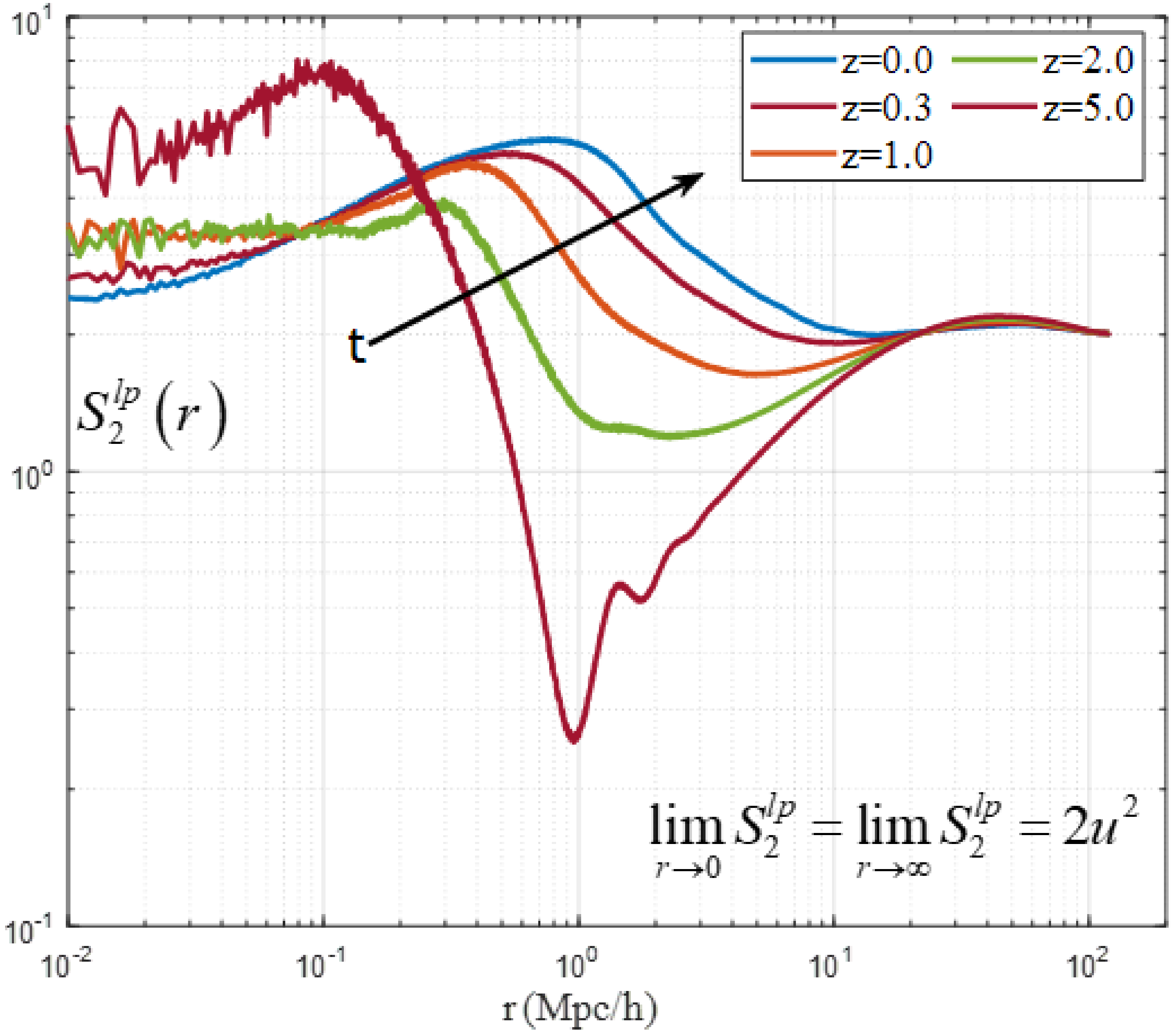
$$S_2^l(2r) = \frac{1}{12r^5}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\sigma_d^2(r)r^4\right)\right)$$

For irrotational flow:

$$S_2^l(r) = \frac{4}{3}\int_0^\infty E_u(k)\left(1 - 3j_0(kr) + 6\frac{j_1(kr)}{kr}\right)dk$$

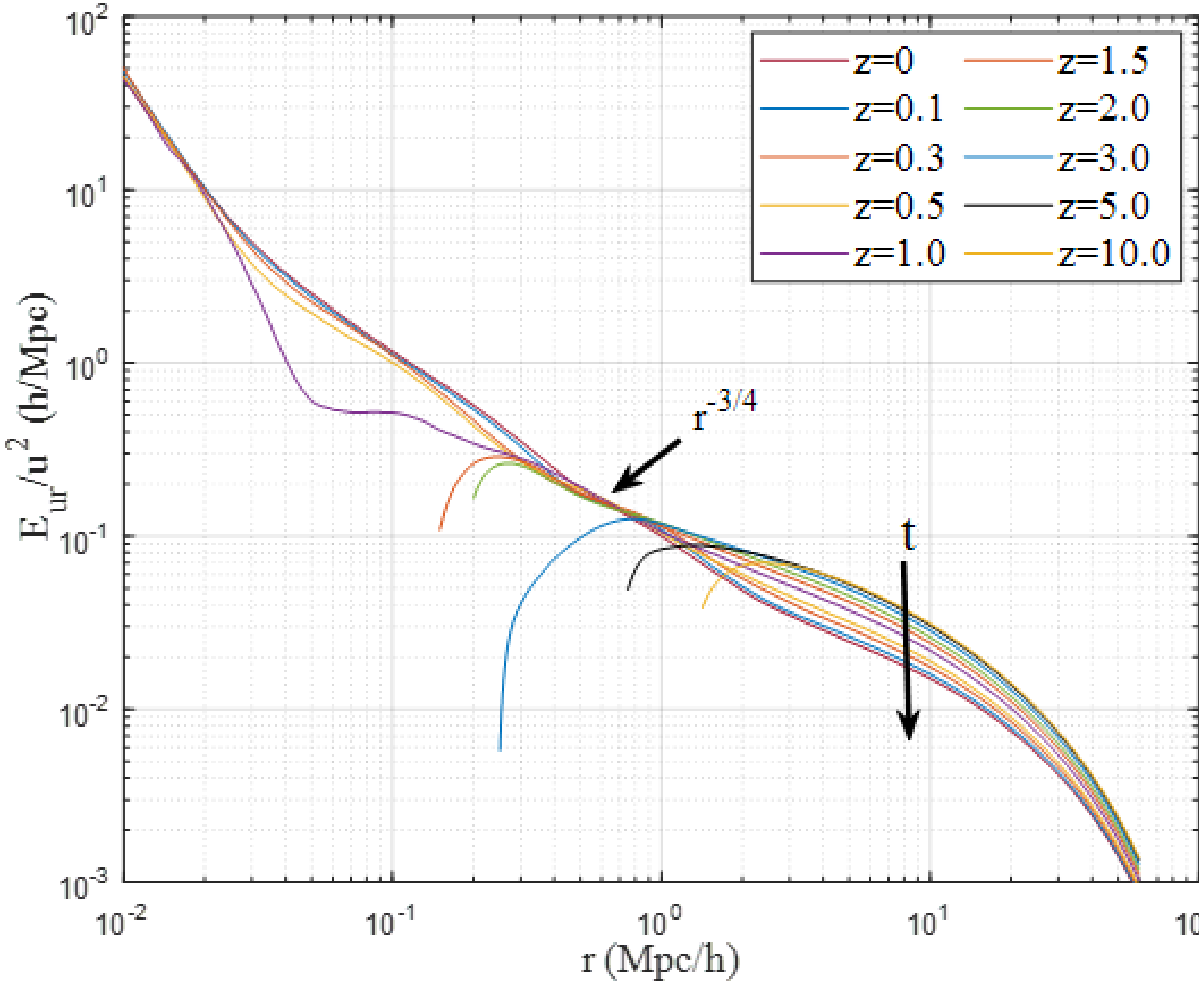
Relation between different structure functions:

$$\frac{\partial\left[rS_2^i(r)\right]}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 S_2^l(r)\right]$$

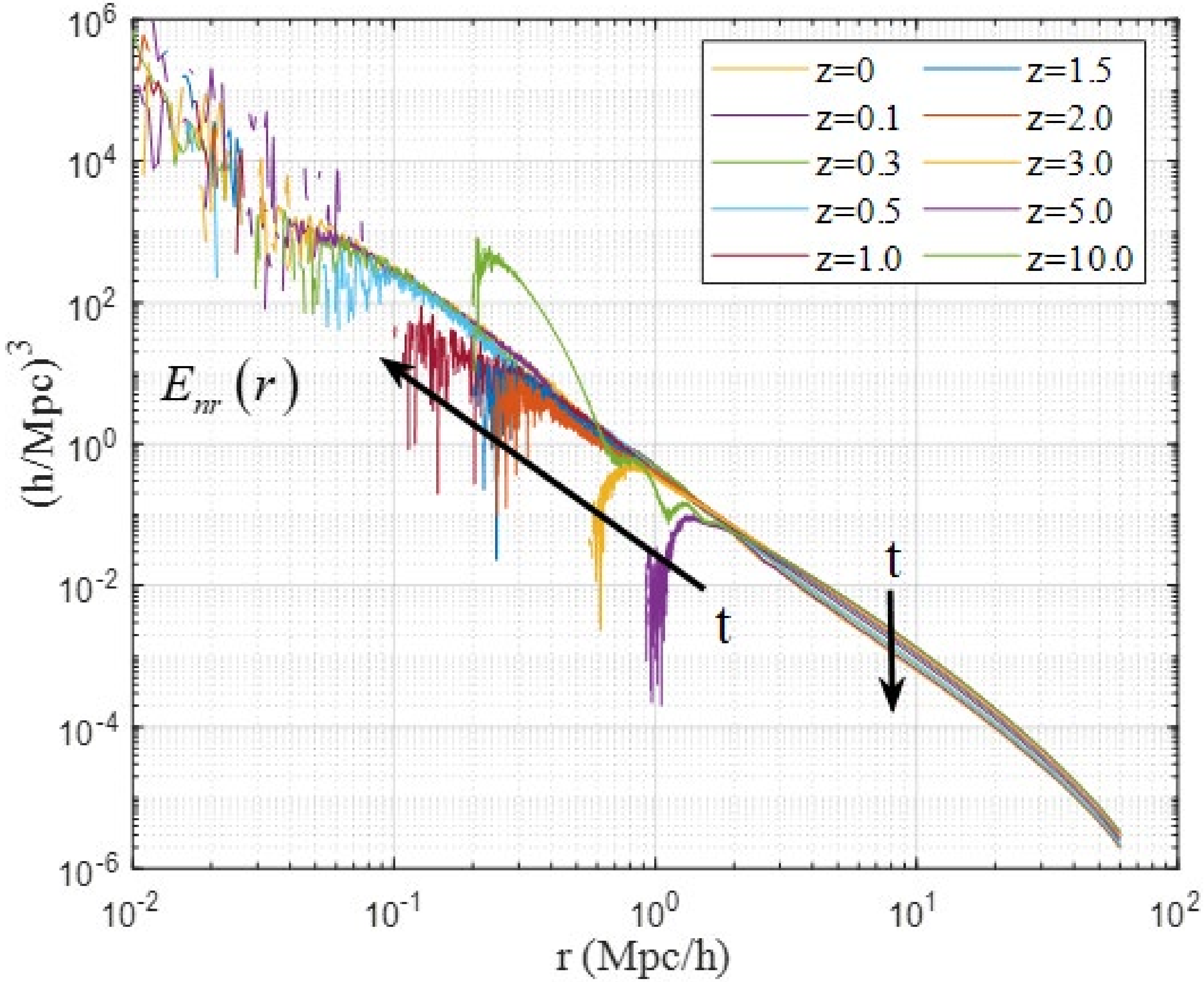


The variation of longitudinal velocity structure function S^{lp}_2 with scale r at different redshifts z

Energy and enstrophy distribution in real space



The real space distribution of energy on scale r at different redshifts



The real space distribution of enstrophy on scale r at different redshifts

Correlation functions of velocity gradients and Kinematic relations

Divergence of velocity:

$$\theta(\mathbf{x}) = \nabla \cdot \mathbf{u}(\mathbf{x})$$

Vorticity (curl):

$$\boldsymbol{\omega}(\mathbf{x}) = \nabla \times \mathbf{u}(\mathbf{x})$$

$$-\nabla^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle = \langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle + \langle \theta \cdot \theta' \rangle = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial R_2}{\partial r} \right]$$

$$-\nabla^2 \langle \mathbf{u} \cdot \mathbf{u}' \rangle = 2\int_0^\infty E_u(k) k^2 \frac{\sin(kr)}{kr} dk$$

Divergence and vorticity correlations:

$$R_\theta + R_{\boldsymbol{\omega}} = \frac{1}{4r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} S_2^i(r) \right)$$

$$R_\theta + R_{\boldsymbol{\omega}} = \frac{1}{96r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 \frac{\partial}{\partial r}\left(S_2^x(r) r^2 \right)\right)\right)$$

$$R_{\boldsymbol{\omega}} = \frac{\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle}{2} = \frac{1}{r^2}\left[r^2 \left(A_2 r - \frac{\partial B_2}{\partial r} \right)\right]_{,r}$$

$$R_\theta = \frac{\langle \theta \cdot \theta' \rangle}{2} = -\frac{1}{2r^2}\left[r^2 \left(4A_2 r + \frac{\partial A_2}{\partial r} r^2 + \frac{\partial B_2}{\partial r} \right)\right]_{,r}$$

For incompressible flow or constant divergence flow:

Vorticity correlation (divergence is zero):

$$\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial R_2}{\partial r} \right] = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial (r^3 L_2)}{\partial r} \right)\right]$$

$$R_{\boldsymbol{\omega}}(r) = \frac{1}{2}\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle = \int_0^\infty E_u(k) k^2 \frac{\sin(kr)}{kr} dk$$

For irrotational flow:

Divergence correlation (vorticity is zero):

$$\langle \theta \cdot \theta' \rangle = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial R_2}{\partial r} \right] = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial (r^3 T_2)}{\partial r} \right)\right]$$

$$R_\theta(r) = \frac{1}{2}\langle \theta \cdot \theta' \rangle = \int_0^\infty E_u(k) k^2 \frac{\sin(kr)}{kr} dk$$

Modeling the longitudinal structure function on large scale

Structure function (pairwise velocity dispersion):

$$S_2^{lp}(r) = \left\langle \left(\Delta u_L\right)^2 \right\rangle = 2\left(\left\langle u_L^2 \right\rangle - L_2(r)\right)$$

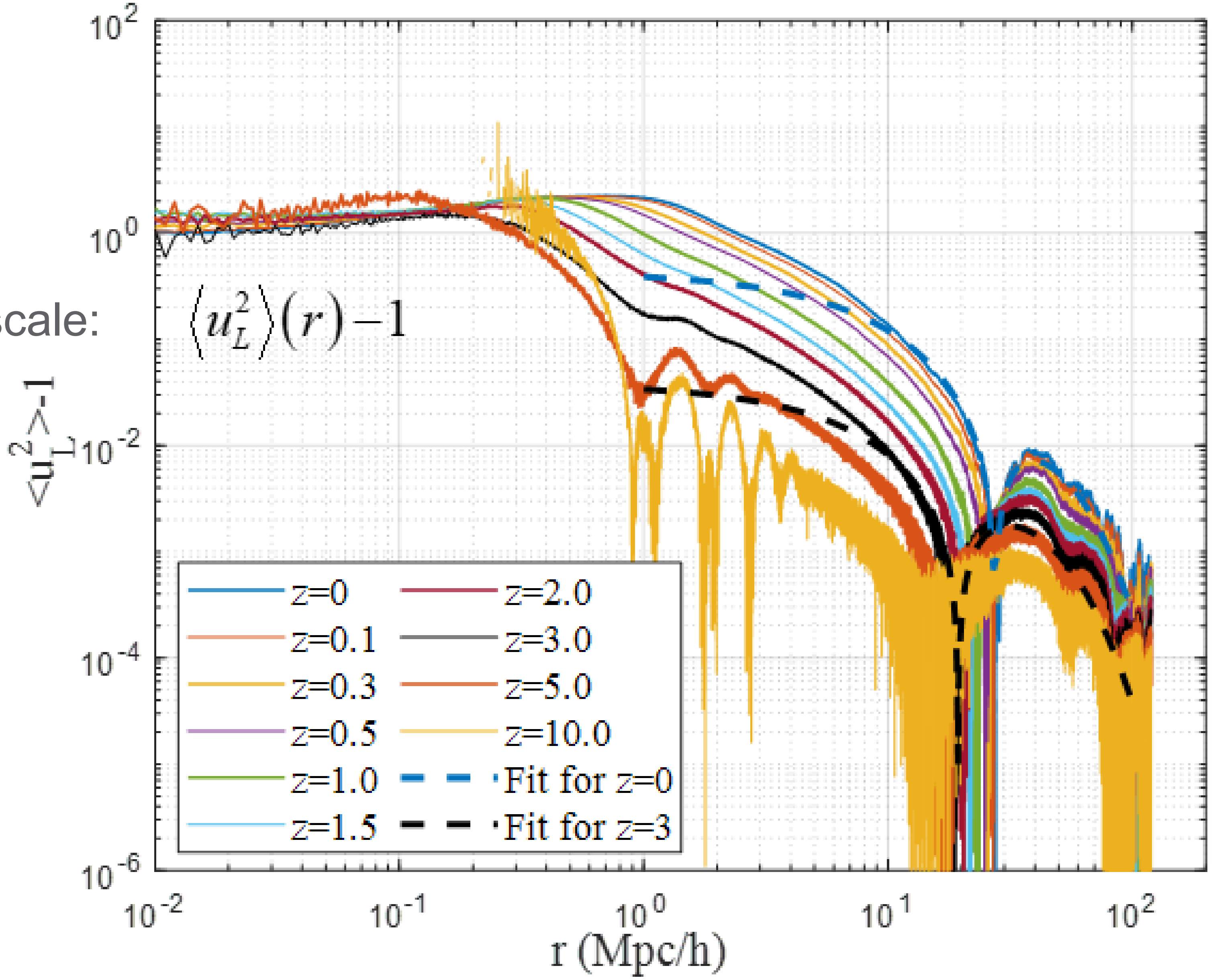
Modeling longitudinal velocity dispersion on large scale:

$$\left\langle u_L^2 \right\rangle = u^2 \left[1 + a_d \exp\left(-\frac{r}{r_{d1}}\right)\left(1 - \frac{r}{r_{d2}}\right)\right]$$

$$a_d = 0.44 a^{7/4}$$

$$r_{d1} = 11.953\, Mpc/h$$

$$r_{d2} = 27.4 a^{1/4}\, Mpc/h$$



The variation of normalized longitudinal velocity dispersion

Pacific Northwest NATIONAL LABORATORY

Modeling the longitudinal structure function on small scale (two-thirds 2/3 law)

Second order structure function (pairwise velocity dispersion):

$$S_2^{lp}(r) = \left\langle (\Delta u_L)^2 \right\rangle = 2\left(\left\langle u_L^2 \right\rangle - L_2(r)\right) \quad \text{with} \quad \lim_{r\to 0} S_2^{lp} = 2u^2$$

For hydrodynamic turbulence: $\lim_{r\to 0} S_2^{lp} = 0$

Construct reduced structure function that is purely determined by the rate of energy cascade ε_u:

$$S_{2r}^{lp} = S_2^{lp}(r) - 2u^2 \ \left(m^2/s^2\right) \quad \text{and} \quad \varepsilon_u : \left(m^3/s^2\right)$$

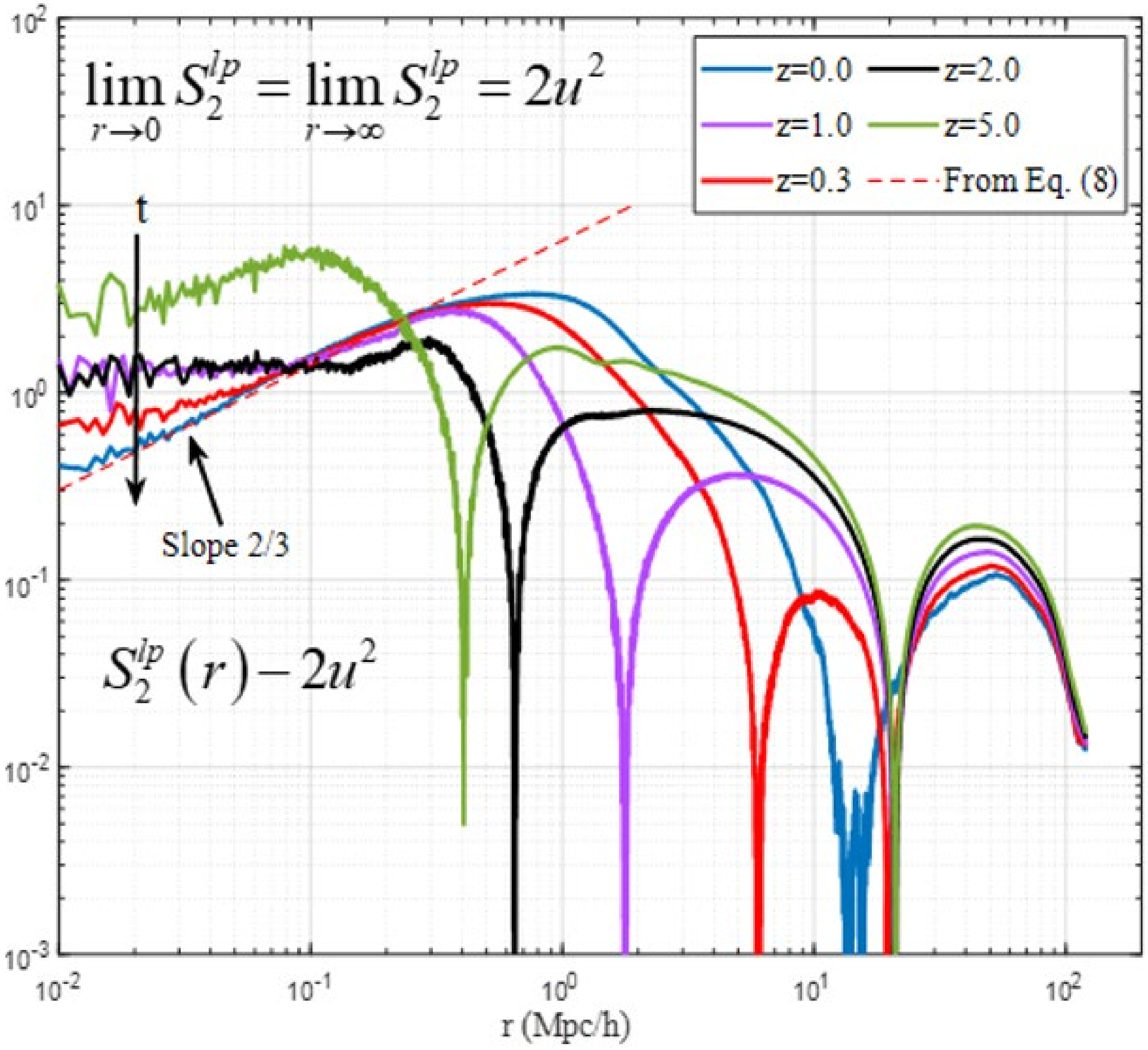
Dimensional analysis leads to two-thirds law for S_{2r}^{lp}

$$S_{2r}^{lp} \propto \left(-\varepsilon_u\right)^{2/3} r^{2/3} \quad \text{or} \quad S_{2r}^{lp} = a^{3/2}\beta_2^*\left(-\varepsilon_u\right)^{2/3} r^{2/3}$$

By introducing a length scale r_s: upper limit for two-thirds law

$$S_2^{lp}(r) = S_{2r}^{lp} + 2u^2 = u^2\left[2 + \beta_2^*\left(r/r_s\right)^{2/3}\right]$$

$$r_s = -\frac{u_0^3}{\varepsilon_u} = \frac{4}{9}\frac{u_0}{H_0} = \frac{2}{3}u_0 t_0 \approx 1.58\,Mpc/h \quad \text{and} \quad \beta_2^* \approx 9.5$$



Variation of normalized reduced longitudinal structure function and two-thirds law

Two-thirds law might be used to predict dark matter particle properties

Modeling the longitudinal structure function on small scale (one-fourth ¼ law)

1/4 law for (modified) structure function on small scale:

Also see slides for additional information.

Potential energy for a sphere of radius r:

$$U(r) = -\frac{\int_0^r \frac{G}{y} M(y)\rho(y)4\pi y^2 dy}{\int_0^r \rho(x)4\pi x^2 dx}$$

Virial theorem

$$2T(r) + \gamma U(r) = 0$$

$$\xi(r) = a^m (r/r_\xi)^{-n} \Rightarrow \rho(y) = \rho_0(1+\xi(y)) \approx \rho_0 a^m (y/r_\xi)^{-n}$$

$$U(r) = -\frac{3-n}{5-2n}\frac{GM(r)}{r} = -\frac{3a^m H_0^2 r^2}{2(5-2n)}\left(\frac{r}{r_\xi}\right)^{-n}$$

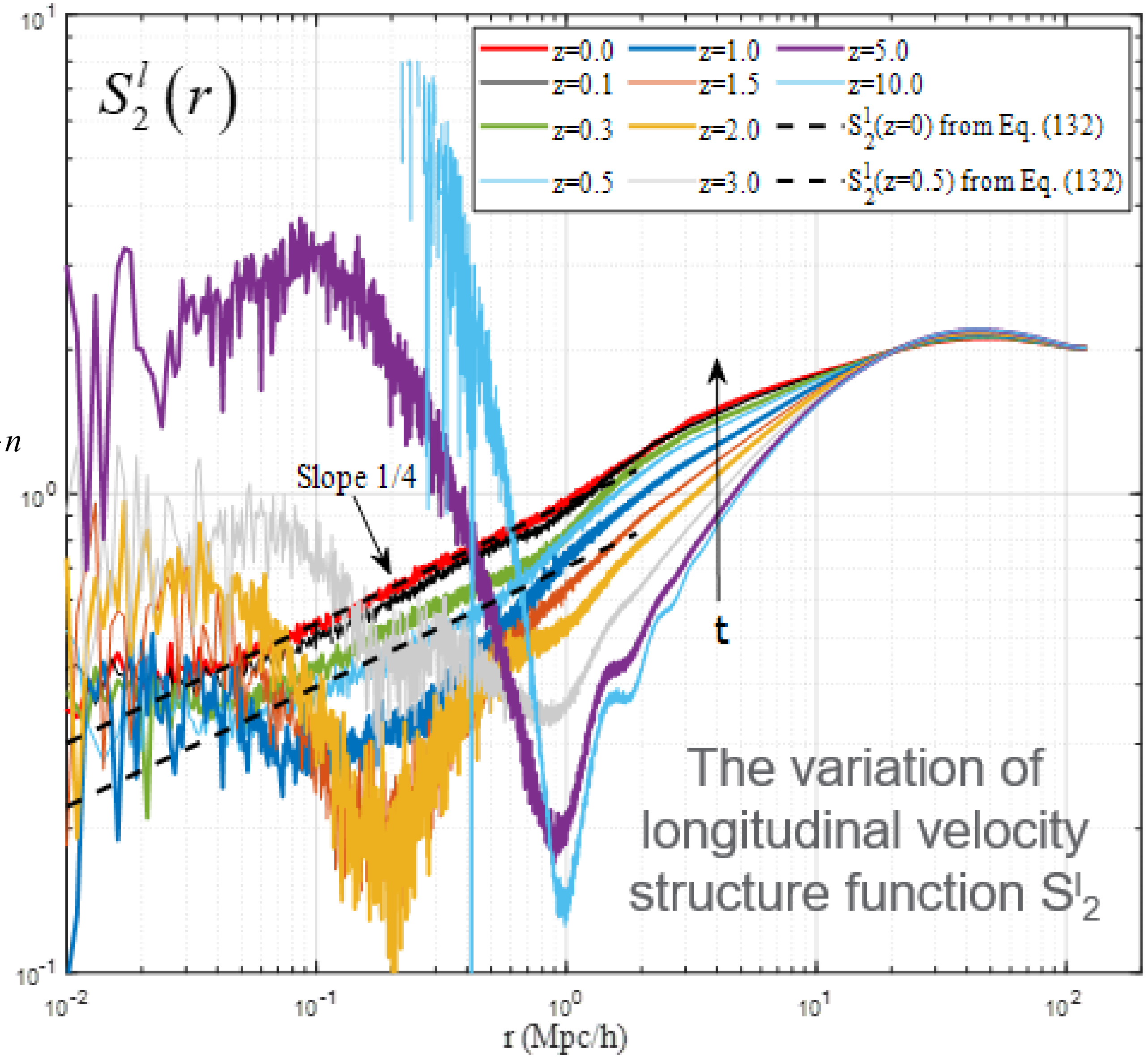
Use virial theorem:

$$\sigma_d^2(r) = \frac{\gamma(H_0 r)^2}{2r^5}\frac{\int_0^r (1+\xi(y))(1+\bar{\xi}(y))y^4 dy}{(1+\bar{\xi}(r))}$$

$$\sigma_d^2(r) = \frac{\gamma a^m (H_0 r)^2}{2(5-2n)}\left(\frac{r}{r_\xi}\right)^{-n}$$

Use kinematic relation

$$S_2^l(r) = \frac{\gamma(6-n)(8-n)}{24(5-2n)2^{2-n}} a^m (H_0 r)^2 \left(\frac{r}{r_\xi}\right)^{-n} \Rightarrow S_2^l(r) \propto r^{1/4}$$



Modeling velocity correlation functions on small scale

1/4 law for (modified) longitudinal structure function can be used to derive all other velocity correlations on small scale:

$$S_2^l = 2u^2 \left(r/r_1 \right)^n \quad \text{with} \quad n \approx 1/4$$

$$r_1(a) \approx r_1^* a^{-3} \quad \text{and} \quad r_1^* \approx 19.4\, Mpc/h$$

Using kinematic relations on small scale:

Longitudinal correlation

$$L_2(r) = u^2 \left[1 - \left(\frac{r}{r_1} \right)^n \right]$$

Transverse correlation

$$T_2 = u^2 \left[1 - \frac{2+n}{2} \left(\frac{r}{r_1} \right)^n \right]$$

Total correlation

$$R_2 = u^2 \left[3 - (3+n) \left(\frac{r}{r_1} \right)^n \right]$$

Velocity dispersion function for energy contained below scale r:

$$\sigma_d^2(r) = \frac{24 \cdot 2^n}{(4+n)(6+n)} u^2 \left(\frac{r}{r_1} \right)^n \approx 1.0745 u^2 \left(\frac{r}{r_1} \right)^n$$

Total structure function

$$S_2^i(r) = 2(3+n) u^2 \left(\frac{r}{r_1} \right)^n$$

Structure function for enstrophy

$$S_2^x(r) = \frac{6n(3+n) \cdot 2^n}{(4+n)(2+n)} u^2 \left(\frac{r}{r_1} \right)^n = 0.6063 u^2 \left(\frac{r}{r_1} \right)^n$$

Vorticity correlation

$$R_{\boldsymbol{\omega}} = \frac{1}{2} \left\langle \boldsymbol{\omega}(\mathbf{x}) \cdot \boldsymbol{\omega}(\mathbf{x}') \right\rangle = \frac{n(1+n)(3+n)}{2r^2} u^2 \left(\frac{r}{r_1} \right)^n$$

Velocity & vorticity spectrum

$$E_u(k) = C u^2 r_1^{-n} k^{-(1+n)} \qquad E_{\boldsymbol{\omega}}(k) = C u^2 r_1^{-n} k^{(1-n)}$$

Proportional constant

$$C = -\frac{2(3+n)\Gamma\left((n+3)/2\right)}{2^{1-n}\Gamma(3/2)\Gamma(-n/2)} = 0.4485$$

Modeling the velocity correlations on entire range

- Correlation functions are modelled on both large and small scales
- Need smooth and differentiable velocity correlations for the entire range of scales
- Correlations of vorticity and divergence can be obtained as derivatives of velocity correlations

$$f_1(r) = R_{2s}(r) = 3 - (3+n)\left(\frac{r}{r_1}\right)^n$$

Correlation function on small scale

$$f_2(r) = R_{2l}(r) = a_0 \exp\left(-\frac{r}{r_2}\right)\left(3 - \frac{r}{r_2}\right)$$

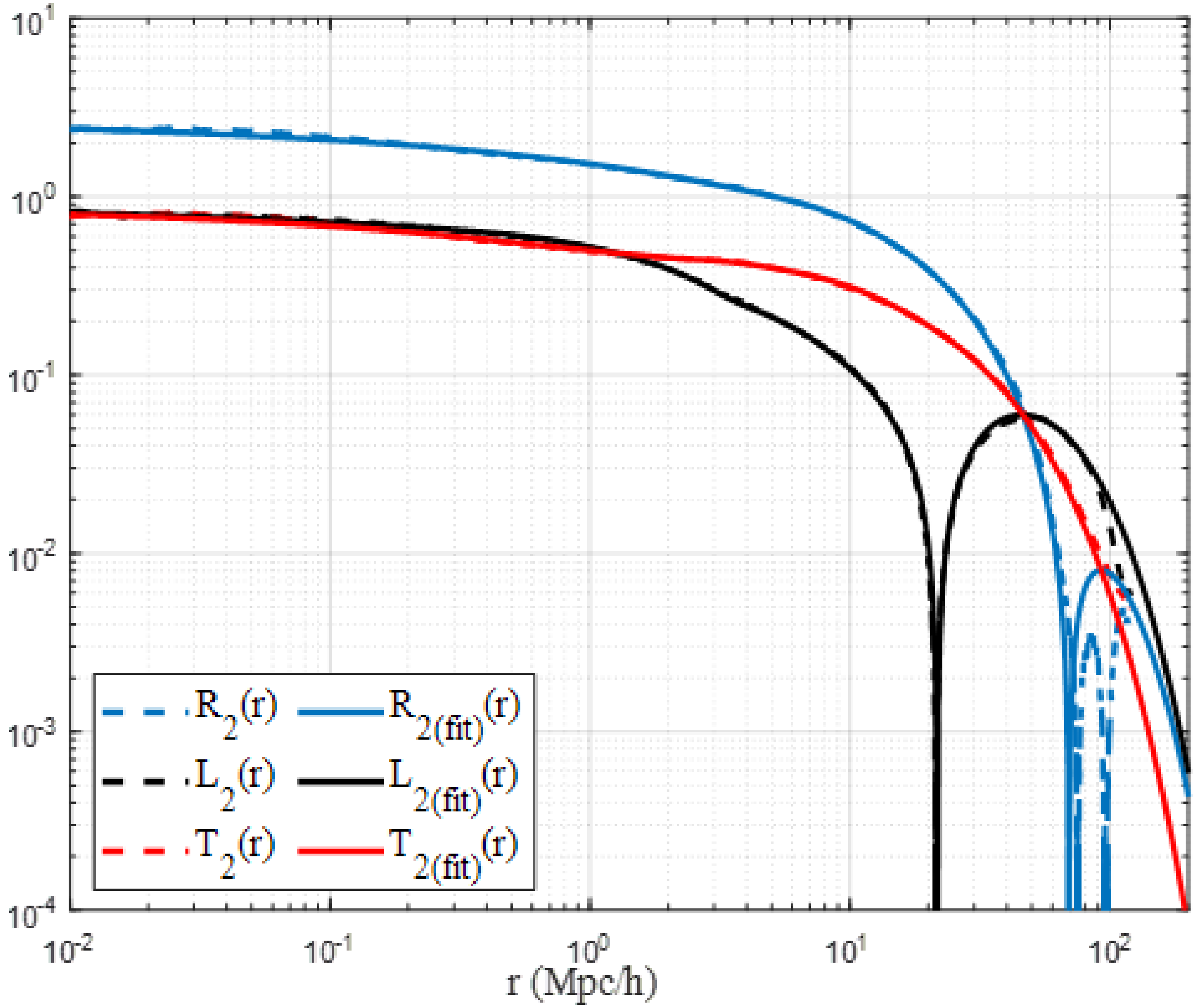
Correlation function on large scale

$$s(r) = \frac{1}{1 + x_b e^{-(r-x_c)/x_a}}$$

Interpolation function for smooth connection

$$R_{2(fit)}(r) = R_{2s}\left(1 - s(r)\right)^{n1} + R_{2l}\left(s(r)\right)^{n2}$$

Final fitted correlation function is obtained by parameter optimization using correlations from N-body simulation



The fitted velocity correlation functions compared to original correlations from N-body simulation

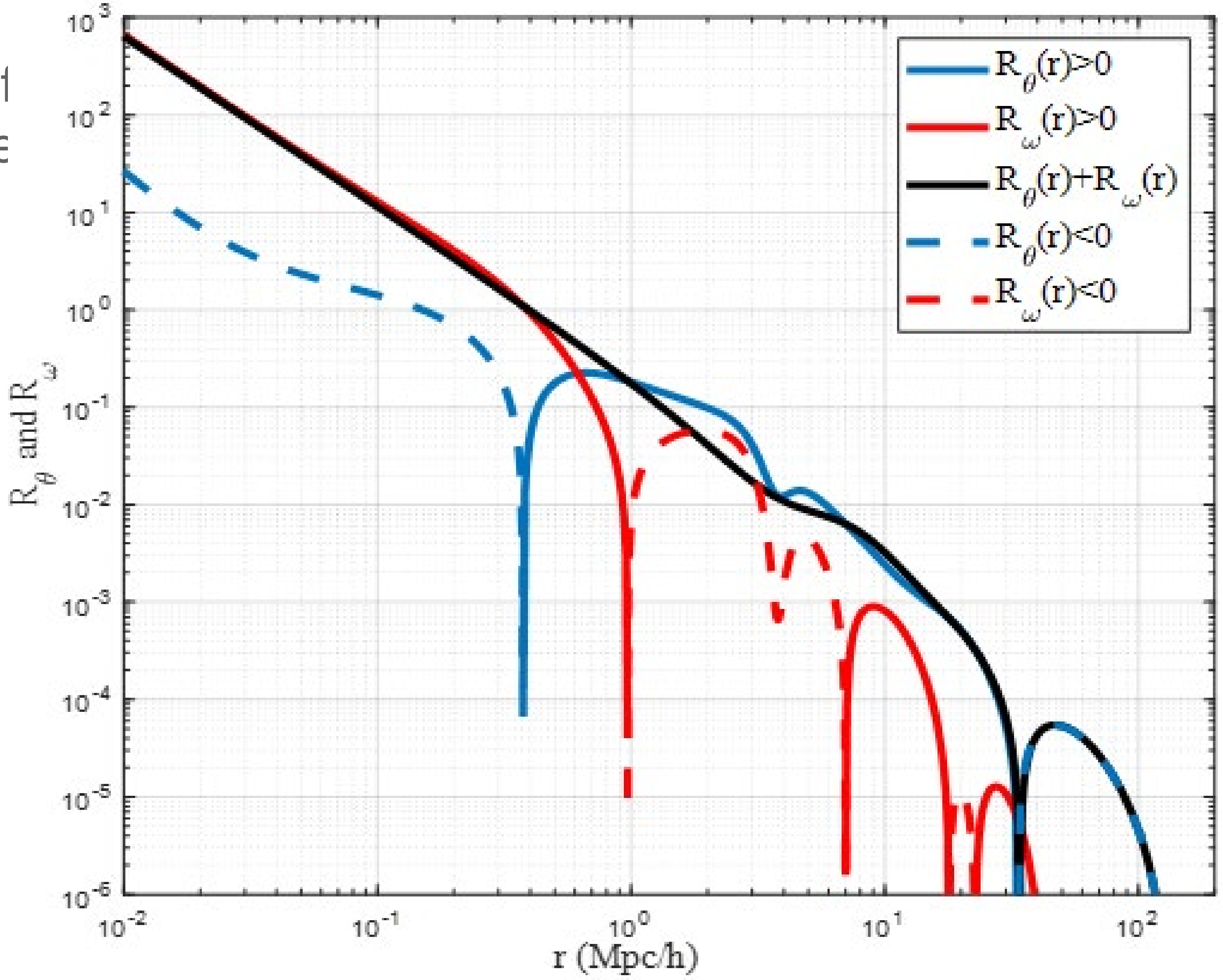
Modeling divergence and vorticity correlations on entire range of scales

- With correlation functions modelled on entire range of scales, correlations of divergence and vorticity can be obtained using kinematic relations.

- Divergence is negatively correlated on scale $r > 30\,Mpc/h$

- Vorticity is negatively correlated for scale r between 1Mpc/h and 7Mpc/h (pair of particles mostly from different halos) and positively correlated on small scale (pair of particles from the same halo).

- Vorticity is dominant on small scale while divergence dominant on large scale.



Variation of correlation functions of divergence and vorticity with scale *r* at z=0

Summary and keywords

Velocity correlation tensor	Longitudinal velocity	Two-thirds law / one-fourth law
Kinematic relations	Transverse velocity	Spectrum functions
Correlation functions	Structure functions	Dispersion functions

- Identify connections with homogeneous isotropic turbulence for the development of the statistical theory in terms of correlation, structure, dispersion, and spectrum functions
- Identify the nature of peculiar velocity in dark matter flow: constant divergence flow on small scale and irrotational flow on large scale.
- Develop kinematic relations between different statistical measures
- The limiting correlation coefficient of velocity $\rho=1/2$ on the smallest scale ($r=0$) is a unique feature of dark matter flow ($\rho=1$ for incompressible flow) along with the implications for particle annihilation
- On large scale, the transverse velocity correlation has an exponential form with a comoving length scale $r_2=21.3$Mpc/h. All correlation/structure/dispersion/spectrum functions for velocity, density, and potential can be derived analytically using kinematic relations for irrotational flow.
- On small scale, the longitudinal structure function follows a one-fourth law $S_2^l \sim r^{1/4}$, along with other correlation/structure/dispersion/spectrum functions obtained from kinematic relations for constant divergence flow.

Scale and redshift dependence of density and velocity distributions in dark matter flow

Xu Z., 2022, arXiv:2202.06515 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2202.06515

Introduction

Review:
Statistical theory in hydrodynamic turbulence

- Velocity fluctuation and distributions
- Incompressible on all scales
  - Divergence-free
  - Constant density

- N-body simulations are invaluable tools for DMF:
  - Velocity fluctuation and distributions
  - Density is non-uniform (density fluctuation/distributions)

- Fundamental problems when projecting N-body density/velocity field onto structured grid:
  - N-body fields are sampled discrete locations of particles.
  - The sampling has a poor quality at locations with low particle density

Goal 1: Density distributions and two-point statistics

Goal 2: Velocity distributions and redshift and scale dependence

Halo-based non-projection approach:

- Instead of projecting, analysis is performed by the statistics over all pairs on different scales to maximumly preserve the information from N-body simulation

- Based on the halo description, divide all particles into halos and out-of-halo particles, whose distributions evolve differently

- Scale and redshift dependence of distributions can be studied by the variation of generalized kurtosis for a given distribution.

One-point probability distributions of density field

- Projecting particle field onto structured grid involves information loss and numerical noise.
- Without projecting onto grid, Delaunay tessellation is used to reconstruct the density field and maximumly preserve information in N-body data.
- Compute the volume V_p occupied by every particle

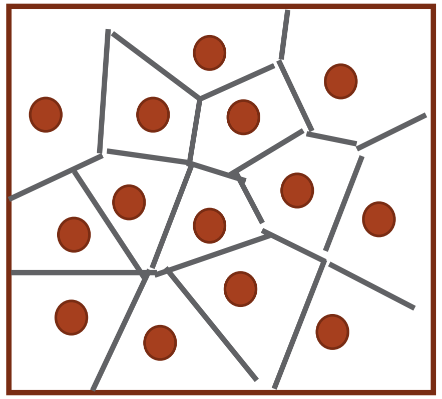
$$\rho(\mathbf{x}) = m_p / V_p \qquad \delta(\mathbf{x}) = \frac{\rho(\mathbf{x})}{\rho_0} - 1$$

Particle density

Particle density contrast

$$\eta(\mathbf{x}) = \log\left(1+\delta(\mathbf{x})\right) = \log\left(\frac{\rho(\mathbf{x})}{\rho_0}\right)$$

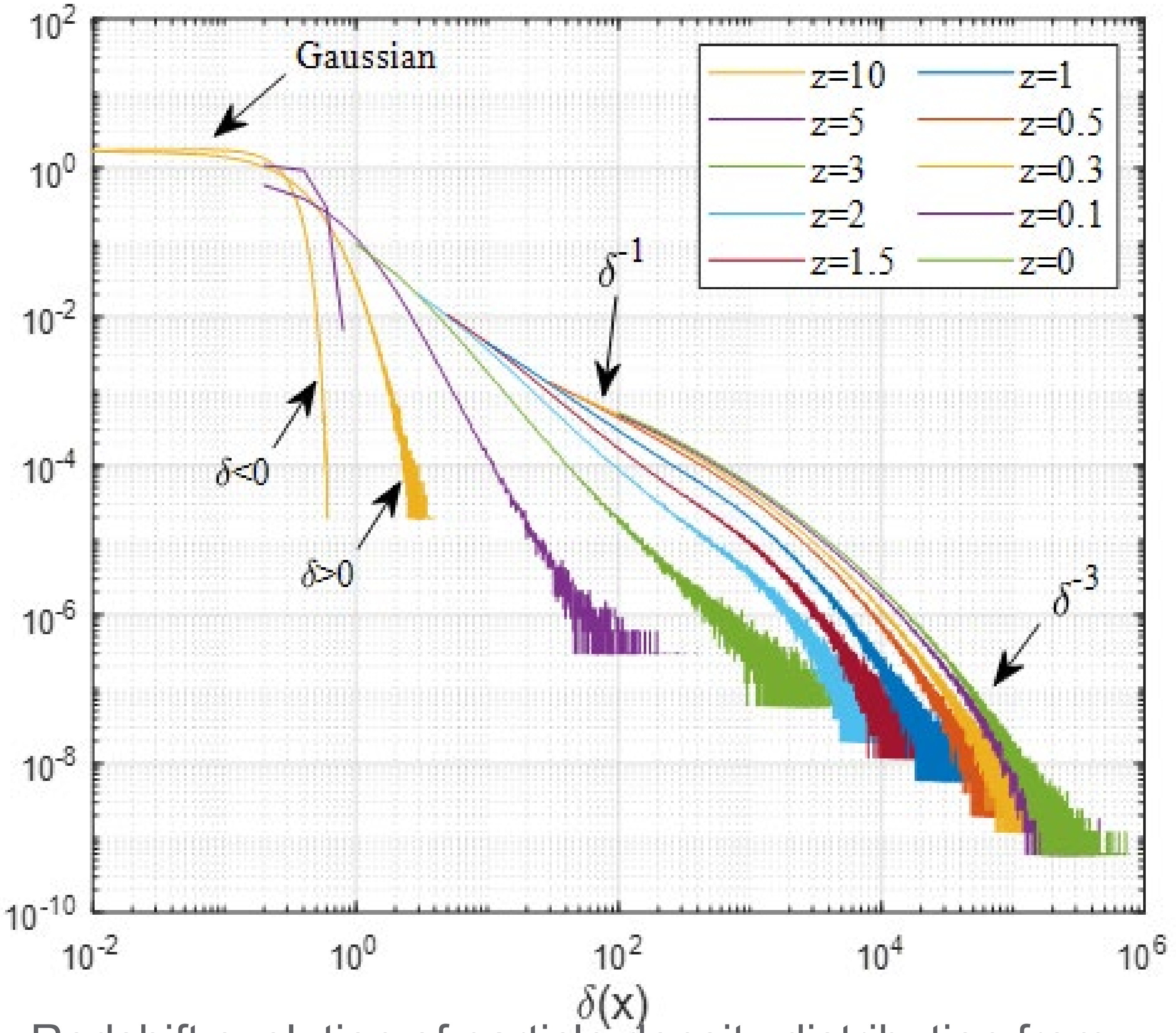
Particle log-density



Delaunay tessellation

$$\left\langle \frac{1}{1+\delta(\mathbf{x})} \right\rangle = 1 \qquad \left\langle e^{-\eta(\mathbf{x})} \right\rangle = 1$$

Constraints for density contrast and log-density



Redshift evolution of particle density distribution from z=10 to z=0. Density evolves from initial Gaussian to an asymmetric distribution with a long tail $\sim\delta^{-3}$

Probability distributions of log-density field

- Gaussian distribution of log-density at high redshift.
- Bimodal distribution of log-density at low redshift.
- Two peaks corresponds to contributions from particles in all halos and particles out-of-halo.
- Best fitted bimodal distribution at z=0 showing fraction of particles in halos is about 60%, consistent with inverse mass cascade theory.

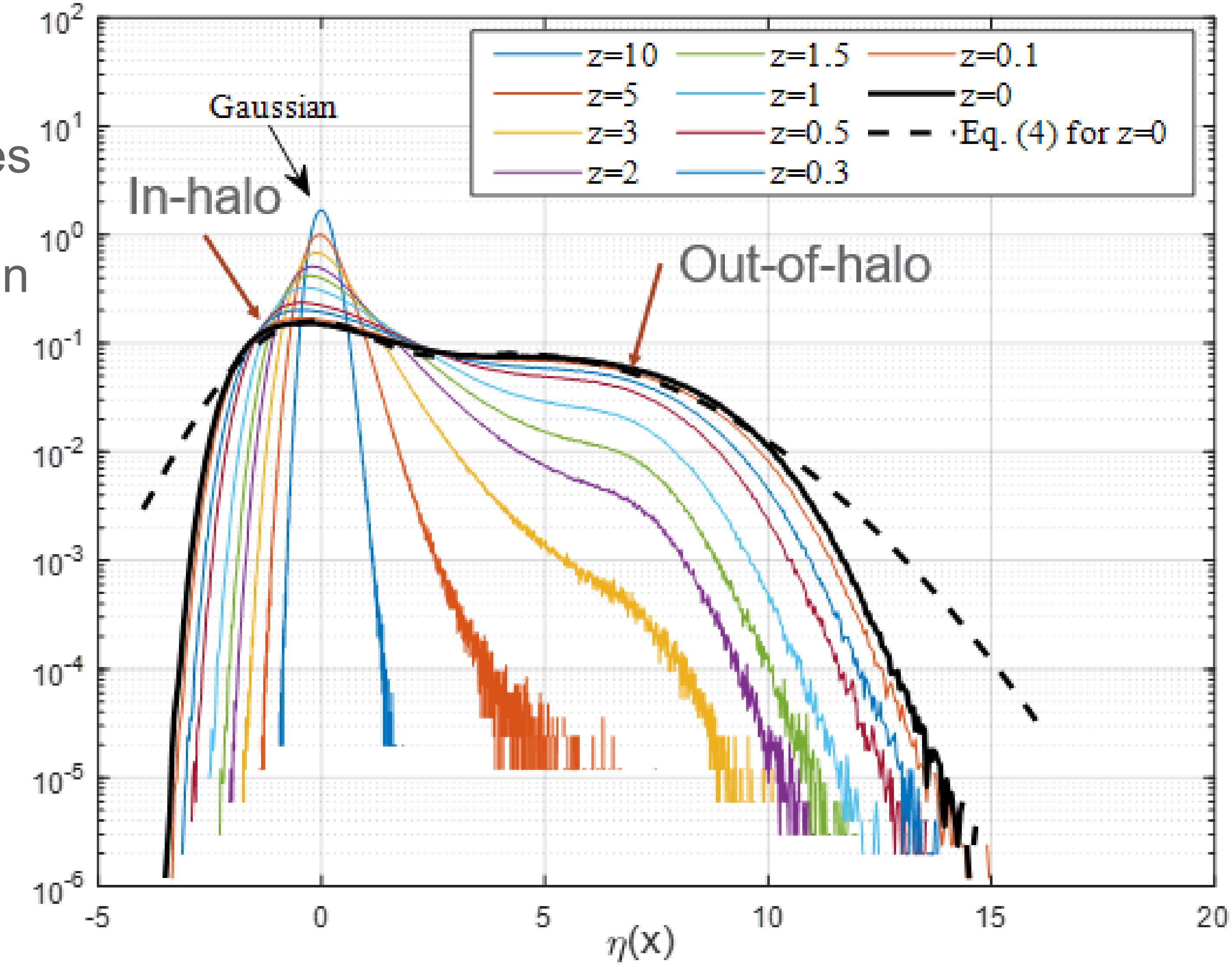
$$f(\eta)=\frac{c_1}{\sqrt{2\pi}\sigma_1}\exp\left[\frac{(\eta-\mu_1)^2}{2\sigma_1^2}\right]+\frac{1-c_1}{\sqrt{2\pi}\sigma_2}\exp\left[\frac{(\eta-\mu_2)^2}{2\sigma_2^2}\right]$$

$c_1=0.404 \qquad c_2=1-c_1=0.596$

$\mu_1=-0.30 \qquad \mu_2=4.256$

$\sigma_1=1.212 \qquad \sigma_2=2.979$

Particles in halos should have an average density close to Δ_c, the critical density ratio $18\pi^2$, such that the mean density for all halo particles $\langle\mu_2\rangle=\log(18\pi^2)\approx 5$



Distribution of log-density at different redshifts z. The log-density evolves from Gaussian to an approximately bimodal distribution at z=0 with two peaks.

Halo-based non-projection approach for particle density

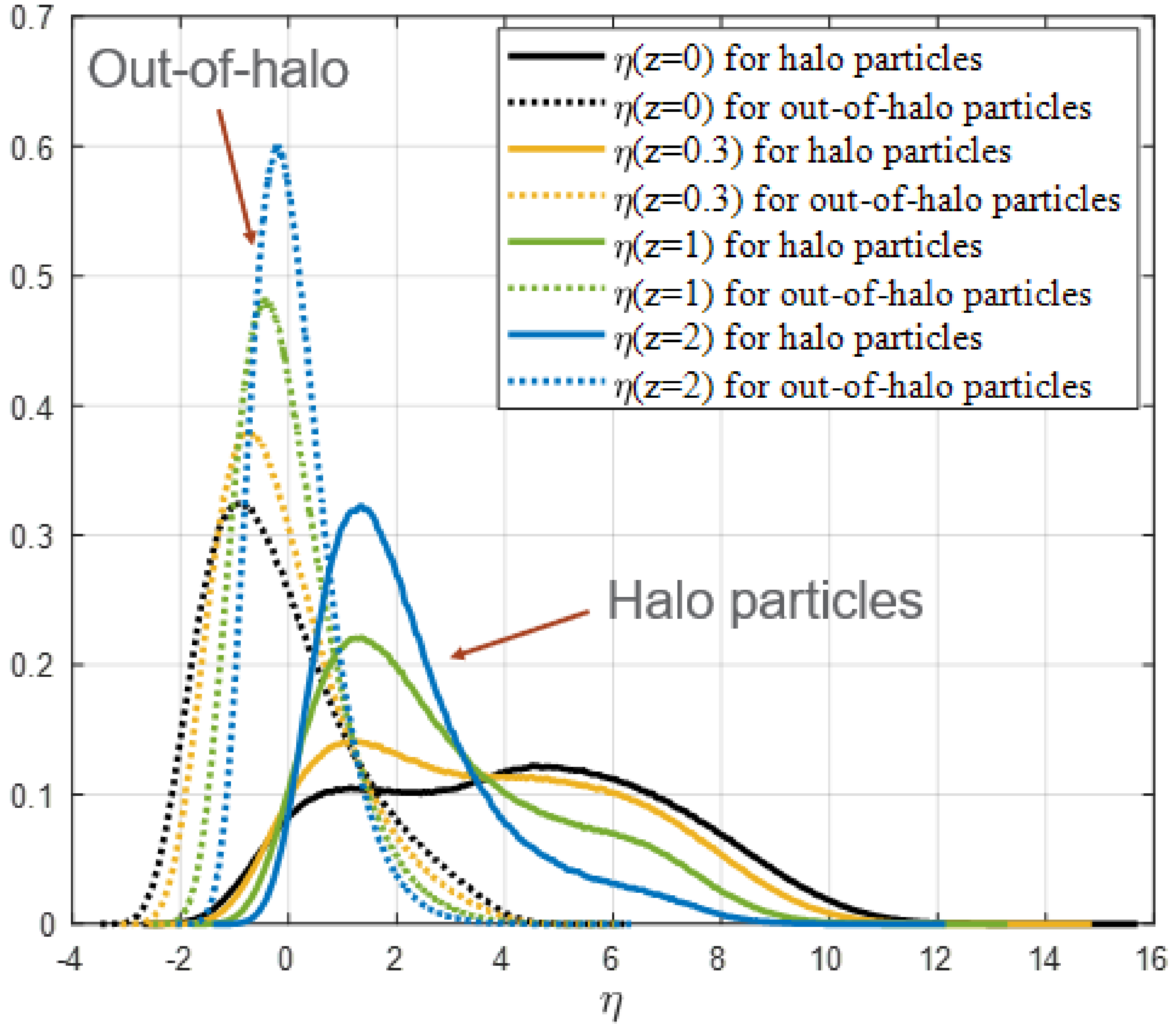
- Checking the density distributions of particles in halos and out-of-halo particles separately.
- Identifying all halos in entire system and dividing all particles into halo and out-of-halo particles.
- For out-of-halo particles, the distribution is relatives Gaussian (or δ is lognormal) with mean density decreasing with time.
- For halo particles, log-density distribution evolves with increasing mean density due to the formation of halos.

Characterizing the time evolution of the shape of distribution by introducing nth order generalized kurtosis:

$$K_n(\tau) = \frac{\left\langle \left(\tau - \langle\tau\rangle\right)^n \right\rangle}{\left\langle \left(\tau - \langle\tau\rangle\right)^2 \right\rangle^{n/2}} = \frac{S_n^{cp}(\tau)}{S_2^{cp}(\tau)^{n/2}}$$ Generalized kurtosis

$$S_n^{cp}(\tau) = \left\langle \left(\tau - \langle\tau\rangle\right)^n \right\rangle$$ nth central moment

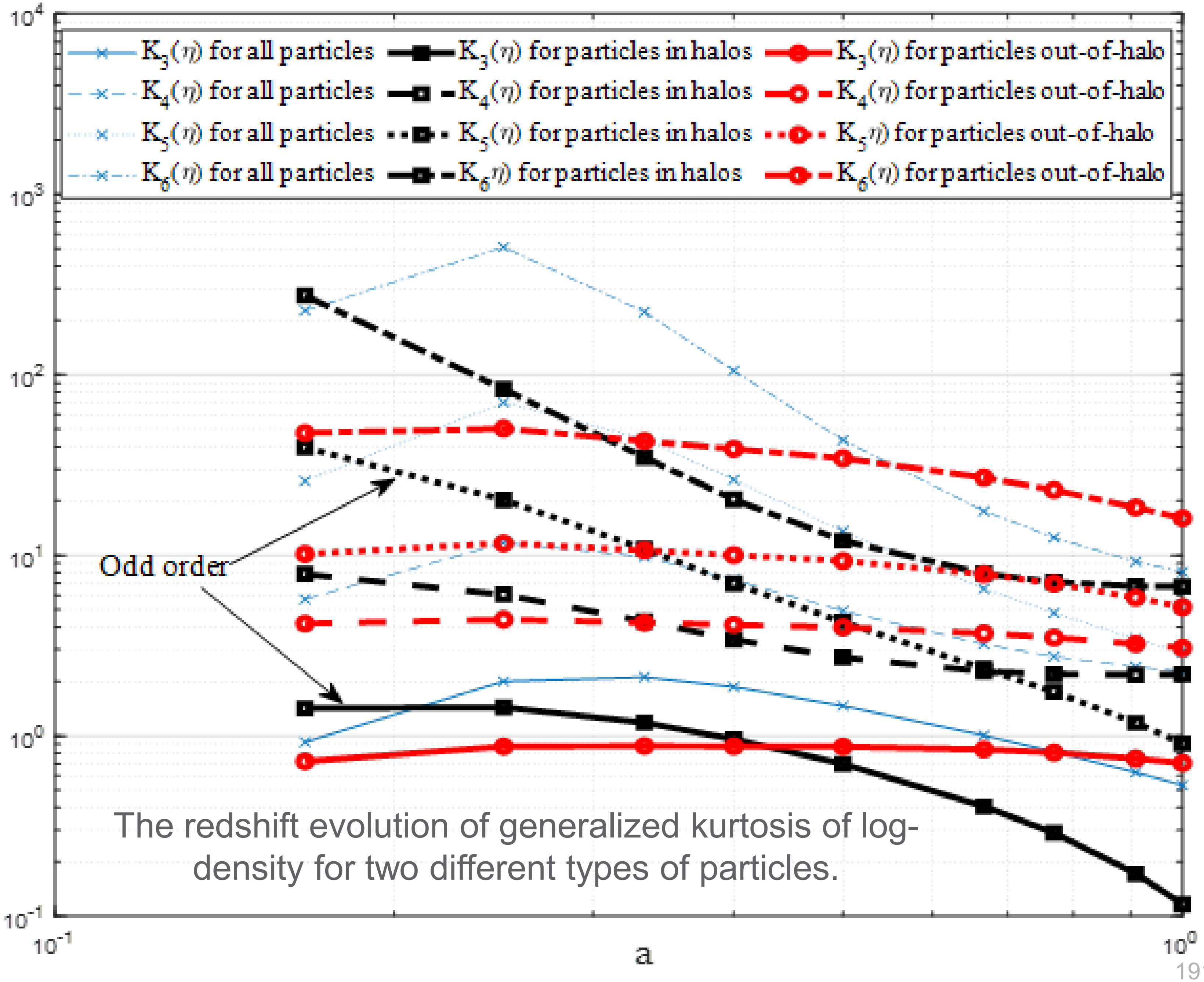
For Gaussian: $K_2 = 1 \quad K_4 = 3 \quad K_6 = 15 \quad K_8 = 105 \quad K_3 = K_5 = 0$



Redshift evolution of log-density distributions for two different types of particles.

Time evolution of comoving particle density field

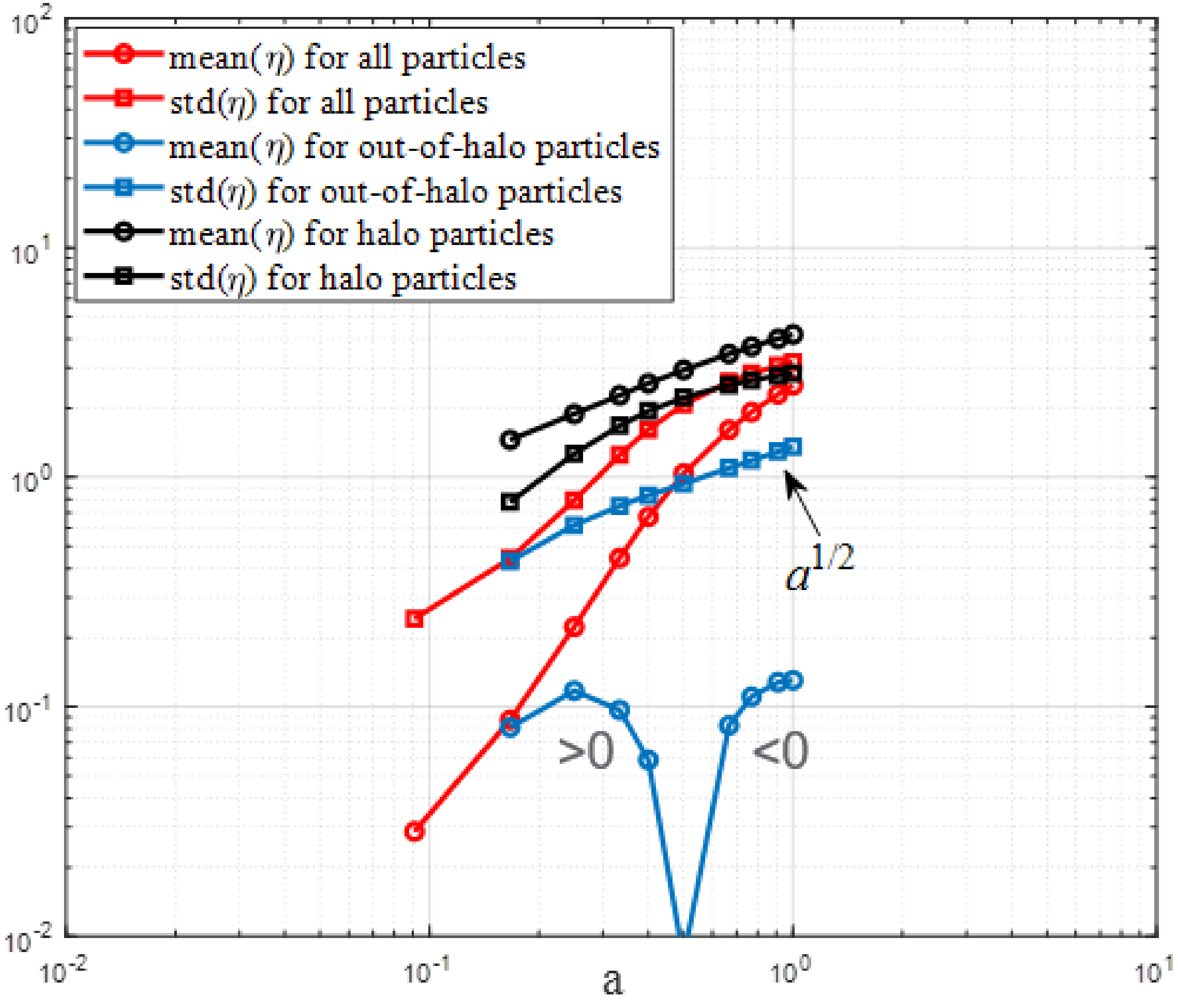
- Distribution of η is always Gaussian for out-of-halo particles.
- Distribution of δ for out-of-halo particles is approximately log-normal
- Distribution of η for halo particles approaching some symmetric non-Gaussian distribution with vanishing odd order kurtosis



The redshift evolution of generalized kurtosis of log-density for two different types of particles.

Time evolution of particle density field

- For out-of-halo particles, the mean log-density decreases with time and $<\eta><0$ after z=1. This reflects less and less out-of-halo particles due to inverse mass cascade.
- For halo particles, mean log-density increasing with time ($<\eta> \sim a^{1/2}$) reflects more and more particles residing in halos
- For halo particles, standard deviation of log-density increasing with time ($\mathrm{std}(\eta) \sim a^{1/2}$)



The variation of mean and standard deviation of log-density with scale factor a.

Two-point statistical measures of density field

Defining two-point density correlation function from radial distribution function g(r) in statistic mechanics, a quantity to measure the averaged particle density from an arbitrary reference particle:

$$dN_p = g(r)\frac{N_p}{V}4\pi r^2 dr$$

N_p/V

mean number density of particles in entire system

$$\int_0^\infty g(r)4\pi r^2 dr = \frac{N_p-1}{N_p}V$$

$$\xi(r) = \langle \delta(\mathbf{x})\delta(\mathbf{x}+\mathbf{r})\rangle = g(r)-1$$

$$\int_0^\infty \xi(r,a)4\pi r^2 dr = -V/N_p < 0$$

Correlation cannot be positive on all scales

Two length scales can be defined from density correlation:

$$l_{\delta 0}(a) = \int_0^\infty \xi(r,a)dr \qquad l_{\delta 1}^2(a) = \int_0^\infty \xi(r,a)rdr$$

On large scale, transverse velocity correlation can be well modelled by exponential function:

$$T_2(r,a) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right) \propto a \qquad a_0(u/u_0)^2 = 0.45a$$

$$r_2 \approx 21.4\,Mpc/h$$

Redshift-independent length scale, might be related to the size of sound horizon

Total velocity correlation

$$R_2(r,a) = \langle \mathbf{u}\cdot\mathbf{u}'\rangle = 2R(r) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(3-\frac{r}{r_2}\right)$$

$$\delta \approx \eta = -\frac{\nabla\cdot\mathbf{u}}{aHf(\Omega_m)}$$

Linear perturbation theory on large scale:

Modeling density correlation on large scale:

$$\xi(r,a) = \frac{1}{(aHf(\Omega_0))^2}\cdot\frac{a_0u^2}{rr_2}\exp\left(-\frac{r}{r_2}\right)\left[\left(\frac{r}{r_2}\right)^2 - 7\left(\frac{r}{r_2}\right)+8\right]$$

Specific potential/kinetic energy from density correlation function

In statistical mechanics, potential energy of any system with particles interacting via a pairwise potential $V_g(r)$ can be related to the radial distribution function g(r):

$$PE = \frac{2\pi\rho_0}{m_p^2}\int_0^\infty r^2\left[g(r)-1\right]V_g(r)\,dr$$

$$P_y(a) = -\frac{2\pi G\rho_0}{a}\int_0^\infty \xi(r,a)\,r\,dr = -\frac{3H_0^2 l_{\delta 1}^2}{4a} < 0$$

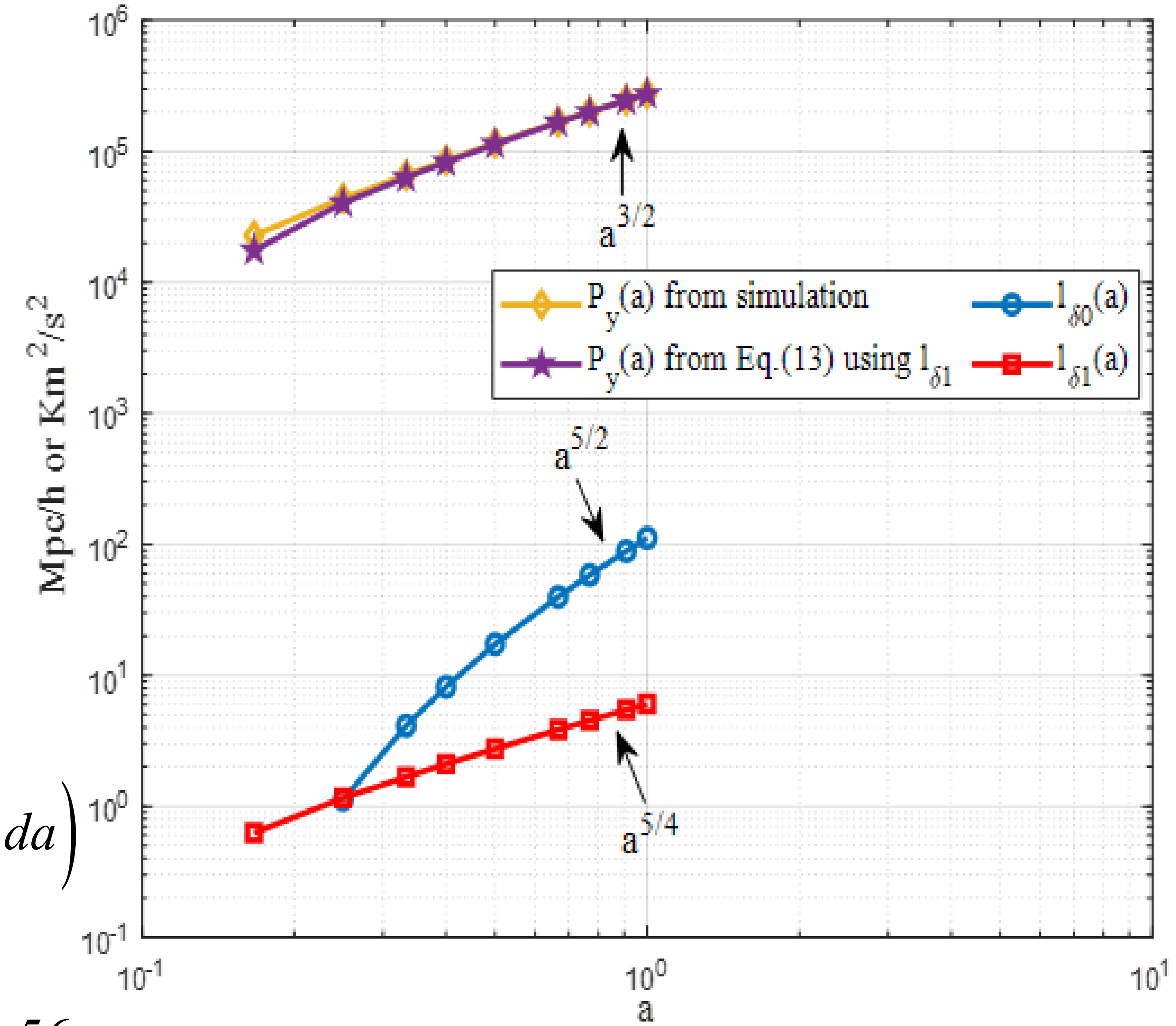
Cosmic energy equation

$$\frac{\partial\left(K_p+P_y\right)}{\partial t} + H\left(2K_p+P_y\right) = 0$$

$$K_p = a^{-2}\int_0^a aP_y\,da - P_y \Rightarrow K_p = \frac{3}{4}H_0^2 a^{-1}\left(l_{\delta 1}^2 - a^{-1}\int_0^a l_{\delta 1}^2\,da\right)$$

Power-law evolution and rate of energy cascade ε_u:

$$K_p = -\varepsilon_u t \quad P_y = \frac{7}{5}\varepsilon_u t \Rightarrow l_{\delta 1}^2(a) = \int_0^\infty \xi(r,a)\,r\,dr = -\frac{56}{45}\frac{\varepsilon_u}{H_0^3}a^{5/2}$$



The variation of two comoving correlation lengths with scale factor a.

Density spectrum/dispersion functions and real space distribution of density fluctuation

Correlation and spectrum form Fourier pair:

$$E_\delta(k,a) = \frac{2}{\pi}\int_0^\infty \xi(r,a)\, kr \sin(kr)\, dr$$

$$\xi(r,a) = \int_0^\infty E_\delta(k,a)\frac{\sin(kr)}{kr}dk$$

Matter spectrum function:

$$P_\delta(k,a) = 2\pi^2 E_\delta(k,a)/k^2$$

The power per logarithmic interval:

$$\Delta_\delta^2(k,a) = E_\delta(k,a)k$$

Density dispersion function (the variance of the density fluctuation on scale r):

$$\sigma_\delta^2(r,a) = \int_{-\infty}^{\infty} E_\delta(k,a) W(kr)^2\, dk$$

First order spherical Bessel function of the first kind

Window function when smoothed with a filter of size r

$$W(x \equiv kr) \qquad W(x) = \frac{3}{x^3}\left[\sin(x) - x\cos(x)\right] = 3\frac{j_1(x)}{x}$$

$$\xi(2r) = \frac{1}{72r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\sigma_\delta^2(r)r^4\right)\right)\right)$$

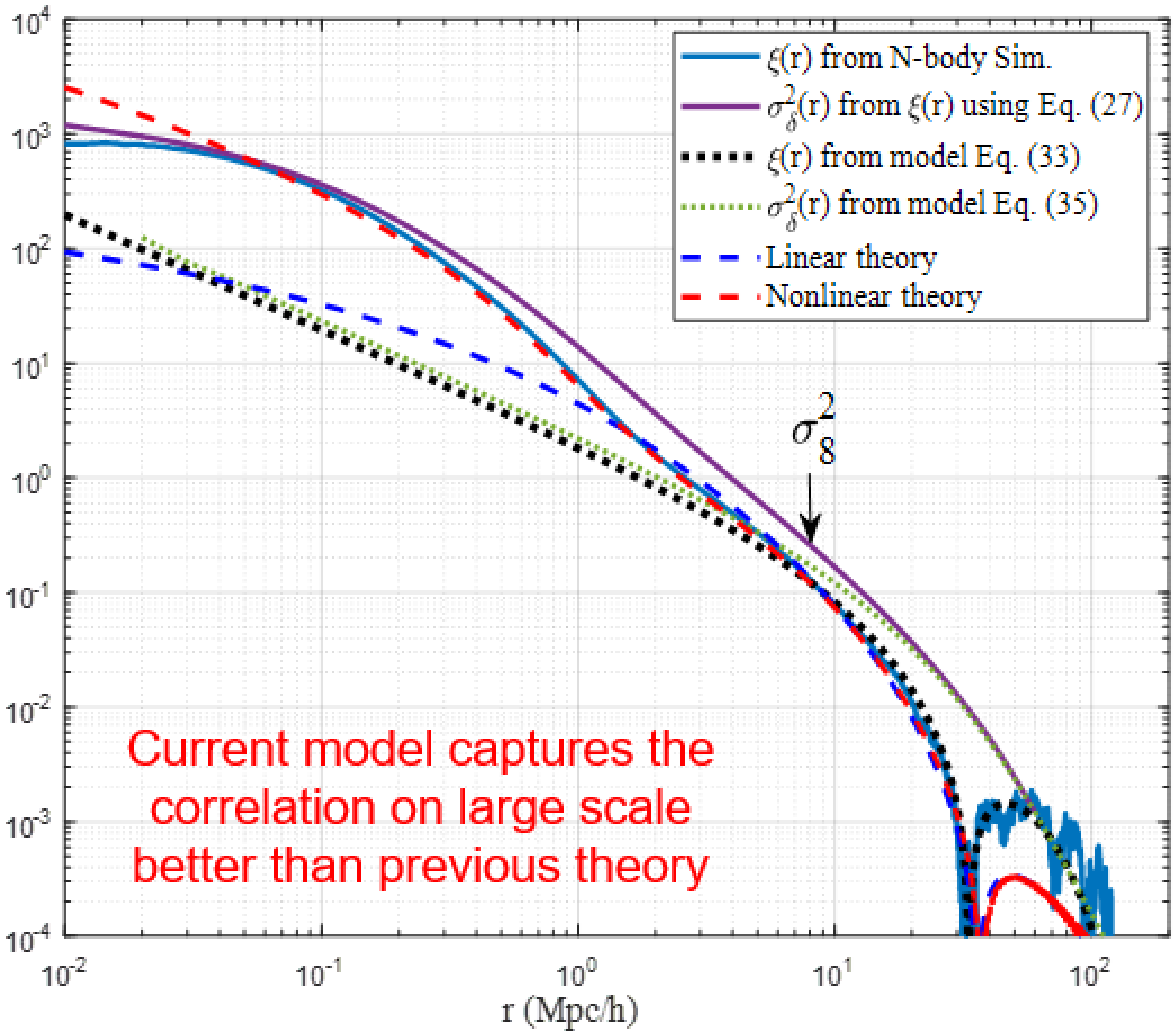
$$E_{\delta r}(r) = -\frac{\partial\sigma_\delta^2(r)}{\partial r}$$

The real-space distribution of density fluctuation in scales [r, r+dr]

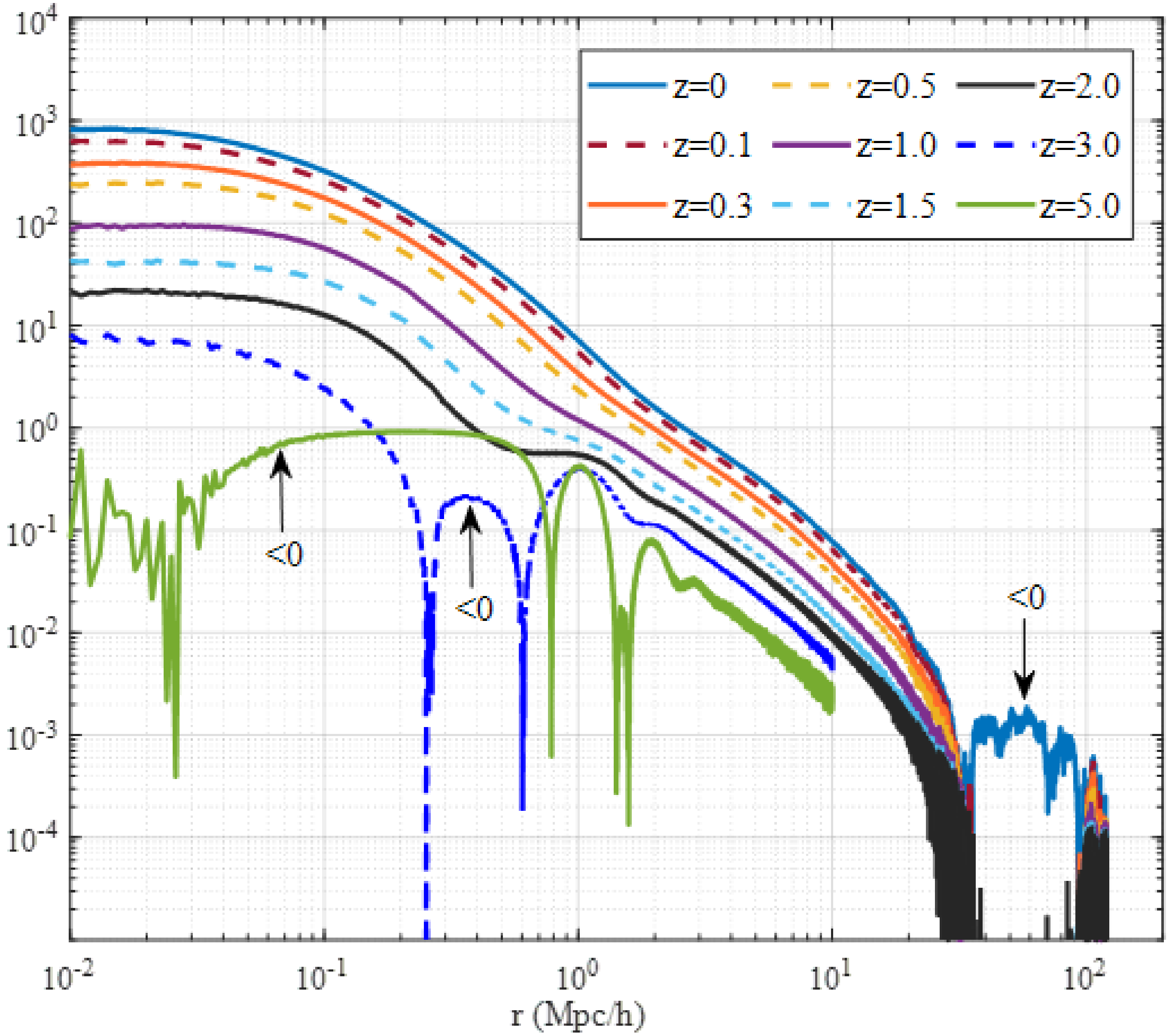
Modeling density dispersion function on large scale:

$$\sigma_\delta^2(r) = \frac{1}{\left(aHf(\Omega_0)\right)^2}\cdot\frac{9a_0u^2}{2r^2}\left\{3\left(\frac{r_2}{r}\right)^4 + \left(\frac{r_2}{r}\right)^2 - \exp\left(-\frac{2r}{r_2}\right)\left[1+\left(\frac{r_2}{r}\right)^2\right]\left[3\left(\frac{r_2}{r}\right)^2 + 6\left(\frac{r_2}{r}\right) + 4\right]\right\}$$

Density correlation function (simulations & models)

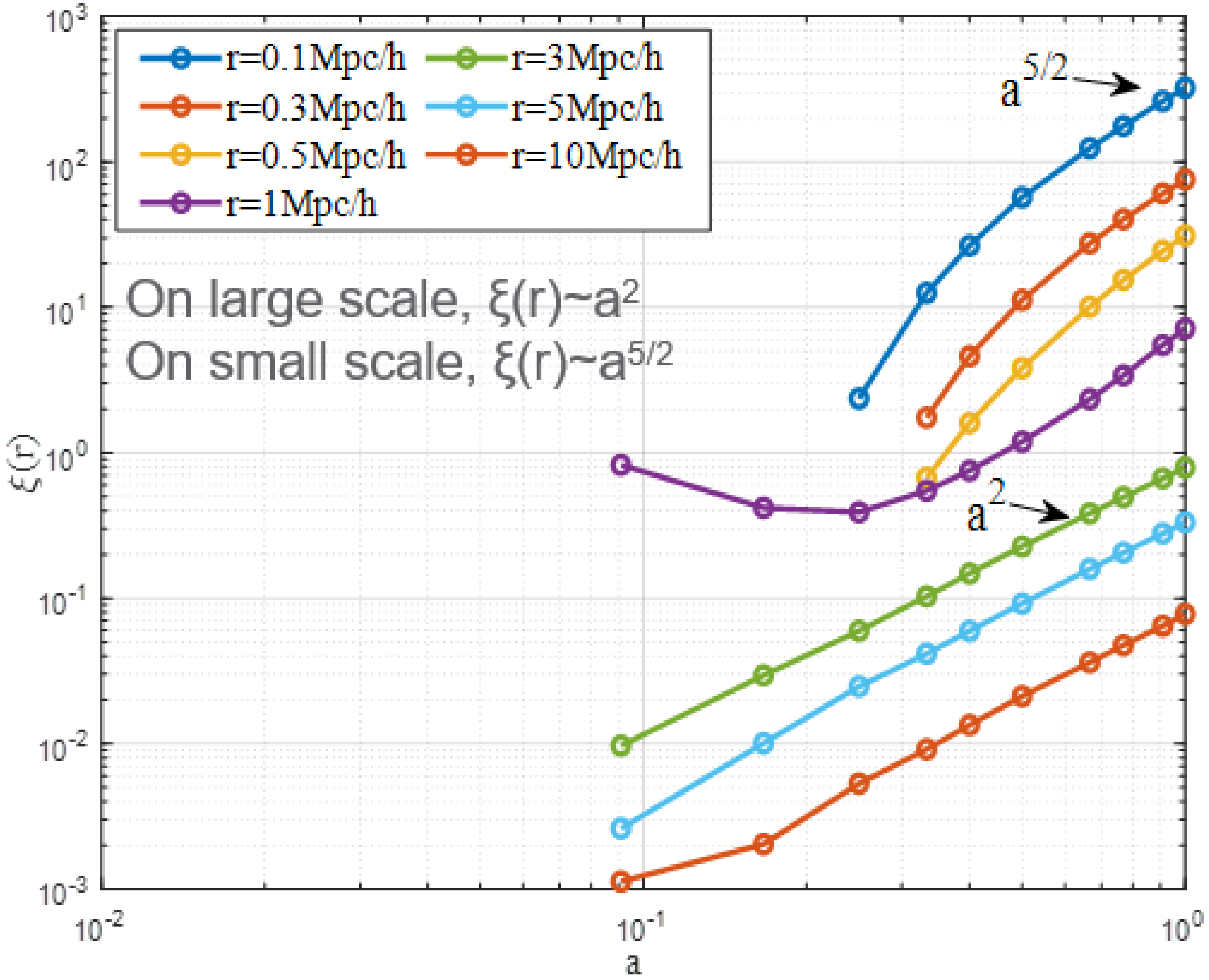


Density correlation function (solid blue) varying with scale r at z=0.

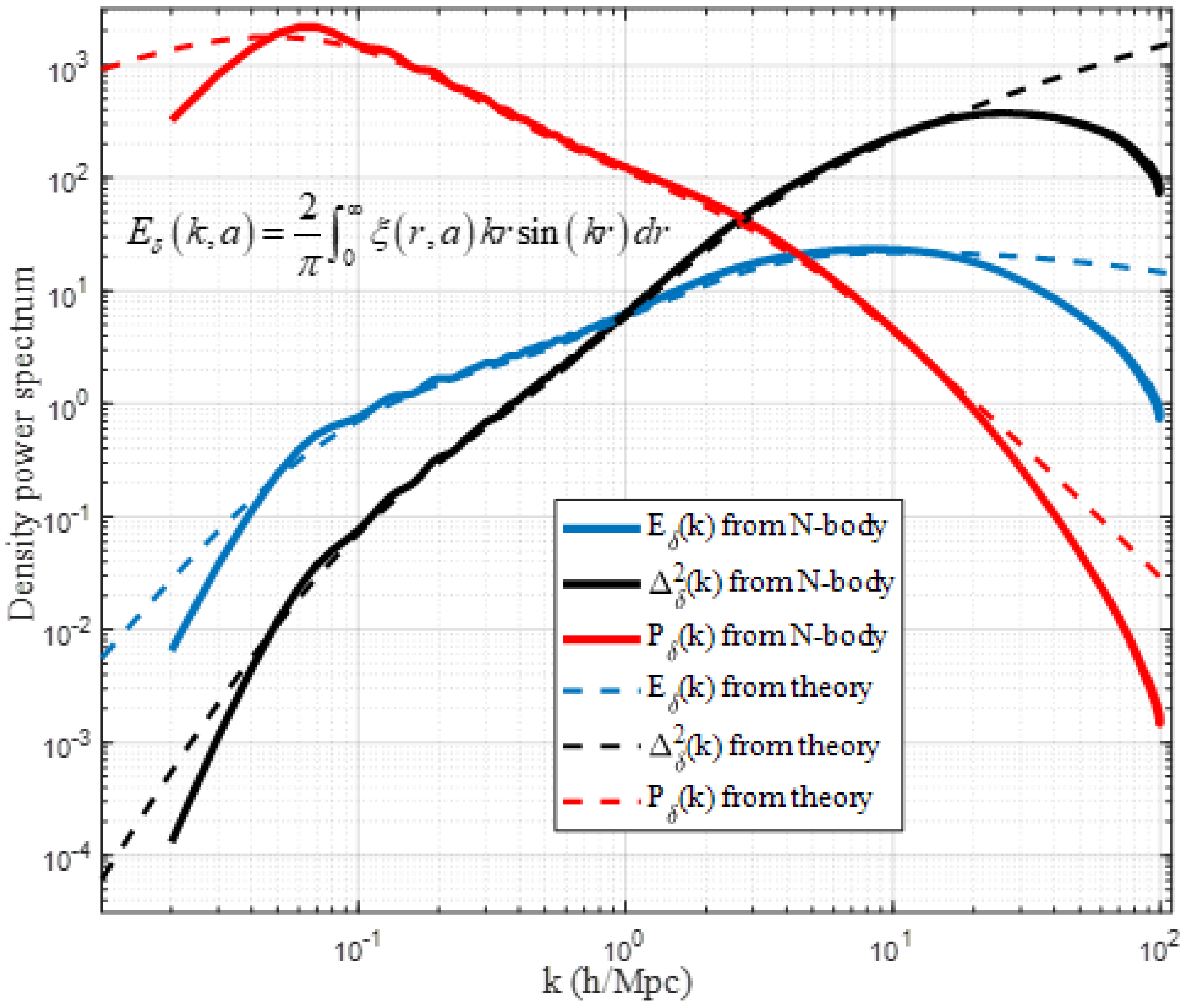


Density correlation function varying with scale r at different redshifts.

Density correlation and spectrum functions (simulation & models)

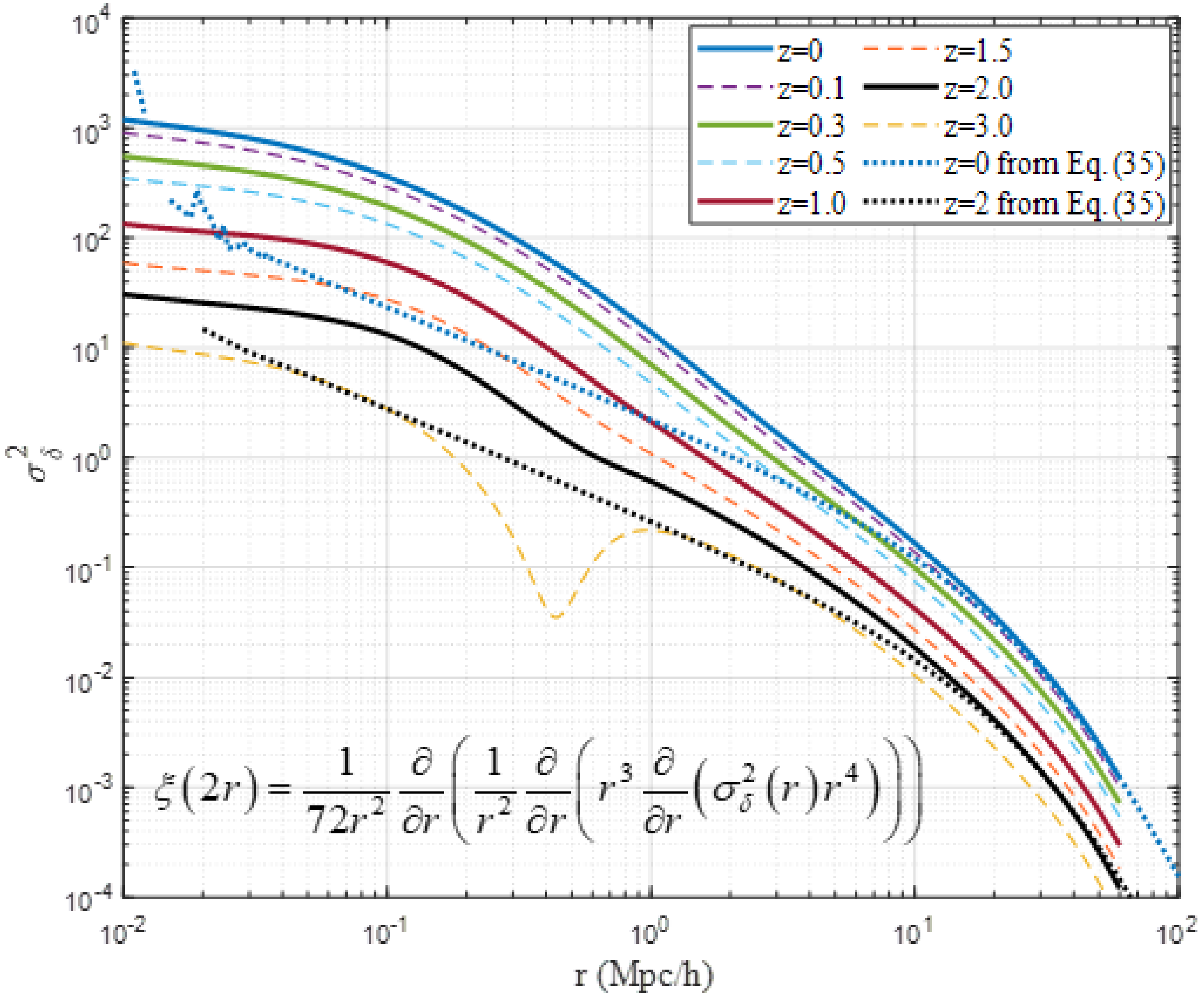


Two-point second order density correlation varying with scale factor a.

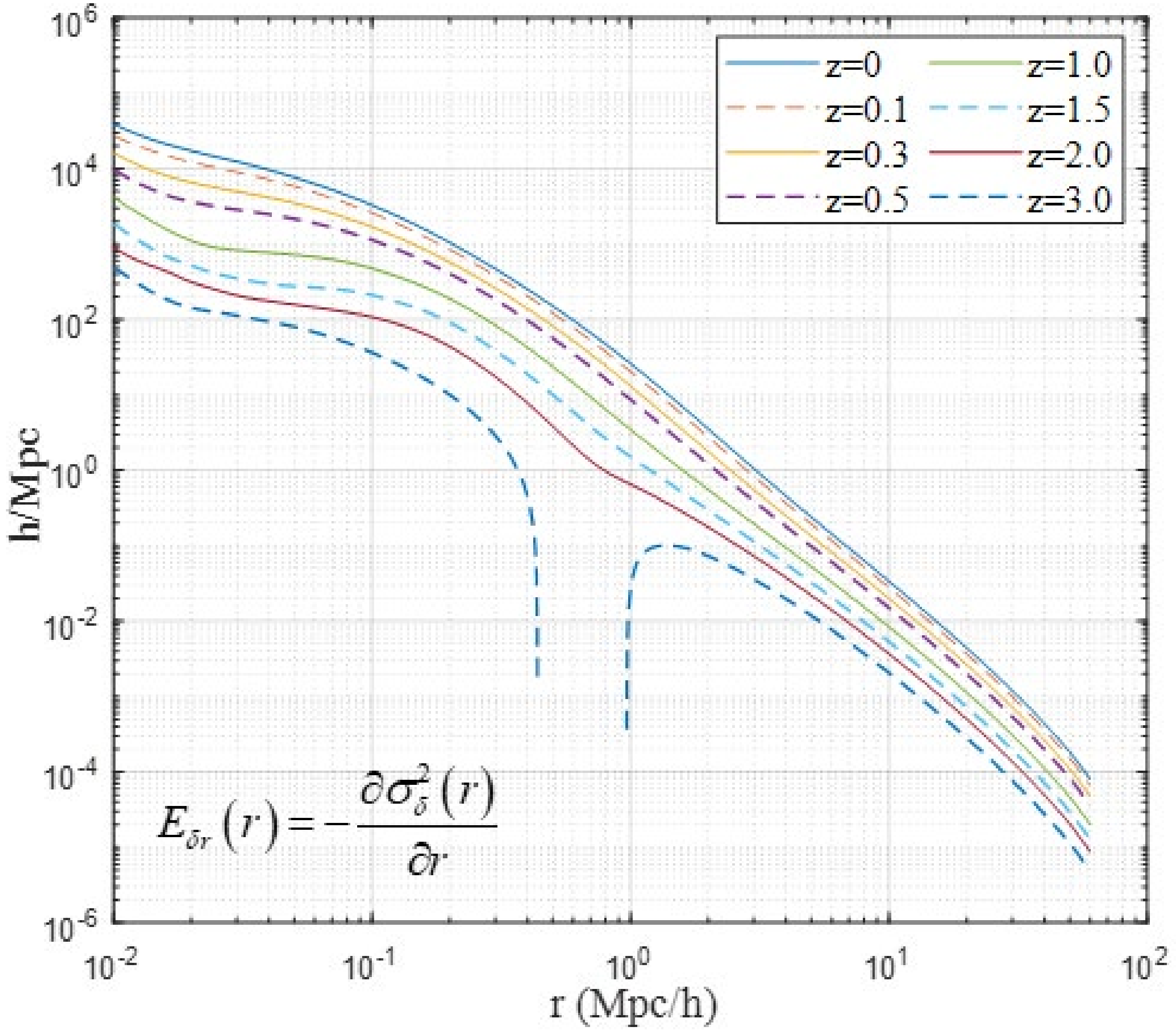


Without projection, density power spectrum can be obtained from Fourier transform of correlation.

Density dispersion function and distribution of density fluctuation

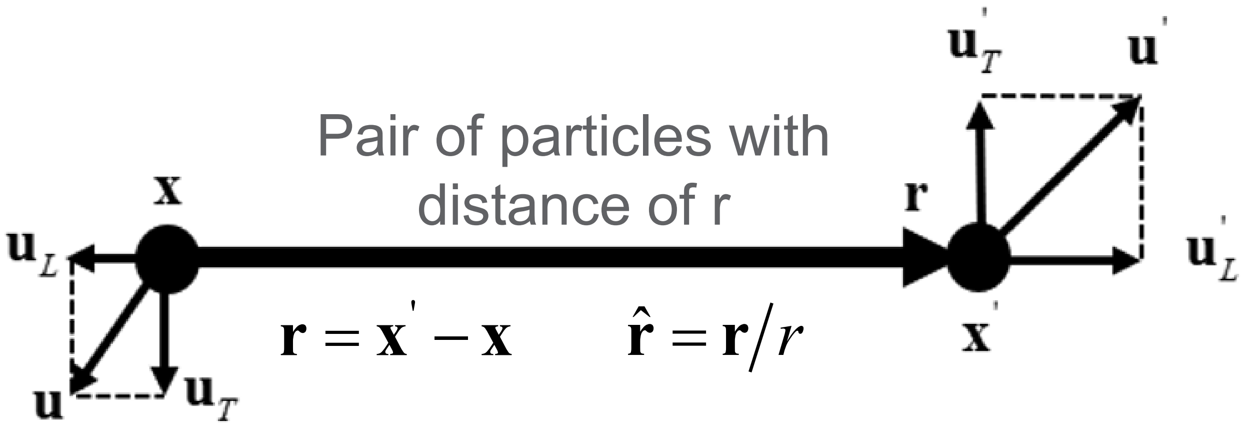


Density dispersion function obtained from density correlation and compared with models.



Distribution of density fluctuation on scale r obtained from density dispersion function

Characterizing distributions of velocity fields



Longitudinal velocity:	Transverse velocity:
$u_L = \mathbf{u} \cdot \hat{\mathbf{r}} = u_i \hat{r}_i$	$\mathbf{u}_T = -(\mathbf{u} \times \hat{\mathbf{r}} \times \hat{\mathbf{r}})$
$u'_L = \mathbf{u}' \cdot \hat{\mathbf{r}} = u'_i \hat{r}_i$	$\mathbf{u}'_T = -(\mathbf{u}' \times \hat{\mathbf{r}} \times \hat{\mathbf{r}})$

Velocity difference or Pairwise velocity: $\Delta u_L = u'_L - u_L$

Velocity sum: $\sum u_L = u'_L + u_L$

We focus on the distribution of seven types of velocities:

Scale-dependent velocities (dependent on r):

- Longitudinal velocity: u'_L and u_L
- Pairwise velocity: $\Delta u_L = u'_L - u_L$
- Velocity sum: $\sum u_L = u'_L + u_L$

Based on halo-based non-projection approach,

Redshift-dependent velocities (dependent on z):

- Velocity of all particles in entire system: $\mathbf{u}_p$
- Velocity of all halo particles: $\mathbf{u}_{hp}$
- Velocity of all out-of-halo particles: $\mathbf{u}_{op}$
- Velocity of all halos: $\mathbf{u}_h$

Redshift dependence of velocity distributions

The scale and redshift variation can be studied by Introducing generalized Kurtosis:

$$K_n(\Delta u_L, r) = \frac{\left\langle \left(\Delta u_L - \langle \Delta u_L \rangle\right)^n \right\rangle}{\left\langle \left(\Delta u_L - \langle \Delta u_L \rangle\right)^2 \right\rangle^{n/2}} = \frac{S_n^{cp}(\Delta u_L, r)}{S_2^{cp}(\Delta u_L, r)^{n/2}}$$

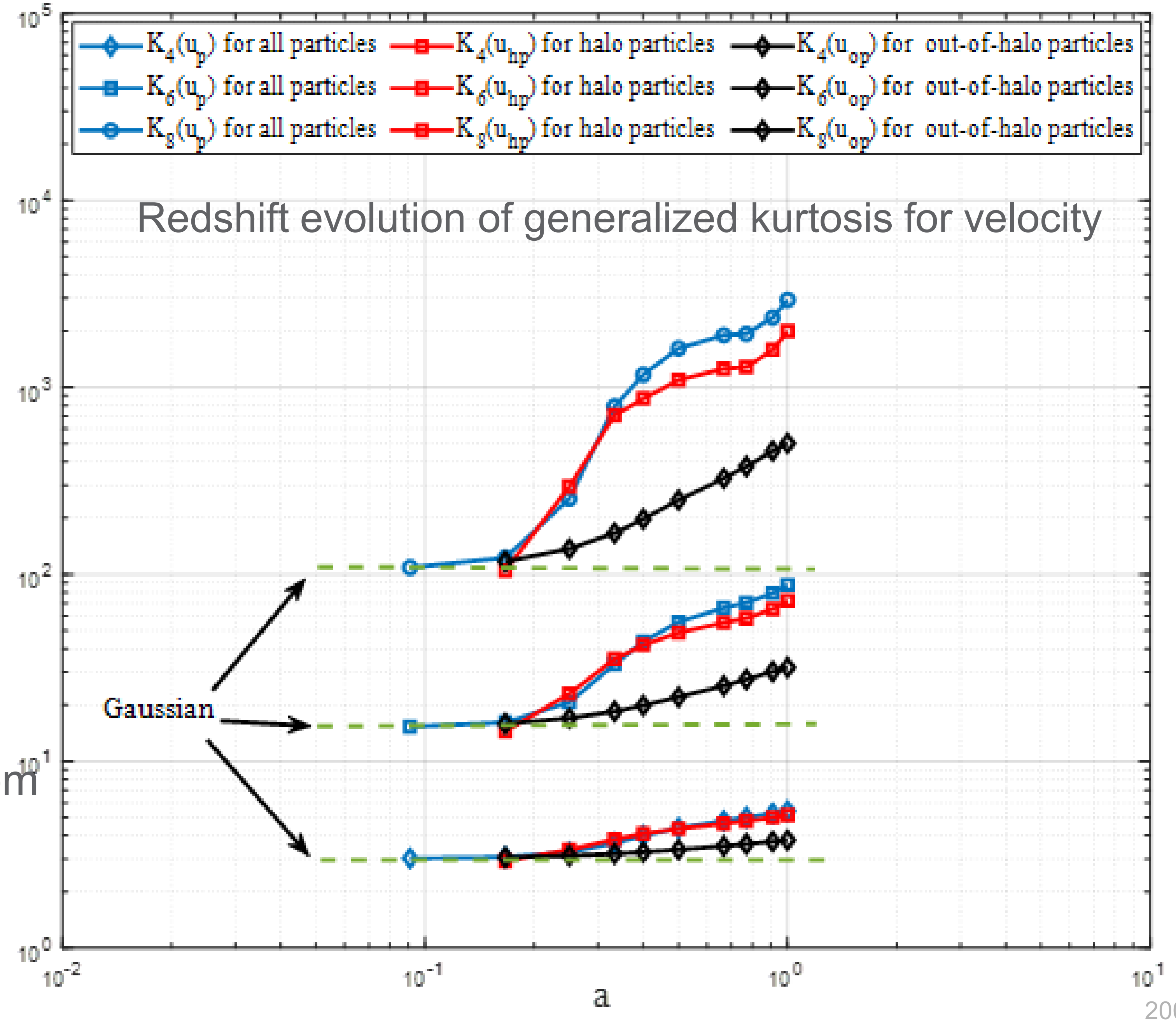
The central moment of order *n*:

$$S_n^{cp}(\Delta u_L, r) = \left\langle \left(\Delta u_L - \langle \Delta u_L \rangle\right)^n \right\rangle$$

The *n*th order longitudinal structure function:

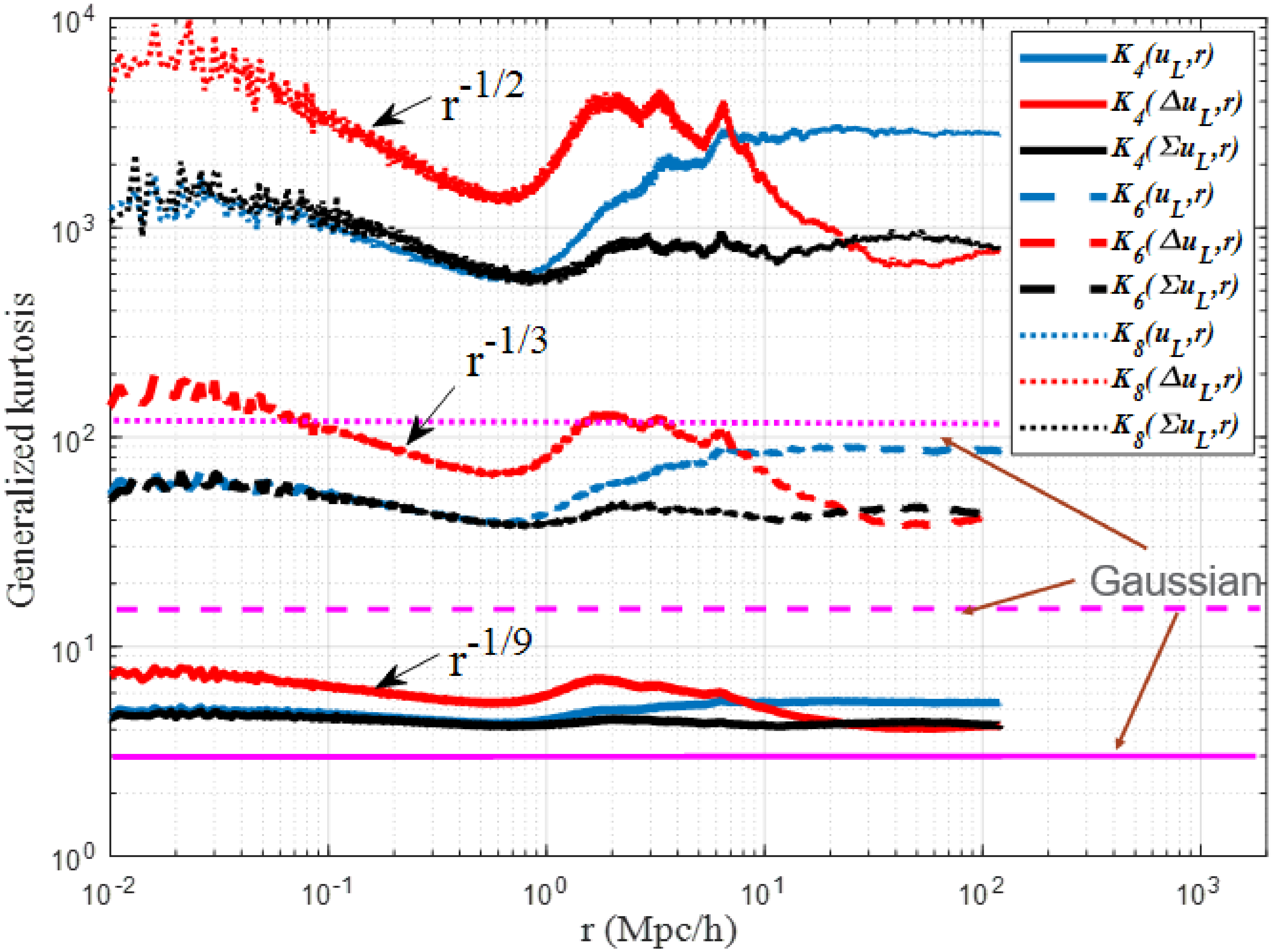
$$S_n^{lp}(r) = \left\langle (\Delta u_L)^n \right\rangle = \left\langle \left(u_L^{'} - u_L\right)^n \right\rangle$$

- All velocities are initially Gaussian.
- Velocity distribution of halo particles deviates from Gaussian much faster than out-of-halo particles due to stronger gravitational interaction in halos.
- All velocities become non-Gaussian with time to maximize system entropy



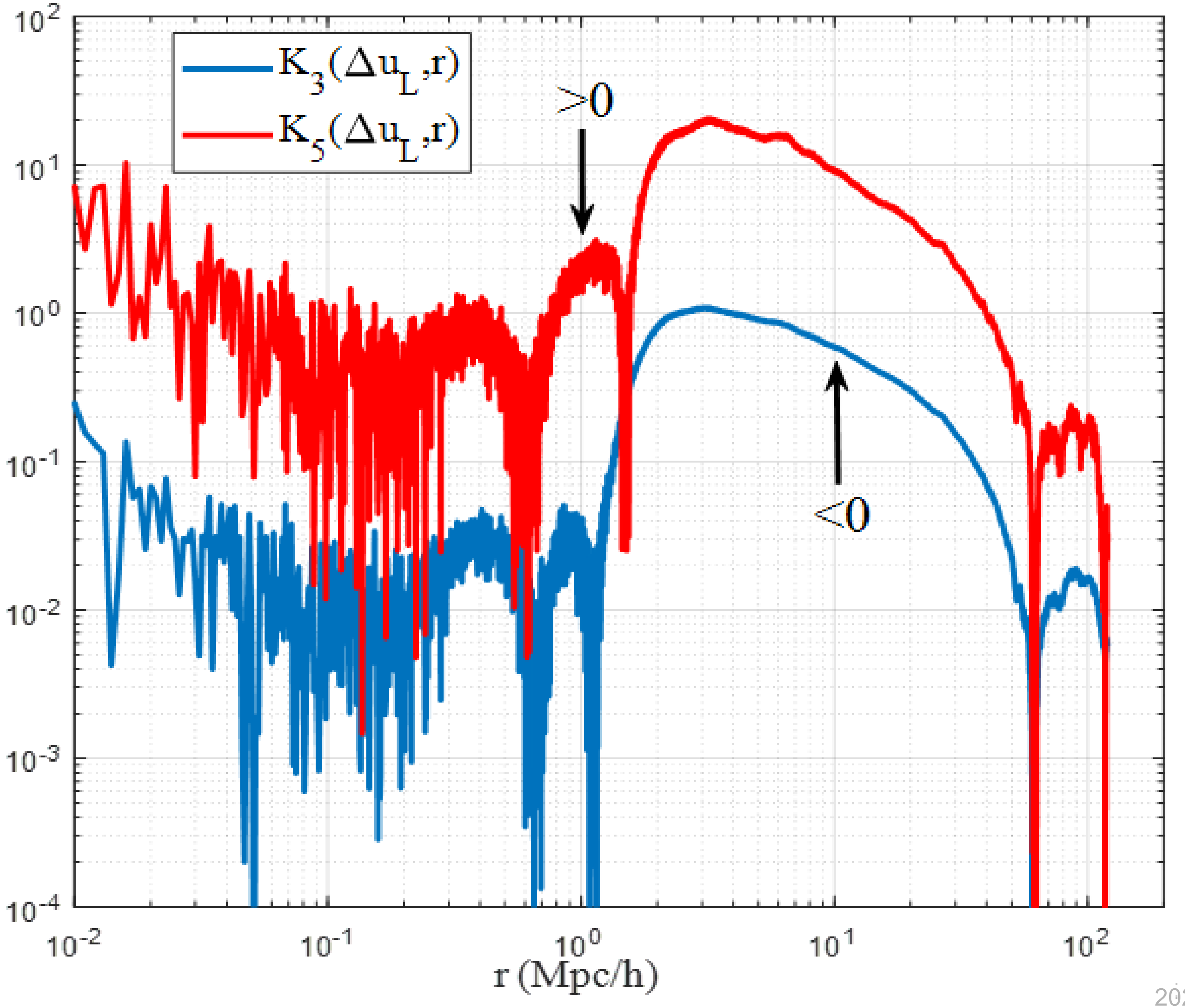
Scale-dependence of velocity distributions

- Even order generalized kurtosis (4th, 6th, and 8th order) at z=0.
- Velocity of fully developed dark matter flow is never Gaussian on any scale due to long-range gravity despite that they can be initially Gaussian.
- For incompressible flow with short range force, distribution is nearly Gaussian on large scale and non-Gaussian on small scale due to viscous force.
- On small scale, distribution of ΣuL approaches the distribution of uL with ρ_L=0.5.
- On large scale, distribution of ΣuL approaches the distribution of ΔuL with ρ_L=0.



Scale-dependence of velocity distributions

- On both small and large scales, generalized kurtosis approaches constant such that there exist unique (limiting) probability distributions that are independent of scale r.
- While on the intermediate scale around 1Mpc/h, all three velocity distributions exhibit the greatest value of generalized kurtosis of different order.
- Third order kurtosis (skewness) vanishes on both small and large scales, where distributions are symmetric.
- The negative skewness on the intermediate scale (distribution skews toward positive side) can be an important signature of inverse cascade of kinetic energy.



First moment of velocity fields and pair conservation equation

Pair conservation equation relates the pairwise velocity with density correlation

$$\frac{\langle \Delta u_L \rangle}{Har} = -\frac{\left(1+\bar{\xi}(r,a)\right)}{3\left(1+\xi(r,a)\right)} \frac{\partial \ln\left(1+\bar{\xi}(r,a)\right)}{\partial \ln a}$$

For large scale in linear regime, average correlation

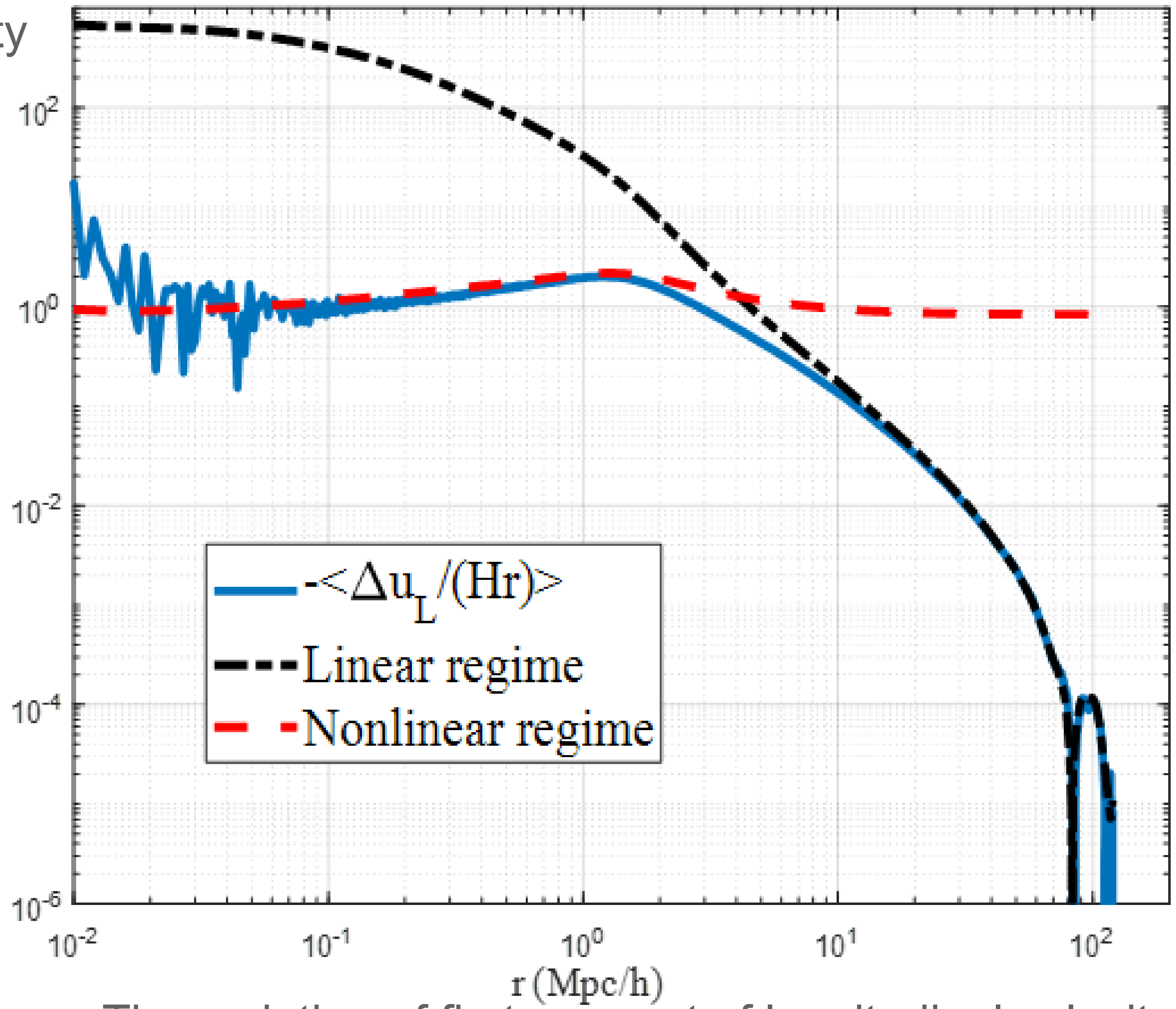
$\bar{\xi} \ll 1$ and $\partial \ln \bar{\xi} / \partial \ln a = 2$

$$\frac{\langle \Delta u_L \rangle}{Har} = -\frac{2\bar{\xi}(r,a)\left(1+\bar{\xi}(r,a)\right)}{3\left(1+\xi(r,a)\right)} \approx -\frac{2}{3}\bar{\xi}(r,a)$$

For small scale in non-linear regime,

$\xi(r,a) \propto a^{\alpha} r^{\gamma}$ and $\partial \ln \bar{\xi} / \partial \ln a = \alpha$

Stable clustering hypothesis $\frac{\langle \Delta u_L \rangle}{Har} = -1$ ➡ $\alpha = \gamma + 3$



The variation of first moment of longitudinal velocity (mean pairwise velocity) with scale r

First moment of velocity fields

On small scale:

$$\langle \Delta u_L \rangle = -Har \quad \langle u_L \rangle = Har/2 \qquad \langle \Sigma u_L \rangle = 0$$

A better relation to fit the simulation data:

$$\langle \Delta u_L \rangle = -Har - ua^{-5/3} \left(r/r_t \right)^{5/2}$$

On large scale:

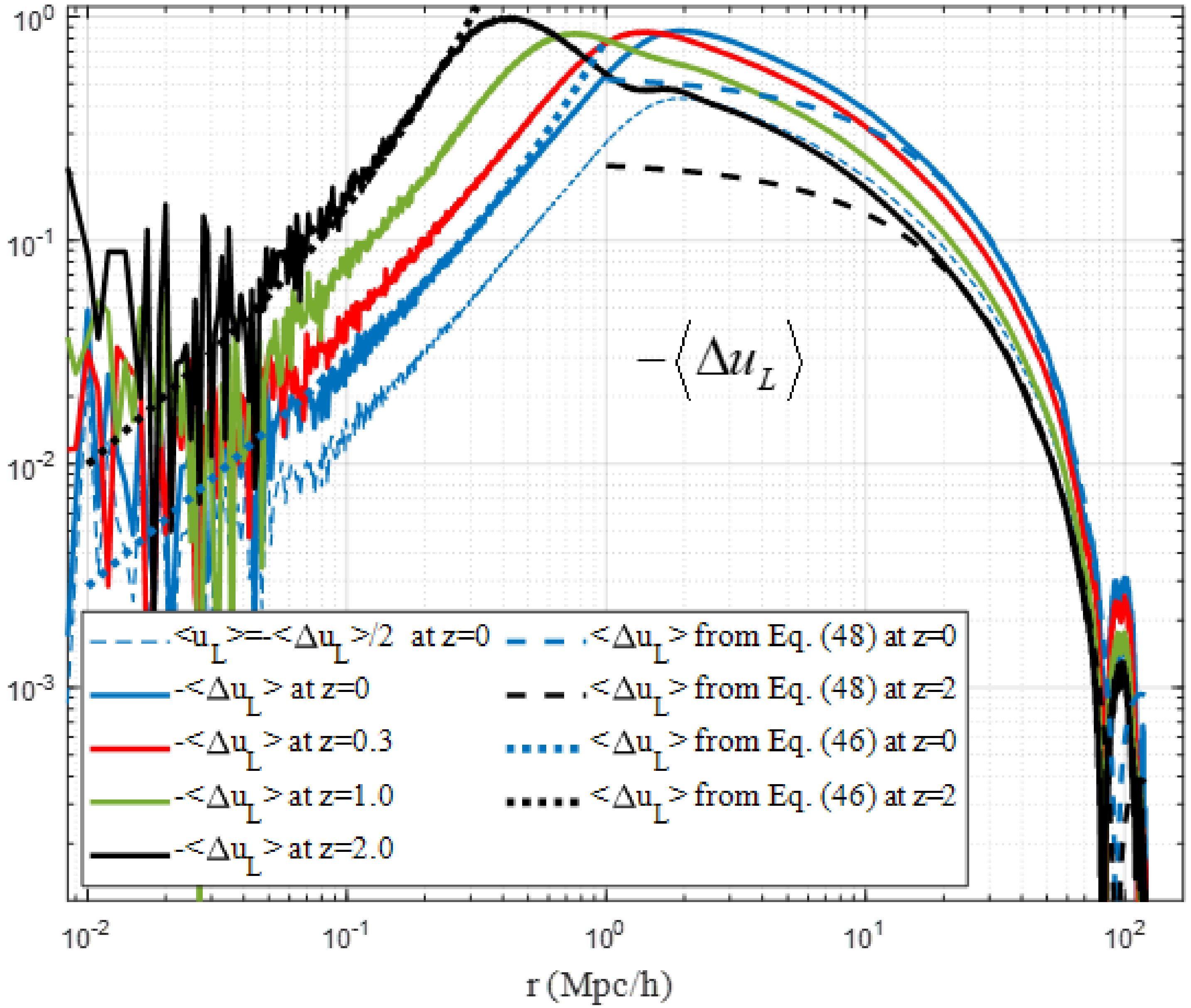
$$\langle \Delta u_L \rangle = -\frac{2Ha}{r^2} \int_0^r \xi(y) y^2 dy$$

From pair conservation equation

$$R_2 = \langle \mathbf{u} \cdot \mathbf{u}' \rangle$$

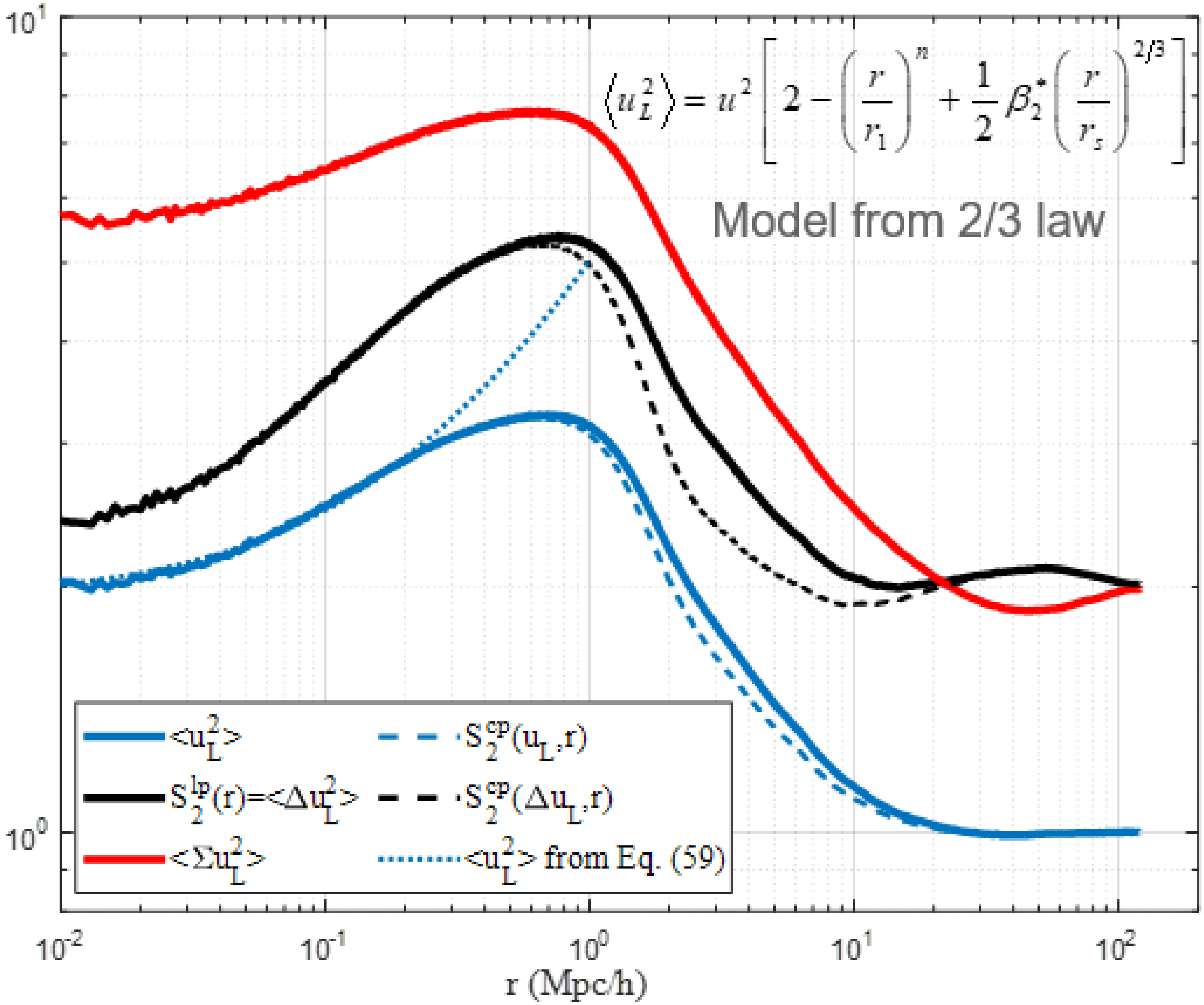
Total velocity correlation

$$\langle \Delta u_L \rangle = \frac{2}{aHf(\Omega_0)} \frac{\partial R_2}{\partial r} = \frac{2a_0 u^2}{aHr_2} \exp\left(-\frac{r}{r_2}\right)\left(\frac{r}{r_2} - 4\right)$$

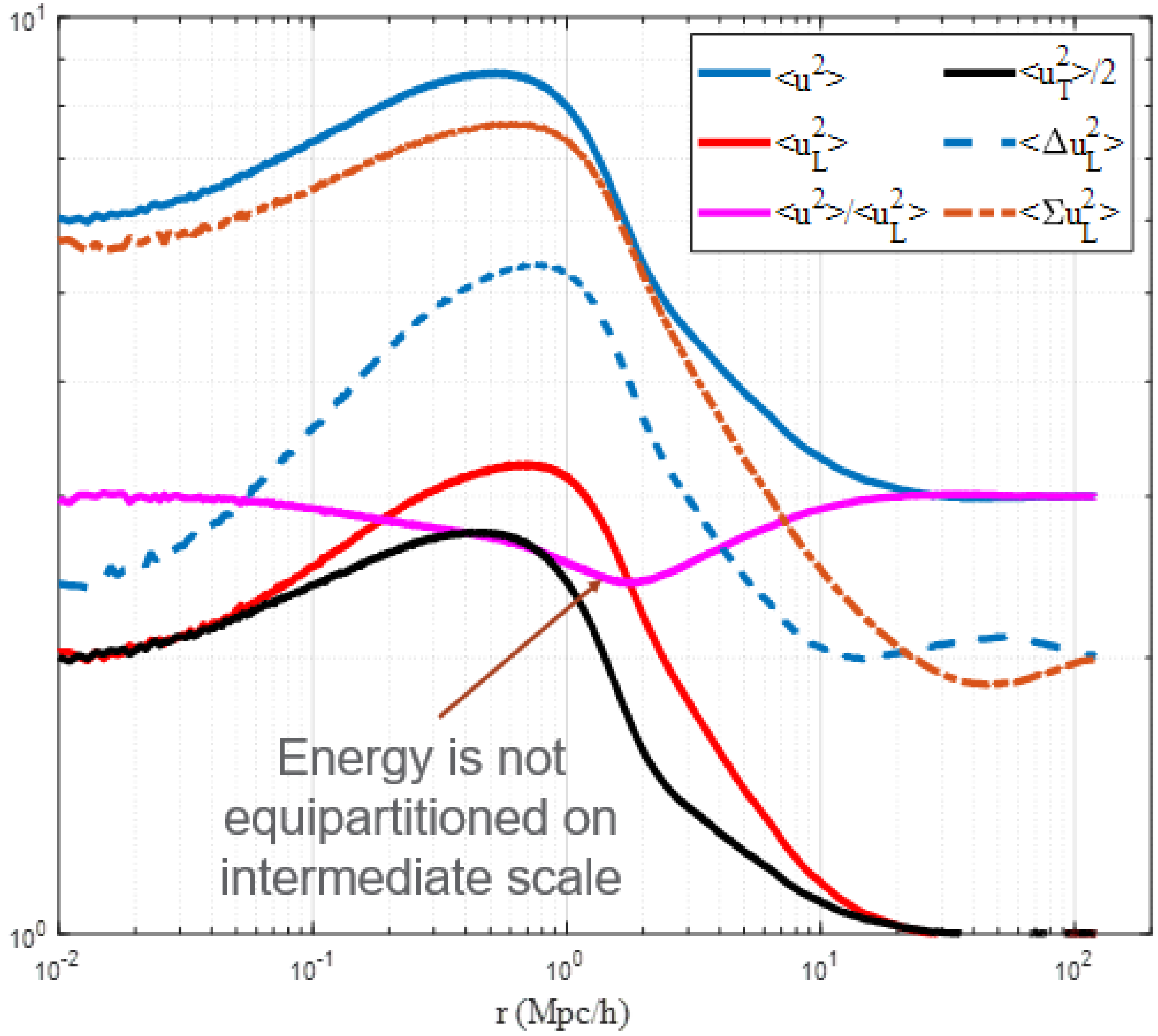


Mean velocity difference (pairwise velocity, normalized by u) varying with scale r at different redshift z

Second moment of velocity fields

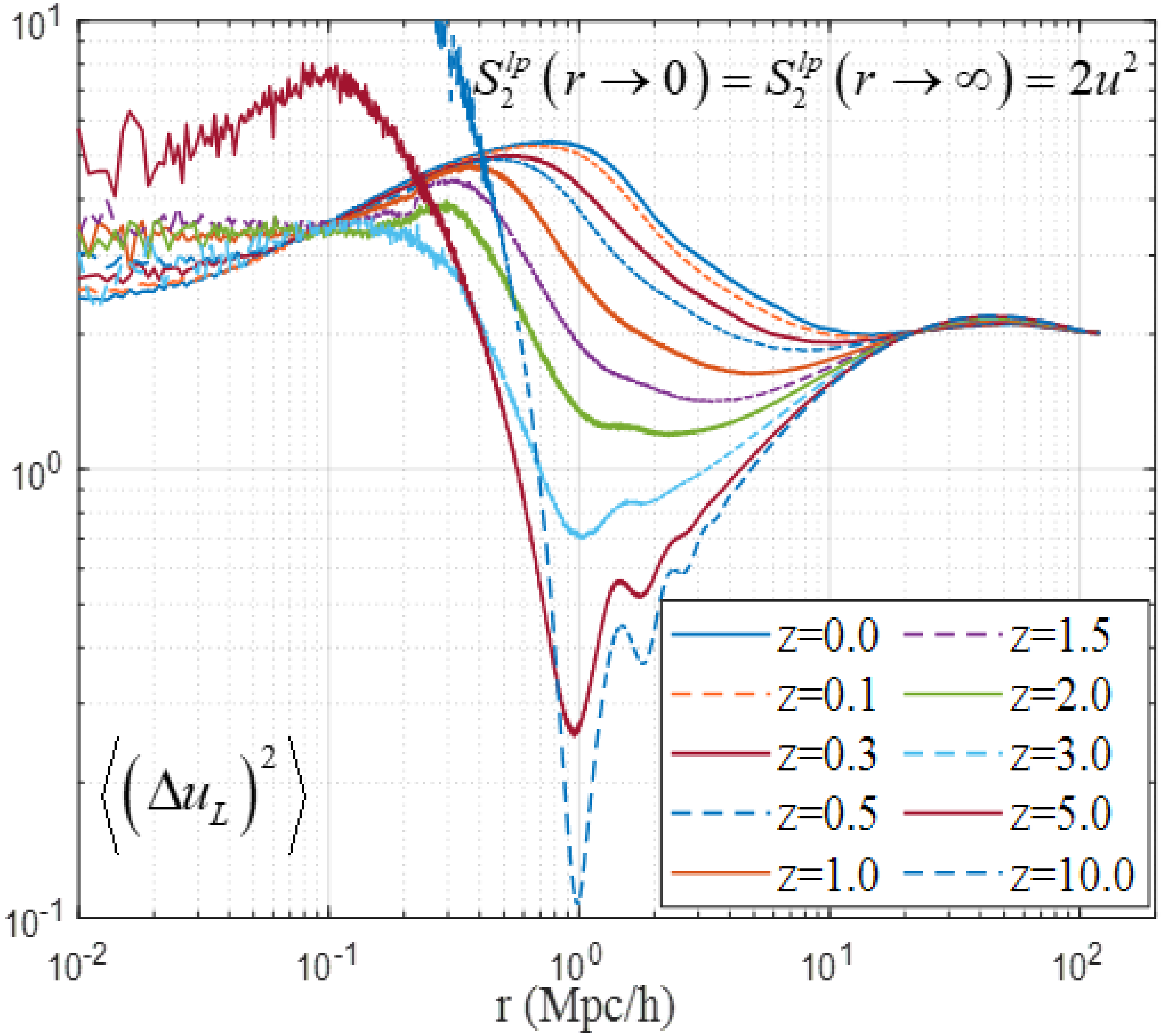


Increase of velocity dispersions with r for $r<r_t$ (pair of particles are more likely from same halos) is mostly due to the increase of velocity dispersion with halo size.

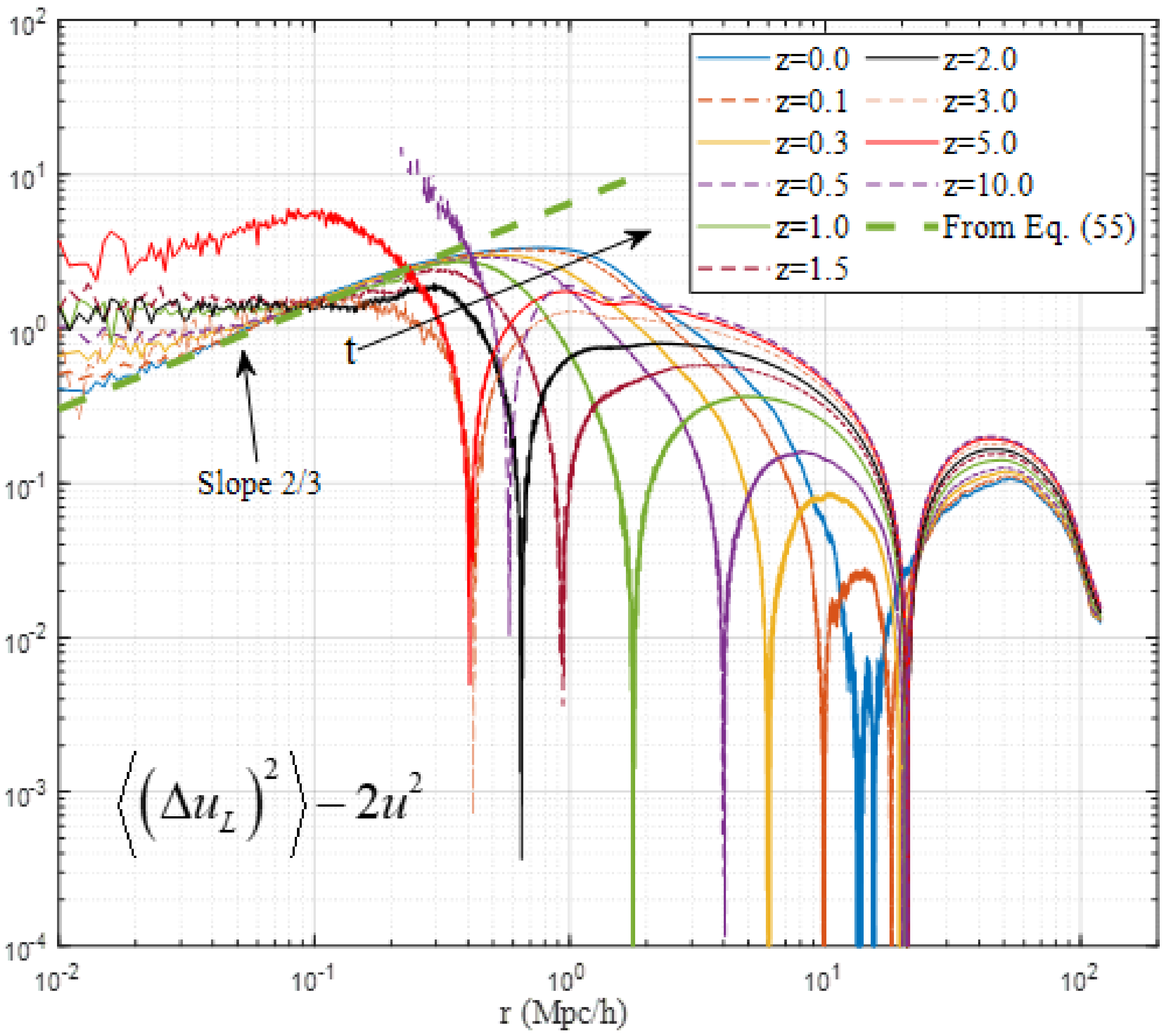


Second moment of velocity (normalized by u^2) varying with scale r at z=0

Second moment of pairwise velocity (pairwise dispersion) and the two-thirds law



Second order longitudinal structure function (pairwise velocity dispersion)



Reduced second order longitudinal structure function (pairwise velocity dispersion) and two-thirds law

Two-thirds law for higher even order structure functions and generalized stable clustering (GSCH)

Original scaling for incompressible flow does not apply for dark matter flow.

All even order reduced structure functions follow the same scaling of two-thirds law.

$$S_{2n}^{lp}(r)=u^{2n}\left[2^{n}K_{2n}(\Delta u_L,0)+\beta_{2n}^{*}(r/r_s)^{2/3}\right]$$

$$r_s=-\frac{u_0^3}{\varepsilon_u}=\frac{4}{9}\frac{u_0}{H_0}=\frac{2}{3}u_0t_0\approx 1.58\,Mpc/h$$

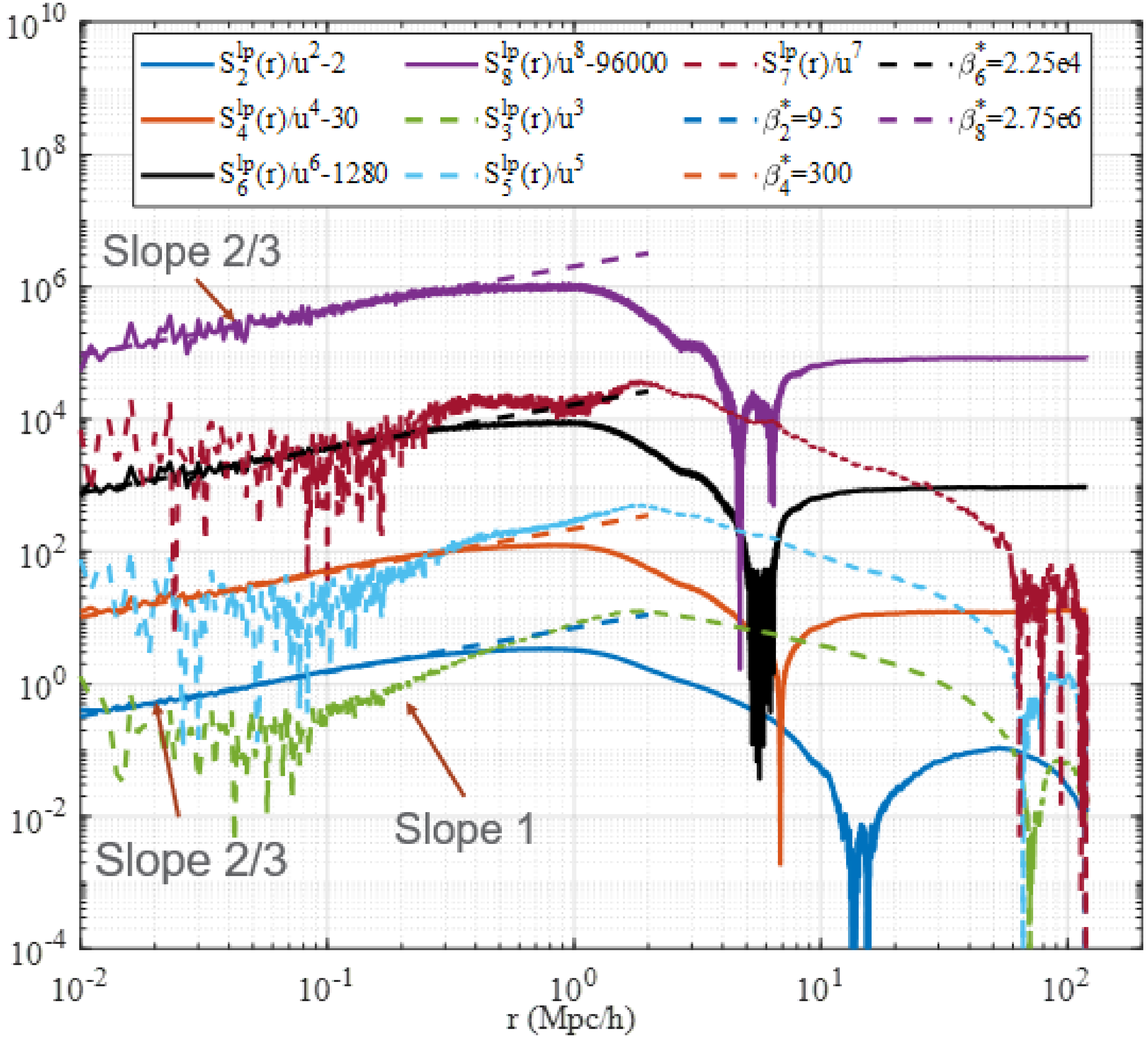
$$-\varepsilon_u=\frac{3}{2}\frac{u_0^2}{t_0}=\frac{9}{4}u_0^2H_0=4.6\times10^{-7}\,m^2/s^3$$

$$\beta_2^*=9.5 \qquad \beta_4^*=300 \qquad \beta_6^*=2.25\times10^4$$

$$\beta_8^*=2.75\times10^6 \qquad \beta_{2n}^*\approx10^{1.826n-1.003}$$

All odd order structure functions follow linear law from generalized stable clustering hypothesis

$$S_{2n+1}^{lp}(r)=(2n+1)S_1^{lp}(r)S_{2n}^{lp}(r)\propto r^1$$



Comparison of velocity fields between incompressible and dark matter flow

Quantity	Incompressible flow	SG-CFD
$\langle u_L \rangle = \langle \mathbf{u} \cdot \hat{\mathbf{r}} \rangle$	0 for all scale r	$\lim_{r\to 0,\infty} \langle u_L \rangle = 0$, varying with r
$\langle u_L^2 \rangle$	u_0^2 for all scale r	$\lim_{r\to 0} \langle u_L^2 \rangle = 2u_0^2$, $\lim_{r\to\infty} \langle u_L^2 \rangle = u_0^2$
$\langle u_L^3 \rangle$	0 for all scale r	$\lim_{r\to 0,\infty} \langle u_L^3 \rangle = 0$, varying with r
PDF of u_L	Gaussian	Non-gaussian on all scales
$\langle \Delta u_L \rangle$	0 for all scale r	$\lim_{r\to 0,\infty} \langle \Delta u_L \rangle = 0$, varying with r
$\langle \Delta u_L^2 \rangle$	$\lim_{r\to 0} \langle \Delta u_L^2 \rangle = 0, \lim_{r\to\infty} \langle \Delta u_L^2 \rangle = u_0^2$	$\lim_{r\to 0} \langle \Delta u_L^2 \rangle = 2u_0^2$, $\lim_{r\to\infty} \langle \Delta u_L^2 \rangle = 2u_0^2$
$K_3(\Delta u_L)$	$\lim_{r\to 0} K_3(\Delta u_L) = -0.4$, $\lim_{r\to\infty} K_3(\Delta u_L) = 0$	$\lim_{r\to 0,\infty} K_3(\Delta u_L) = 0$, varying with r
$K_4(\Delta u_L)$	$\lim_{r\to 0} K_4(\Delta u_L) \approx 4$ (depends on Re), $\lim_{r\to\infty} K_4(\Delta u_L) = 3$ (Gaussian)	$\lim_{r\to 0} K_4(\Delta u_L) = 7.5$, $\lim_{r\to\infty} K_4(\Delta u_L) = 4.2$
$\langle \Sigma u_L \rangle$	0 on all scales	0 on all scales
$\langle \Sigma u_L^2 \rangle$	$\lim_{r\to 0} \langle \Sigma u_L^2 \rangle = 4u_0^2, \lim_{r\to\infty} \langle \Sigma u_L^2 \rangle = 2u_0^2$	$\lim_{r\to 0} \langle \Delta u_L^2 \rangle = 6u_0^2$, $\lim_{r\to\infty} \langle \Delta u_L^2 \rangle = 2u_0^2$

Modeling velocity distributions on small scale

- On small scale, velocities u_L and Σu_L should have the same limiting distribution.

- On small scale both should follow a X distribution to maximize system entropy.

Maximum entropy distribution:

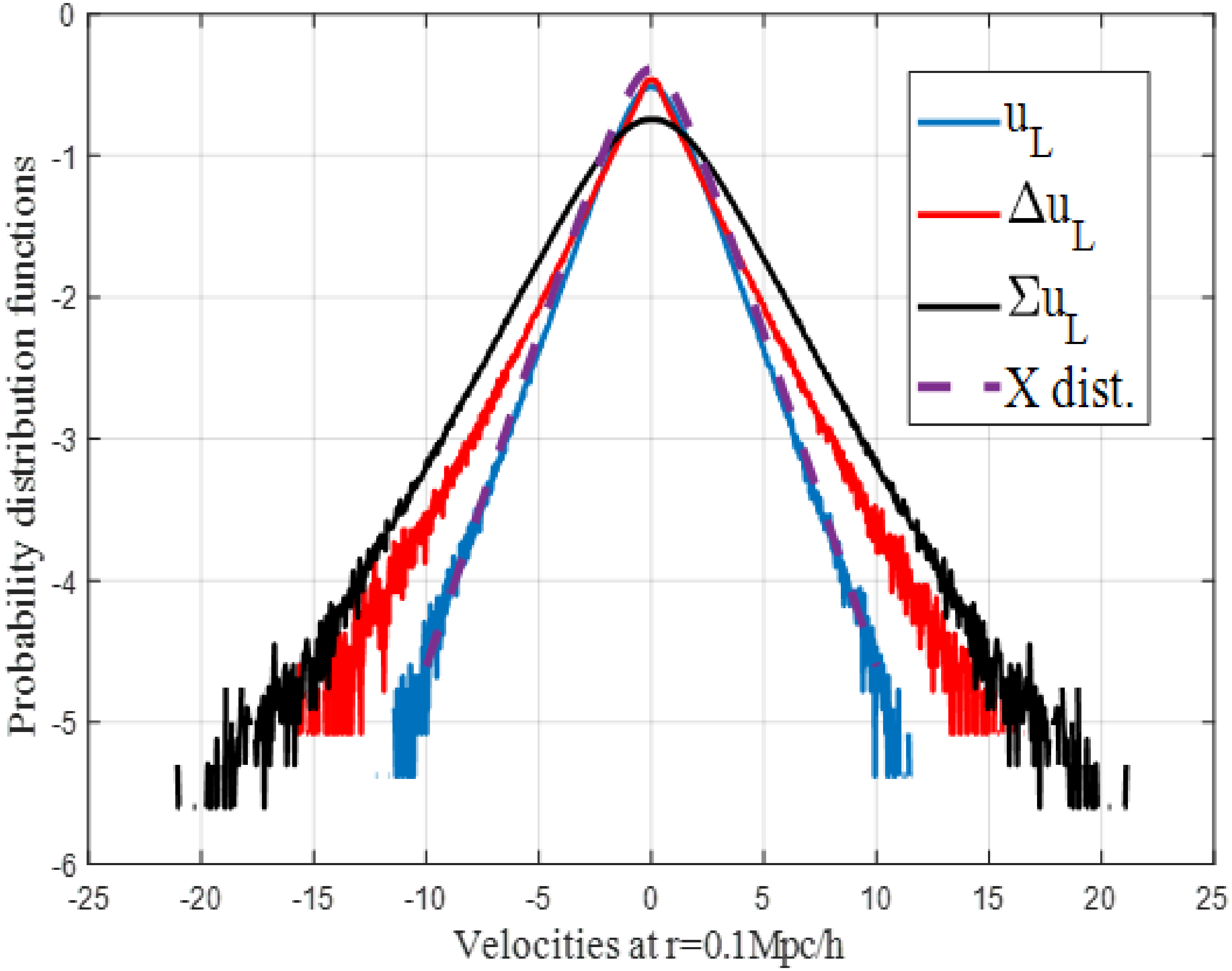
$$X(v) = \frac{1}{2\alpha v_0} \frac{e^{-\sqrt{\alpha^2 + (v/v_0)^2}}}{K_1(\alpha)}$$

Shape parameter: α;
Velocity scale: v_0;

The mth order generalized kurtosis of X distribution:

$$K_m(X) = \left(\frac{2K_1(\alpha)}{K_2(\alpha)}\right)^{m/2} \frac{\Gamma((1+m)/2)}{\sqrt{\pi}} \cdot \frac{K_{(1+m/2)}(\alpha)}{K_1(\alpha)}$$

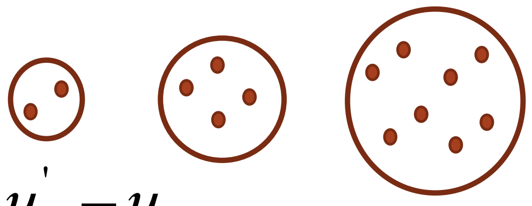
- The shape of velocity distribution changes with redshift z such that α is redshift-dependent.
- Kurtosis K_m is only dependent on α and also redshift-dependent



Distributions of velocities on scale of r=0.1Mpc/h at z=0

Distribution of pairwise velocity on small scale

- On small scale, velocities u_L and Σu_L follows X distribution.
- Distribution of pairwise velocity Δu_L is different with moment estimated.
- Pairs of particles with same r can be from halos of different size.



$$\Delta u_L = u_L^{'} - u_L$$

Key: correlation between two longitudinal velocities decreases with halo size:

$$\rho_{cor}\left(m_h\right) = \sigma_h^2 / \sigma^2$$

Double-λ halo mass function:

$$f\left(\nu\right) = f_{D\lambda}\left(\nu\right) = \frac{\left(2\sqrt{\eta_0}\right)^{-q}}{\Gamma\left(q/2\right)} \nu^{q/2-1} \exp\left(-\frac{\nu}{4\eta_0}\right)$$

$$P_{\Delta uL}\left(x\right) = \int_0^\infty \frac{1}{\sqrt{2\pi}\sqrt{2\left(1-\rho_{cor}\right)}\sigma} e^{-x^2/\left[4\left(1-\rho_{cor}\right)\sigma^2\right]} \beta_p f\left(\nu\right) \nu^p d\nu$$

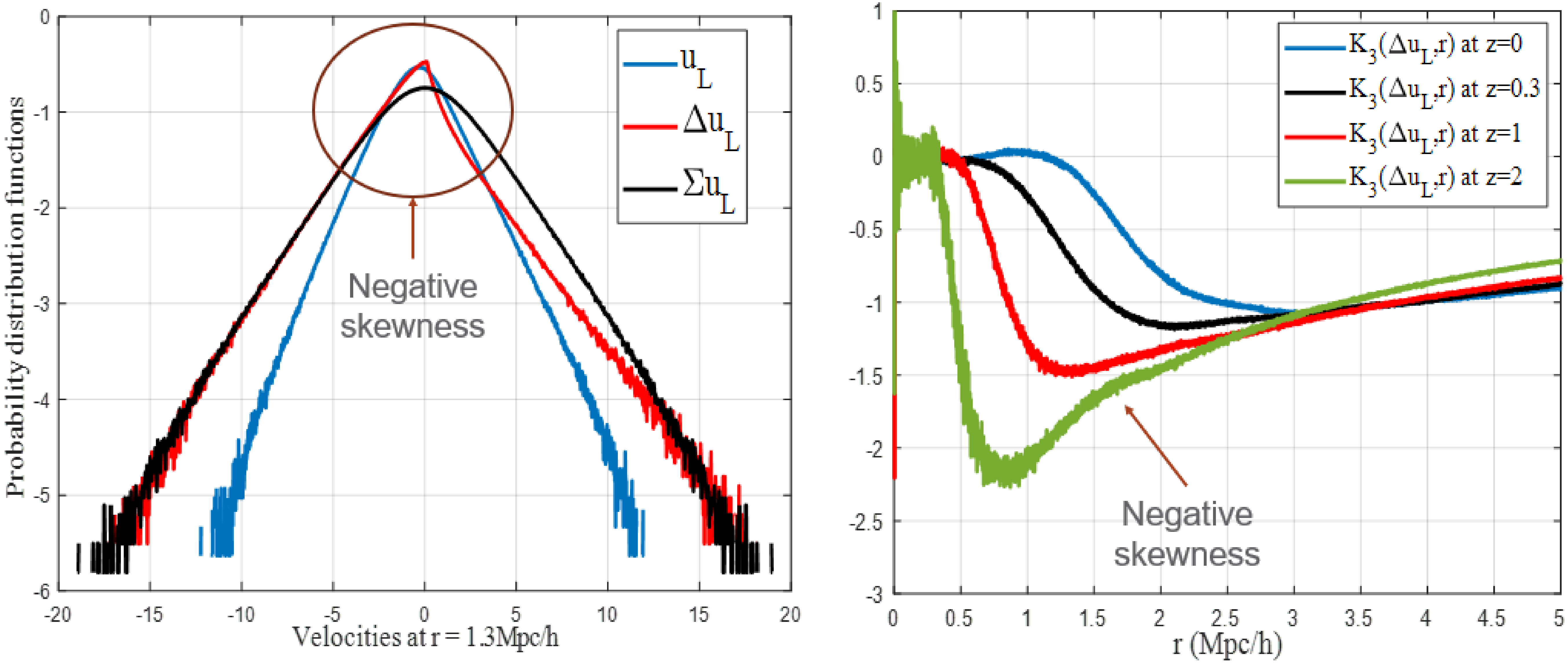
The limiting distributions of velocity fields on small and large scales

	Velocity fields	Distribution	4th Kurtosis	6th Kurtosis	8th Kurtosis
$r \to 0$	u_L, Σu_L	N-body, z=0, Fig. 14	**4.8**	**57**	**1200**
$r \to 0$	Δu_L	N-body, z=0, Fig. 14	7.5	160	6000
$r \to 0$	u_L, Σu_L	$X\left(x\right)$	4.6	48.9	944.8
$r \to 0$	Δu_L	Eq. (80)	7.7	159.24	6356
$r \to \infty$	Δu_L, Σu_L	N-body, z=0, Fig. 14	**4.181**	**41.46**	**670.8**
$r \to \infty$	u_L	N-body, z=0 Fig. 14	5.39	85.78	2800
$r \to \infty$	Δu_L, Σu_L	Logistic (Eq. (82))	4.2	279/7	685.8
$r \to \infty$	u_L	$P_{uL}\left(x\right)$(Eq. (85))	5.4	78.4	2269.8
	Exponential??	Laplace distribution	6	90	2520
		Gaussian distribution	**3**	**15**	**105**

Generalized kurtosis:

$$K_{2n}\left(\Delta u_L\right) = \frac{(2n)!}{n!2^n} \frac{\Gamma\left(n+p+q/2\right)\left[\Gamma\left(p+q/2\right)\right]^{n-1}}{\left[\Gamma\left(1+p+q/2\right)\right]^n}$$

Velocity distributions on intermediate scale



Distribution of Σu_L is symmetric, while the distribution of Δu_L is non-symmetric with non-zero (negative) skewness and skew toward positive side. This is a necessary feature of inverse energy cascade.

Modeling velocity distributions on large scale

- Distribution of Δu_L on large scale is usually assumed to be exponential in literature (non-smooth).
- This seems not agree with N-body simulation
- On large scale, Both Σu_L and Δu_L can be modelled by a logistic distribution.

Logistic distribution for both velocities:

$$P_{\Delta u_L}(x) = \frac{1}{4s}\operatorname{sech}^2\left(\frac{x}{2s}\right)$$

Reduce to exponential at large velocity:

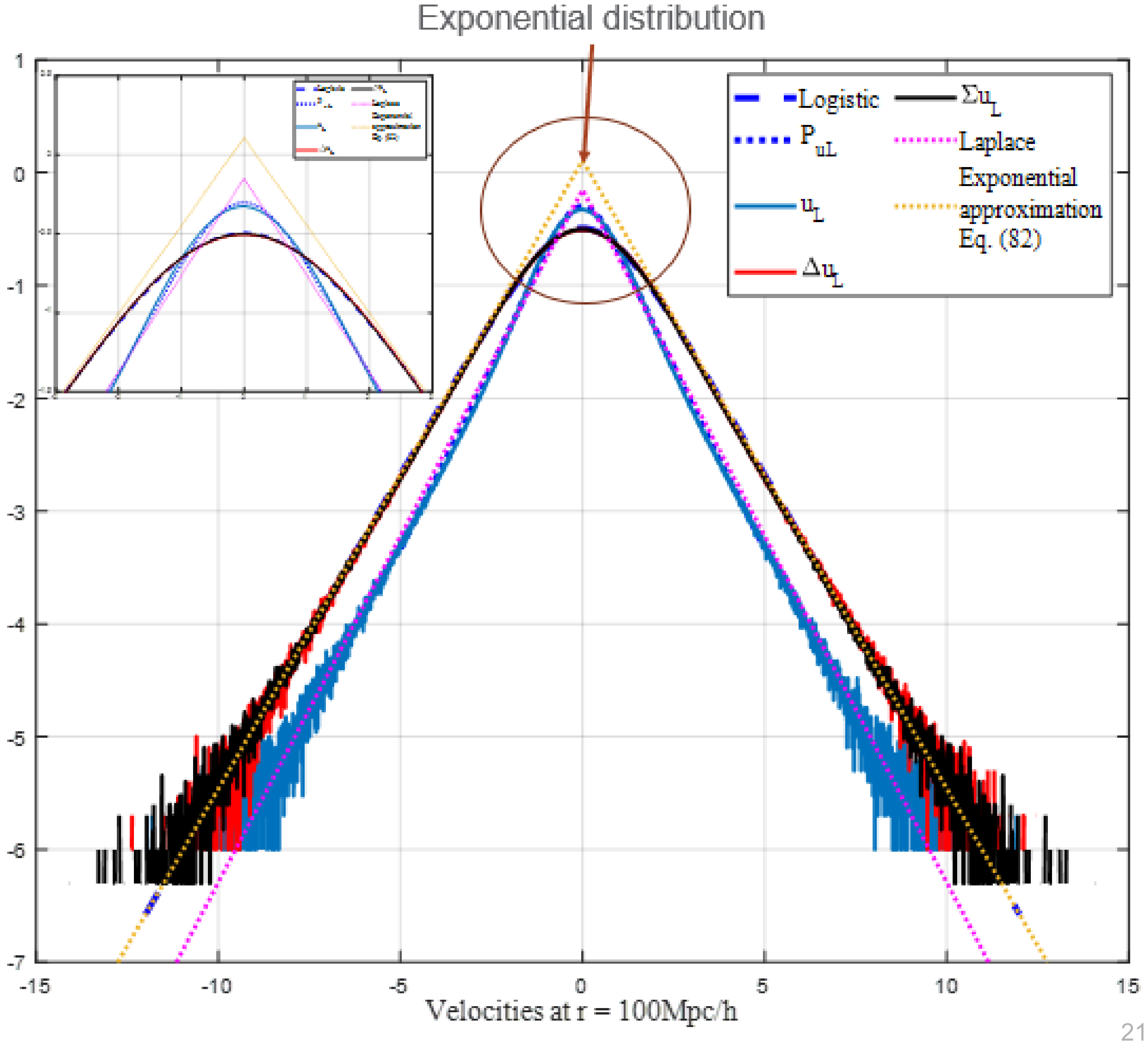
$$P_{\Delta u_L}(x \to \infty) \approx \frac{1}{s}\exp\left(-\frac{x}{s}\right)$$

Longitudinal velocity u_L should satisfy for ρ_L=0:

$$P_{\Delta u_L}(z) = \int_{-\infty}^{\infty} P_{u_L}(x) P_{u_L}(z-x)\,dx$$

$$MGF_{P_{u_L}}(t) = \sqrt{\frac{\pi st}{\sin(\pi st)}}$$

Moment generating function for u_L



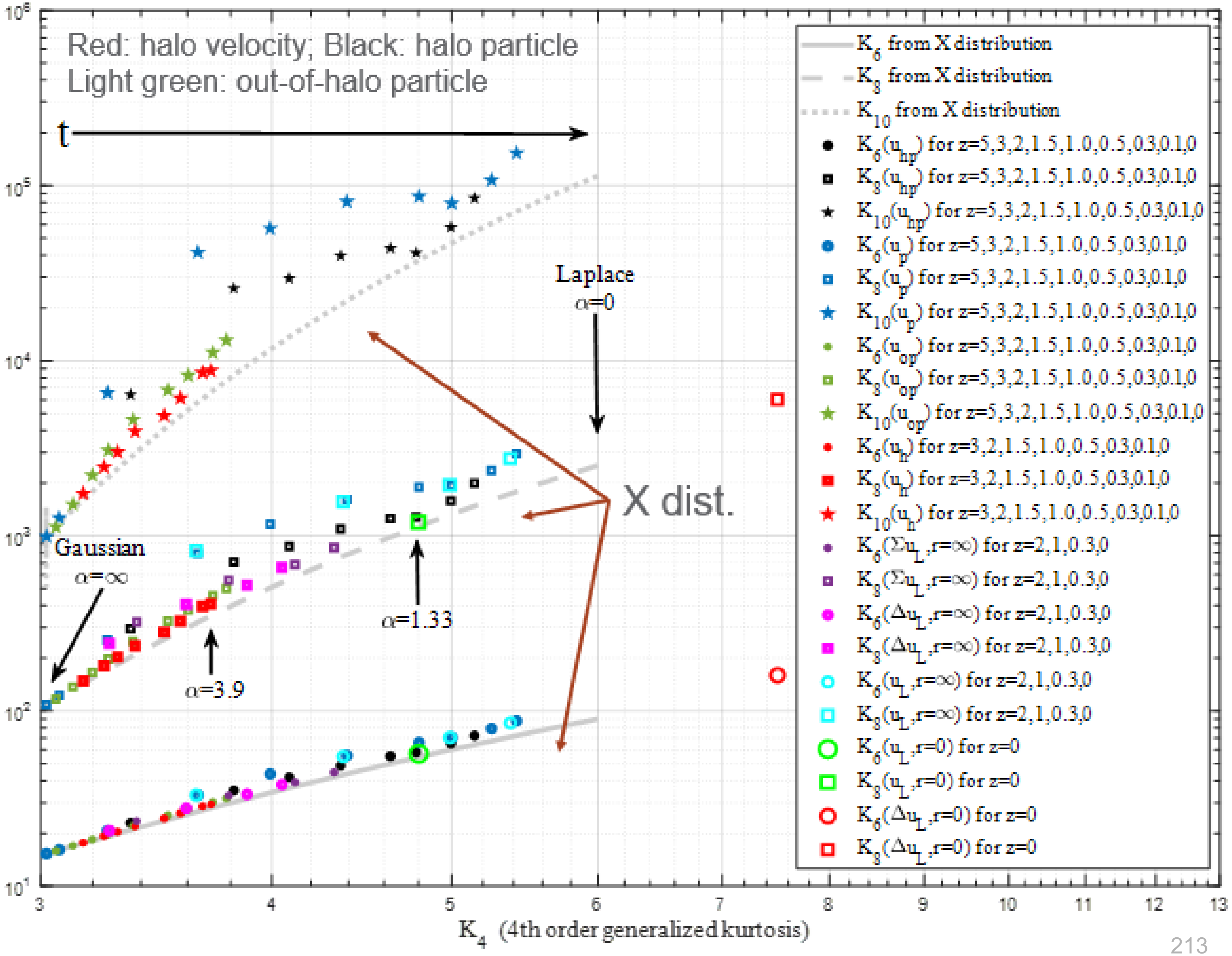
The redshift evolution of velocity distributions

- Distribution of different types of velocities changes due to redshift evolution of α.
- Shape parameter α decreases with time.
- Most velocities follows the X distribution to maximize system entropy
- Halo velocity and out-of-halo particle velocity evolves much slower than halo particle velocity due to weaker gravity on large scale.

Generalized kurtosis of X distribution:

$$K_m(X) = \left(\frac{2K_1(\alpha)}{K_2(\alpha)}\right)^{m/2} \frac{\Gamma((1+m)/2)}{\sqrt{\pi}} \cdot \frac{K_{(1+m/2)}(\alpha)}{K_1(\alpha)}$$

Plot K4 vs. K6, K4 vs. K8, and K4 vs. K10;



Summary and keywords

Delaunay tessellation	Pairwise velocity	Skewness
Generalized kurtosis	Velocity sum	Generalized stable clustering
Two-thirds law	X distribution	Pair conservation equation

- A halo-based non-projection approach is proposed to study the scale and redshift dependence of density and velocity distributions in dark matter flow.
- A two-thirds law for pairwise velocity was established, i.e. S_2^{lp}-$2u^2 \sim \varepsilon_u r^{2/3}$, where r is the separation between pair of particles and ε_u is the constant rate of energy cascade.
- Two-thirds law can be generalized to all even moments of pairwise velocity, while odd moments ~r
- The distributions of longitudinal velocity u_L, pairwise velocity Δu_L, and velocity sum Σu_L, are analytically modeled on both small and large scales
- Fully developed velocity fields are never Gaussian on any scale despite that they can be initially Gaussian.
- Delaunay tessellation is used to reconstruct the density field from N-body simulation, which results in an asymmetric density distribution with a long tail.
- Density correlation is obtained by directly counting all pairs on a given scale r along with simple analytical models for all second order density statistics.

The statistical theory of dark matter flow (high order)

Xu Z., 2022, arXiv:2202.02991 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2202.02991

Introduction

Review:
Statistical theory in hydrodynamic turbulence

- Kinematic relations between statistical measures (2^{nd} and 3^{rd} order)
- Dynamic relations between statistical measures of different order (from NS equations of velocity)
- Reynolds decomposition
- Closure problem, eddy viscosity, etc…

Current statistical theory of dark matter flow is not satisfactory:

- Dark matter flow is intrinsically complex with different nature of flow on different scales, i.e. a constant divergence flow on small scale and an irrotational flow on large scale.
- The kinematic and dynamic relations need to be developed separately for both types of flow on different scales.
- Dynamic equations of velocity (Jeans' equation) are not self-closed. No dynamic relations can be derived without a self-closed dynamics for velocity evolution.

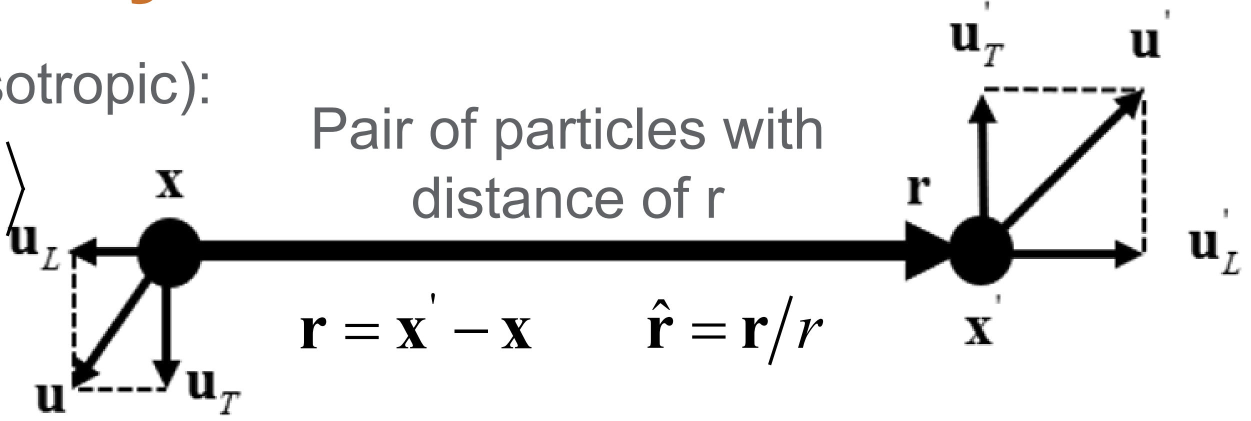
- Existing work mostly focus on the 1st and 2nd order velocity statistics, while the peculiar velocity field contains much richer information beyond the second order.
- Finally, very challenging to explore high order statistics, as that inherently involves tensor and vector calculus of great complexity.

- ❖ Most kinematic relations between statistical measures (2^{nd})
  Need to extend to high and arbitrary order
- ❖ Develop self-consistent dynamic equation for velocity field
- ❖ Develop dynamic relations between statistical measures of different order
- ❖ Derive the "eddy" (artificial) viscosity from velocity fluctuation

Two-point third order velocity correlation tensors

Third order velocity correlation tensor (homogeneous and isotropic):

$$Q_{ijk}\left(\mathbf{x},\mathbf{r}\right)=Q_{ijk}\left(\mathbf{r}\right)=Q_{ijk}\left(r\right)=\left\langle u_i\left(\mathbf{x}\right)u_j\left(\mathbf{x}\right)u_k\left(\mathbf{x}'\right)\right\rangle=\left\langle u_i u_j u_k'\right\rangle$$



General form of isotropic third order tensor:

$$Q_{ijk}\left(r\right)=A_3\left(r\right)r_i r_j r_k+B_3\left(r\right)\left(r_i\delta_{jk}+r_j\delta_{ki}\right)+D_3\left(r\right)r_k\delta_{ij}$$

Divergence of second order tensor:

$$Q_{ijk,k}=\frac{\partial\left\langle u_i u_j u_k'\right\rangle}{\partial r_k}=\left(5A_3+\frac{\partial A_3}{\partial r}r+\frac{2}{r}\frac{\partial B_3}{\partial r}\right)r_i r_j+\left(2B_3+\frac{\partial D_3}{\partial r}r+3D_3\right)\delta_{ij}$$

$$Q_{ijk,k}=\left\langle\left(u_i\left(\mathbf{x}\right)u_j\left(\mathbf{x}\right)\right)\left(\nabla'\cdot u_j\left(\mathbf{x}'\right)\right)\right\rangle=0$$ ⬅ Incompressible flow

$$Q_{ijk,k}=\left\langle\left(u_i\left(\mathbf{x}\right)u_j\left(\mathbf{x}\right)\right)\left(\nabla'\cdot u_j\left(\mathbf{x}'\right)\right)\right\rangle=\theta\left\langle\left(u_i\left(\mathbf{x}\right)u_j\left(\mathbf{x}\right)\right)\right\rangle\neq 0$$ ⬅ Constant divergence flow

Use this to derive Kinematic relations

Longitudinal velocity:

$$u_L=\mathbf{u}\cdot\hat{\mathbf{r}}=u_i\hat{r}_i$$

$$u_L'=\mathbf{u}'\cdot\hat{\mathbf{r}}=u_i'\hat{r}_i$$

Transverse velocity:

$$\mathbf{u}_T=-\left(\mathbf{u}\times\hat{\mathbf{r}}\times\hat{\mathbf{r}}\right)$$

$$\mathbf{u}_T'=-\left(\mathbf{u}'\times\hat{\mathbf{r}}\times\hat{\mathbf{r}}\right)$$

Velocity difference or Pairwise velocity: $\Delta u_L=u_L'-u_L$

Velocity sum: $\sum u_L=u_L'+u_L$

Curl of second order tensor:

$$\nabla\times Q_{mni}\left(r\right)=\varepsilon_{ijk}Q_{mnk,j}=\left(A_3-\frac{1}{r}\frac{\partial B_3}{\partial r}\right)\left(\varepsilon_{imk}r_n r_k+\varepsilon_{ink}r_m r_k\right)=0$$ ⬅ Irrotational flow

Different odd order kinematic relations for incompressible flow and constant divergence flow

Two-point third order velocity correlation functions

Using index contraction of third order tensor to define four scalar correlation functions

Two total correlation functions:

$$R_3(r) = \frac{1}{2} Q_{ijk}\left(\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i\right) = \left\langle u_L \mathbf{u} \cdot \mathbf{u}' \right\rangle = A_3 r^3 + \left(4B_3 + D_3\right) r$$

$$R_{31}(r) = Q_{ijk}\delta_{ij}\hat{r}_k = \left\langle \mathbf{u} \cdot \mathbf{u} u_L' \right\rangle = A_3 r^3 + \left(2B_3 + 3D_3\right) r$$

Longitudinal triple correlation function:

$$L_3(r) = Q_{ijk}\hat{r}_i\hat{r}_j\hat{r}_k = \left\langle u_L^2 u_L' \right\rangle = A_3 r^3 + \left(2B_3 + D_3\right) r$$

Transverse third-order correlation function:

$$T_3(r) = \left\langle u_L \mathbf{u}_T \cdot \mathbf{u}_T' \right\rangle / 2 = \left(R_3 - L_3\right)/2 = B_3 r$$

$$R_3(r) = L_3(r) + 2T_3(r)$$

Relation to third correlation tensor:

$$Q_{iki,k} = Q_{ijk,i}\delta_{jk} = Q_{ikk,i} = \frac{1}{r^2}\left(r^2 R_3\right)_{,r}$$

$$Q_{iik,k} = \frac{1}{r^2}\left(r^2 R_{31}\right)_{,r}$$

Correlation functions of any order (pth order):

$$L_{(p,q)} = \left\langle u^q u_L^{p-q-1} u_L' \right\rangle$$

$$R_{(p,q+1)} = \left\langle u^q u_L^{p-q-2} u_i u_i' \right\rangle = \left\langle u^q u_L^{p-q-2} \mathbf{u} \cdot \mathbf{u}' \right\rangle$$

$$R_{(p,q+1)} = L_{(p,q)} + 2T_{(p,q)}$$

Goal is to identify kinematics relations between correlations functions of same order

Kinematic relations for third order correlation functions

For incompressible flow: $\nabla \cdot \mathbf{u} = 0$

$$R_3 = \frac{1}{2r^3}\left(r^4 L_3\right)_{,r} \qquad T_3 = \frac{1}{4r}\left(r^2 L_3\right)_{,r} \qquad r^2\left(r^2 R_3\right)_{,r} = 2\left(r^4 T_3\right)_{,r} \qquad R_{31}(r) = \left\langle \mathbf{u}\cdot\mathbf{u}u_L^{'}\right\rangle = 0$$

Relations between correlation functions

$$Q_{ijk}(r) = \frac{L_3 - rL_3^{'}}{2}\hat{r}_i\hat{r}_j\hat{r}_k + \frac{2L_3 + rL_3^{'}}{4}\left(\hat{r}_i\delta_{jk} + \hat{r}_j\delta_{ki}\right) - \frac{L_3}{2}\hat{r}_k\delta_{ij}$$

Correlation tensor in terms of correlations

For constant divergence flow: $\nabla \cdot \mathbf{u} = \theta$ Reduced to incompressible flow with Θ=0

$$R_3 + \frac{1}{2}\left\langle u_L^2\right\rangle \theta r = \frac{1}{2r^3}\left(r^4 L_3\right)_{,r}$$

$$\left\langle u^2\right\rangle \approx 3\left\langle u_L^2\right\rangle$$

$$\left\langle u^2\right\rangle \theta = \frac{1}{r^2}\left(r^2 R_{31}\right)_{,r}$$

$$R_3 + \frac{1}{6r}\left(r^2 R_{31}\right)_{,r} = \frac{1}{2r^3}\left(r^4 L_3\right)_{,r}$$

For irrotational flow: $\nabla \times \mathbf{u} = 0$

$$\left(rR_3\right)_{,r} + R_{31} = \frac{1}{r^3}\left(r^4 L_3\right)_{,r} \qquad 3L_3 - R_{31} = 2\left(rT_3\right)_{,r} \qquad 3R_3 - R_{31} = \frac{2}{r^3}\left(r^4 T_3\right)_{,r}$$

Scaling laws for two-point third order velocity structure function (review)

Structure functions as moments of pairwise velocity:

$$S_3^{lp}(r) = \left\langle (\Delta u_L)^3 \right\rangle = \left\langle (u_L' - u_L)^3 \right\rangle = 6L_3(r) - 2\left\langle u_L^3 \right\rangle \qquad S_m^{lp} = \left\langle (\Delta u_L)^m \right\rangle = \left\langle (u_L' - u_L)^m \right\rangle$$

Two-thirds law for even order (reduced) structure function:

$$S_{2n}^{lp}(r) - S_{2n}^{lp}(0) \propto (-\varepsilon_u)^{2/3} r^{2/3}$$

ε_u : rate of energy cascade.

Generalized stable clustering hypothesis (GSCH)

$$S_{2n+1}^{lp}(r) = (2n+1) S_1^{lp}(r) S_{2n}^{lp}(r)$$

$$S_{2n+1}^{lp}(r) = -(2n+1) HarS_{2n}^{lp}(0) = -2^n (2n+1) K_{2n}(\Delta u_L, 0) Haru^{2n} \propto r$$

$K_{2n}(\Delta u_L, 0)$: Generalized kurtosis of the distribution of pairwise velocity

Velocity correlation functions of any (pth) order $L_{(p,q)}$ and $R_{(p,q)}$

Table 2. The velocity correlation functions of different order

p	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
1	$L_{(1,0)}=\langle u_L^{'}\rangle$ or $\langle \Delta u_L\rangle/2$					
2	$L_{(2,0)}=\langle u_L u_L^{'}\rangle$	$R_{(2,1)}=\langle \mathbf{u}\cdot\mathbf{u}^{'}\rangle$				
3	$L_{(3,0)}=\langle u_L^2 u_L^{'}\rangle$	$R_{(3,1)}=\langle u_L\mathbf{u}\cdot\mathbf{u}^{'}\rangle$ or R_3	$L_{(3,2)}=\langle u^2 u_L^{'}\rangle$ or R_{31}			
4	$L_{(4,0)}=\langle u_L^3 u_L^{'}\rangle$	$R_{(4,1)}=\langle u_L^2\mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(4,2)}=\langle u^2 u_L u_L^{'}\rangle$	$R_{(4,3)}=\langle u^2\mathbf{u}\cdot\mathbf{u}^{'}\rangle$		
5	$L_{(5,0)}=\langle u_L^4 u_L^{'}\rangle$	$R_{(5,1)}=\langle u_L^3\mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(5,2)}=\langle u^2 u_L^2 u_L^{'}\rangle$	$R_{(5,3)}=\langle u^2 u_L\mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(5,4)}=\langle u^4 u_L^{'}\rangle$	
6	$L_{(6,0)}=\langle u_L^5 u_L^{'}\rangle$	$R_{(6,1)}=\langle u_L^4\mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(6,2)}=\langle u^2 u_L^3 u_L^{'}\rangle$	$R_{(6,3)}=\langle u^2 u_L^2\mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(6,4)}=\langle u^4 u_L u_L^{'}\rangle$	$R_{(6,5)}=\langle u^4\mathbf{u}\cdot\mathbf{u}^{'}\rangle$

p independent correlation functions

Kinematic relations (for same order p)

Dynamic relations (for different order p)

$$L_{(p,q)}=\langle u^q u_L^{p-q-1}u_L^{'}\rangle \quad R_{(p,q+1)}=\langle u^q u_L^{p-q-2}u_i u_i^{'}\rangle=\langle u^q u_L^{p-q-2}\mathbf{u}\cdot\mathbf{u}^{'}\rangle \quad R_{(p,q+1)}=L_{(p,q)}+2T_{(p,q)}$$

Correlation functions in the limit of small and large scale

For odd order p

$$\lim_{r\to 0}\frac{\left\langle u^q u_L^{p-q-1}\right\rangle}{\left\langle u_L^{p-1}\right\rangle}=\frac{p}{p-q} \qquad \lim_{r\to \infty}\frac{\left\langle u^q u_L^{p-q-1}\right\rangle}{\left\langle u_L^{p-1}\right\rangle}=\frac{p}{p-q}$$

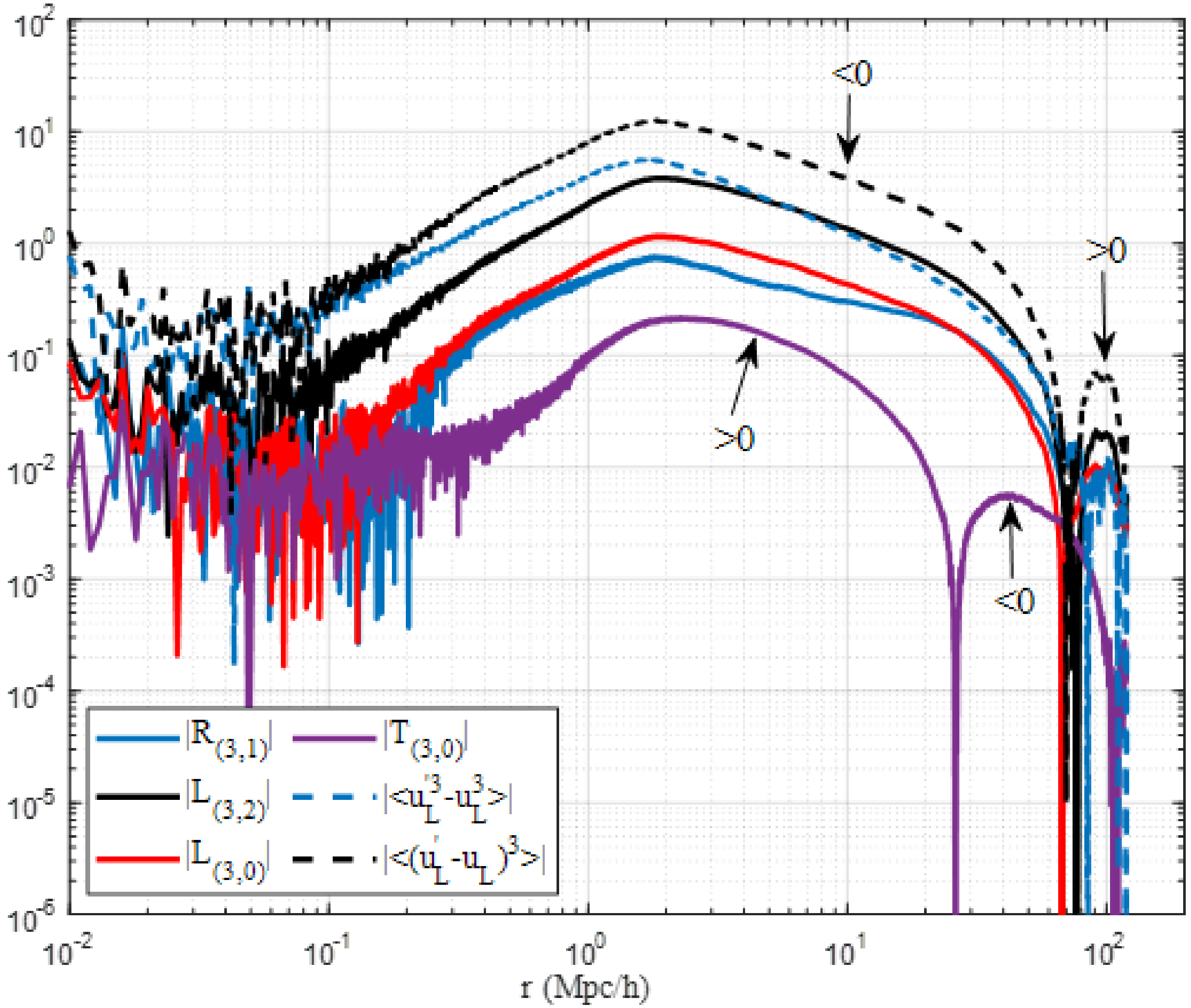
$$\lim_{r\to 0,\infty}\frac{L_{(p,q)}}{L_{(p,0)}}=\lim_{r\to 0,\infty}\frac{\left\langle u^q u_L^{p-q-1}u_L^{'}\right\rangle}{\left\langle u_L^{p-1}u_L^{'}\right\rangle}=\lim_{r\to 0,\infty}\frac{\left\langle u^q u_L^{p-q-1}\right\rangle}{\left\langle u_L^{p-1}\right\rangle}=\frac{p}{p-q}$$

For even order p

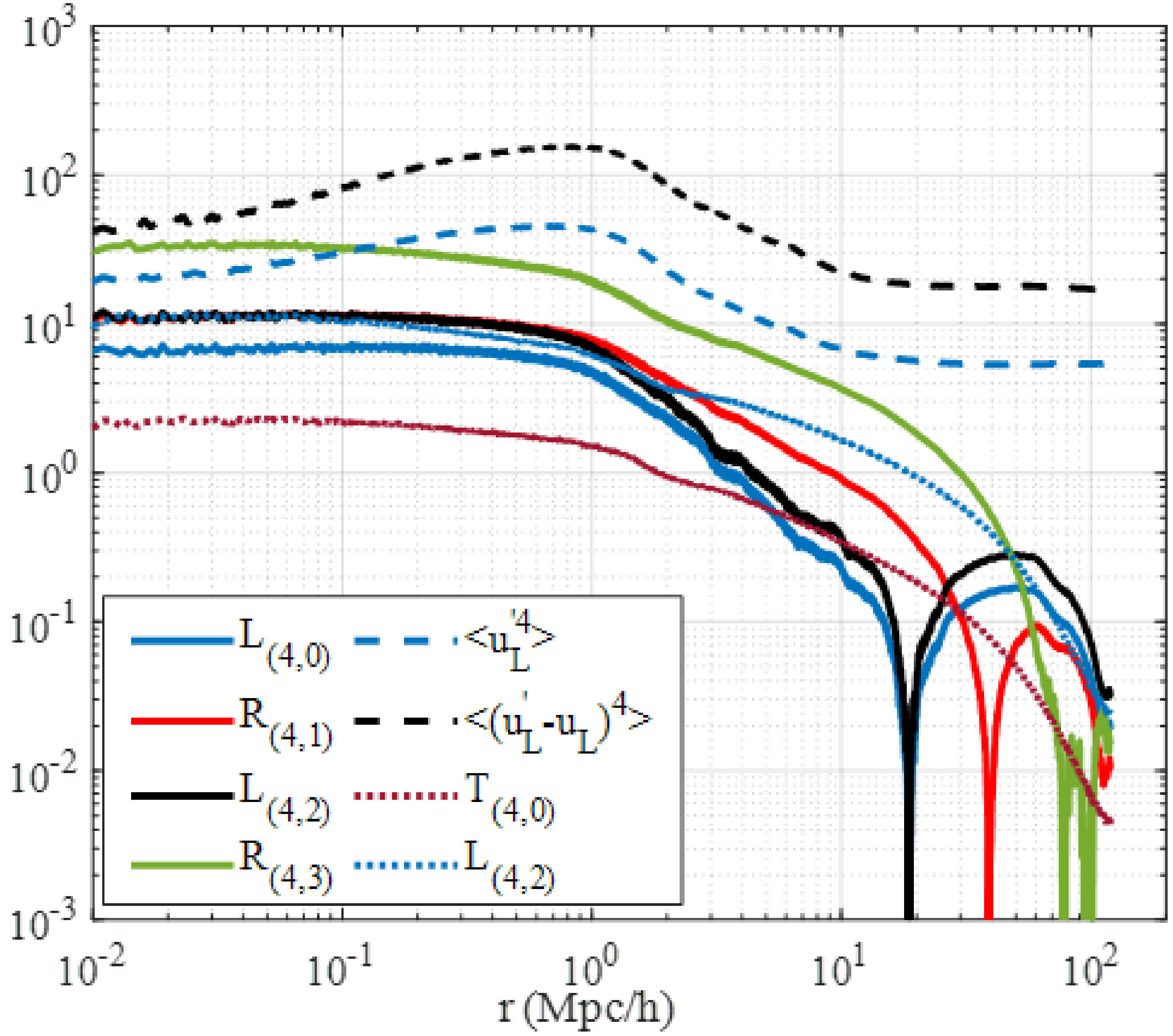
$$\lim_{r\to 0}\frac{R_{(p,q+1)}}{L_{(p,0)}}=\lim_{r\to 0}\frac{\left\langle u^q u_L^{p-q-2}\mathbf{u}\cdot\mathbf{u}^{'}\right\rangle}{\left\langle u_L^{p-1}u_L^{'}\right\rangle}=\frac{p+1}{p-q-1} \qquad \lim_{r\to 0,\infty}\frac{L_{(p,q)}}{L_{(p,0)}}=\lim_{r\to 0,\infty}\frac{\left\langle u^q u_L^{p-q-1}u_L^{'}\right\rangle}{\left\langle u_L^{p-1}u_L^{'}\right\rangle}=\frac{p+1}{p+1-q}$$

- The collisionless nature has effects on the limits of correlations functions oat both small and large scales.
- These results can be confirmed by N-body simulation data

Correlation and structure functions from N-body simulation



Two-point third order velocity correlation and structure functions (normalized by u^3) at z=0



Two-point fourth order velocity correlation and structure functions (normalized by u^4) at z=0

Kinematic relations for correlation functions $L_{(p,q)}$ and $R_{(p,q)}$ of any (pth) order (derivation skipped)

For incompressible flow: $\nabla \cdot \mathbf{u} = 0$

$$\left(p-q-1\right)R_{(p,q+1)} = \frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$$

$$2\left(p-q-1\right)T_{(p,q)} = \frac{1}{r}\left(r^2 L_{(p,q)}\right)_{,r}$$

$$\left(r^2 R_{(p,q+1)}\right)_{,r} = \frac{2}{r^{p-q-1}}\left(r^{p-q+1}T_{(p,q)}\right)_{,r}$$

If Θ=0

For constant divergence flow: $\nabla \cdot \mathbf{u} = \theta$

$$\left(p-q-1\right)R_{(p,q+1)} + \left\langle u^q u_L^{p-q-1}\right\rangle \theta r = \frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$$

If Θ ≠0 and p is even: $\lim_{r\to 0}\left\langle u^q u_L^{p-q-1}\right\rangle = 0$

$$\left(p-q-1\right)R_{(p,q+1)} = \frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$$

If Θ ≠0 and p is odd:

$$\left\langle u^{p-1}\right\rangle \theta r = \frac{1}{r}\left(r^2 L_{(p,p-1)}\right)_{,r}$$

$$\left(p-1\right)R_{(p,1)} + \left\langle u_L^{p-1}\right\rangle \theta r = \frac{1}{r^p}\left(r^{p+1}L_{(p,0)}\right)_{,r}$$

If Θ ≠0 and p=1:

$$\theta = \frac{1}{r^2}\left(r^2 L_{(1,0)}\right)_{,r} = \frac{1}{2r^2}\left(r^2\left\langle \Delta u_L\right\rangle\right)_{,r}$$

Kinematic relations for even order correlations of constant divergence flow should be the same as that of incompressible flow

For irrotational flow: $\nabla \times \mathbf{u} = 0$

$$\left(R_{(p,q+1)}r\right)_{,r} + \left(p-q-2\right)L_{(p,q+2)} = \frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$$

$$\left(p-q\right)R_{(p,q+1)} - \left(p-q-2\right)L_{(p,q+2)} = \frac{2}{r^{p-q}}\left(r^{p-q+1}T_{(p,q)}\right)_{,r}$$

$$\left(p-q\right)L_{(p,q)} - \left(p-q-2\right)L_{(p,q+2)} = 2\left(rT_{(p,q)}\right)_{,r}$$

Kinematic relations validated by N-body simulations

Original Kinematic relations

On small scale, kinematic relations for even order (even p) correlations are the same as those for incompressible flow:

$$H_{(p,q)}^{S}\left(r\right)=\frac{\left(p-q-1\right)}{r^{p-q+1}L_{(p,q)}}\int_{0}^{r}R_{(p,q+1)}r^{p-q}dr=1$$

On small scale, kinematic relations for odd order (odd p) correlations are the same as those for incompressible flow:

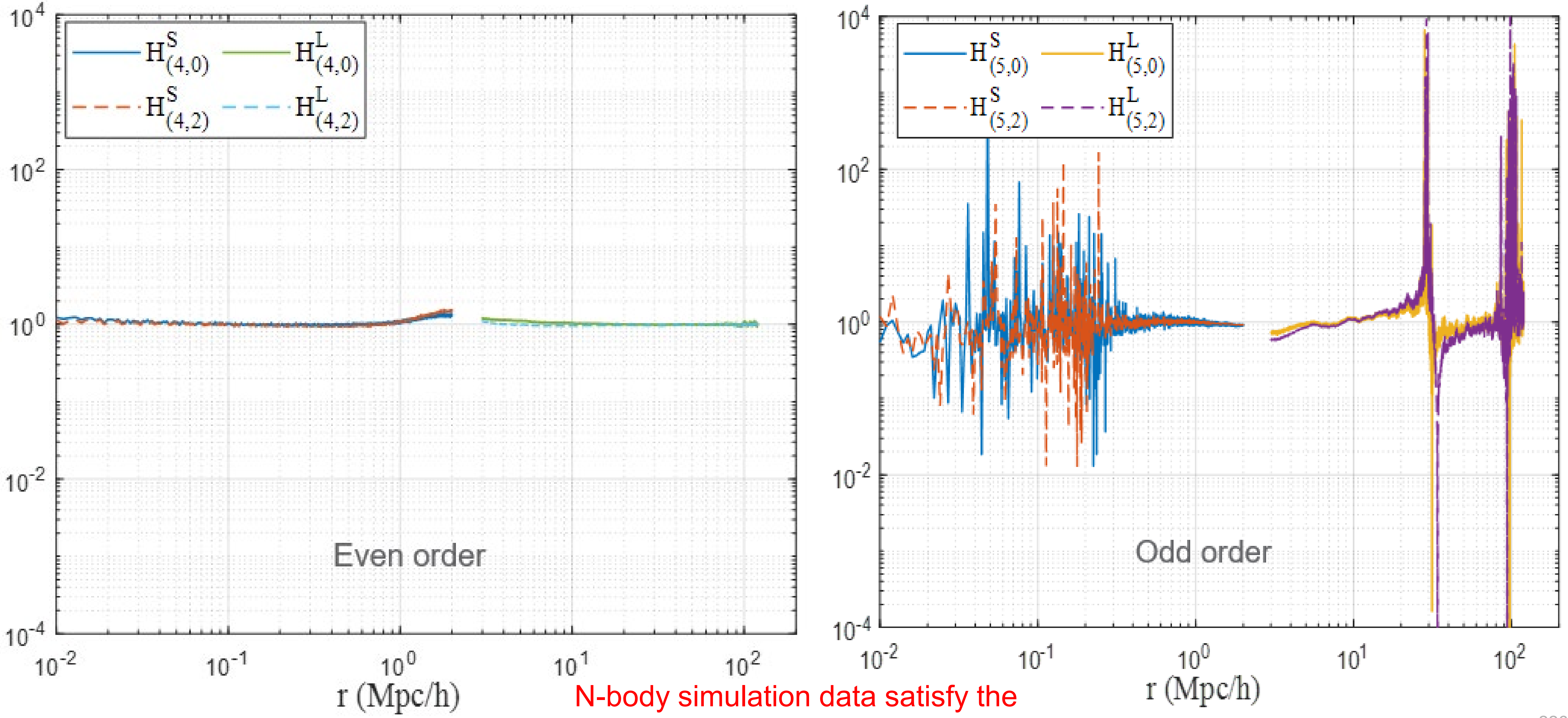
$$H_{(p,q)}^{S}\left(r\right)=\frac{\left(p-q-1\right)}{r^{p-q+1}L_{(p,q)}}\int_{0}^{r}\left(R_{(p,q+1)}-\frac{L_{(p,p-1)}}{p-q}\right)r^{p-q}dr+\frac{1}{\left(p-q\right)}\cdot\frac{L_{(p,p-1)}}{L_{(p,q)}}=1$$

On large scale, kinematic relations for irrotational flow:

$$H_{(p,q)}^{L}\left(r\right)=\frac{1}{2r^{p-q+1}T_{(p,q)}}\int_{0}^{r}\left[\left(p-q\right)R_{(p,q+1)}-\left(p-q-2\right)L_{(p,q+2)}\right]r^{p-q}dr=1$$

- To validate kinematic relations with N-body data, we need to construct equivalent relations.
- Extract high order correlation functions from N-body simulation data
- Dark matter flow is of constant divergence on small scale and irrotational on large scale
- Check the equivalent kinematic relations against simulation data

Kinematic relations validated by N-body simulations



N-body simulation data satisfy the kinematic relations.

Dynamic relations from dynamics on large scale

- Kinematic relations are relations between correlation and structure functions of the same order;
- Dynamic relations are relations between correlation functions of different orders and can only be obtained from the self-closed dynamic evolution of velocity.
- However, closure problem is well known for Jeans' equations which are not self-closed.
- Self-closed dynamic equations of velocity must be introduced on small and large scale.
- Dynamic equations are subsequently converted into dynamic relations.

Self-closed adhesion approximation on large scale : $\nabla \times \mathbf{v} = 0$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v} = c(a)\mathbf{v} + \nu(a)\nabla^2\mathbf{v}$$

Damping

"Artificial " viscosity

Neglect second order

$$\frac{\partial \mathbf{v}}{\partial t} = c(a)\mathbf{v}$$

Zeldovich approximation

Using identity:

$$\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) + (\nabla\times\mathbf{u})\times\mathbf{u}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2a}\nabla(\mathbf{v}\cdot\mathbf{v}) = c(a)\mathbf{v} + \nu(a)\nabla^2\mathbf{v}$$

$$c(a) = \left(\frac{4\pi G\rho_0}{Hf(\Omega_m)} - H\right) = \frac{1}{2}H$$

Matter dominant

$$\frac{\partial v_j}{\partial t} + \frac{1}{2a}\frac{\partial(v_i v_i)}{\partial x_j} = cv_j + \nu\nabla^2 v_j \quad \times v_i' \qquad \text{Index Eq. at location x}$$

$$+ \quad \frac{\partial v_i'}{\partial t} + \frac{1}{2a}\frac{\partial(v_j' v_j')}{\partial x_i'} = cv_i' + \nu\nabla'^2 v_i' \quad \times v_j \qquad \text{Index Eq. at location x'}$$

$$= \frac{\partial\langle v_j v_i'\rangle}{\partial t} + \frac{1}{2a}\left\langle v_i'\frac{\partial(v_k v_k)}{\partial x_j} + v_j\frac{\partial(v_k' v_k')}{\partial x_i'}\right\rangle = c\langle v_j v_i' + v_i' v_j\rangle + \nu\nabla^2\langle v_j v_i' + v_i' v_j\rangle$$

Dynamic relations from dynamics on large scale

Time evolution of the second order correlation tensor Q_{ij}:

$$\frac{\partial Q_{ij}}{\partial t}=\frac{1}{2a}\left(\frac{\partial Q_{kki}}{\partial r_j}+\frac{\partial Q_{kkj}}{\partial r_i}\right)+2cQ_{ij}+2\nu\nabla^2 Q_{ij} \quad \times\delta_{ij}$$

Time evolution of the second order correlation function R_2:

$$\frac{\partial R_2}{\partial t}=2\Gamma(r)+2cR_2+2\nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_2}{\partial r}\right)\right)$$

Fourier transform: E_u: Energy spectrum

$$\frac{\partial E_u}{\partial t}=T(k,t)+2cE_u(k,t)-2\nu k^2 E_u(k,t)$$

$$\Gamma(r)=\frac{1}{2a}\frac{\partial Q_{kki}}{\partial r_i}=\frac{1}{2ar^2}\left(r^2R_{31}\right)_{,r}$$

Real-space energy transfer function

$$T(k)=\frac{2}{\pi}\int_0^\infty \Gamma(r)kr\sin(kr)\,dr$$

Spectral energy transfer function

$$L_{(3,2)}(r)=R_{31}(r)=-2a\nu\frac{\partial R_2}{\partial r}$$

Dynamic relation between 2nd and 3rd correlation functions

Density correlation:

$$\xi(r)=-\frac{1}{\left(aHf(\Omega_m)\right)^2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_2}{\partial r}\right)\right]$$

$$\Gamma(r)=\nu\left(aHf(\Omega_m)\right)^2\xi(r)=\frac{\nu a_0u^2}{rr_2}\exp\left(-\frac{r}{r_2}\right)\left[\left(\frac{r}{r_2}\right)^2-7\left(\frac{r}{r_2}\right)+8\right]$$

Third order correlation:

$$R_{31}=\left\langle u^2u_L^{'}\right\rangle=-\nu Ha^2f(\Omega_m)^2\left\langle\Delta u_L\right\rangle=-\frac{2a_0u^2a\nu}{r_2}\exp\left(-\frac{r}{r_2}\right)\left(\frac{r}{r_2}-4\right)$$

$$T(k)=a_0u^2\frac{16\nu}{\pi r_2}\frac{1}{\left(1+1/(kr_2)^2\right)^3}$$

Modeling high order correlation functions on large scale

The same model can be generalized to high order correlation functions:

$$L_{(3,2)} = R_{31} = \left\langle u^2 u_L^{'} \right\rangle = a_3 u^3 \exp\left(-\frac{r}{r_2}\right)\left(\frac{r}{r_2} - b_3\right)$$

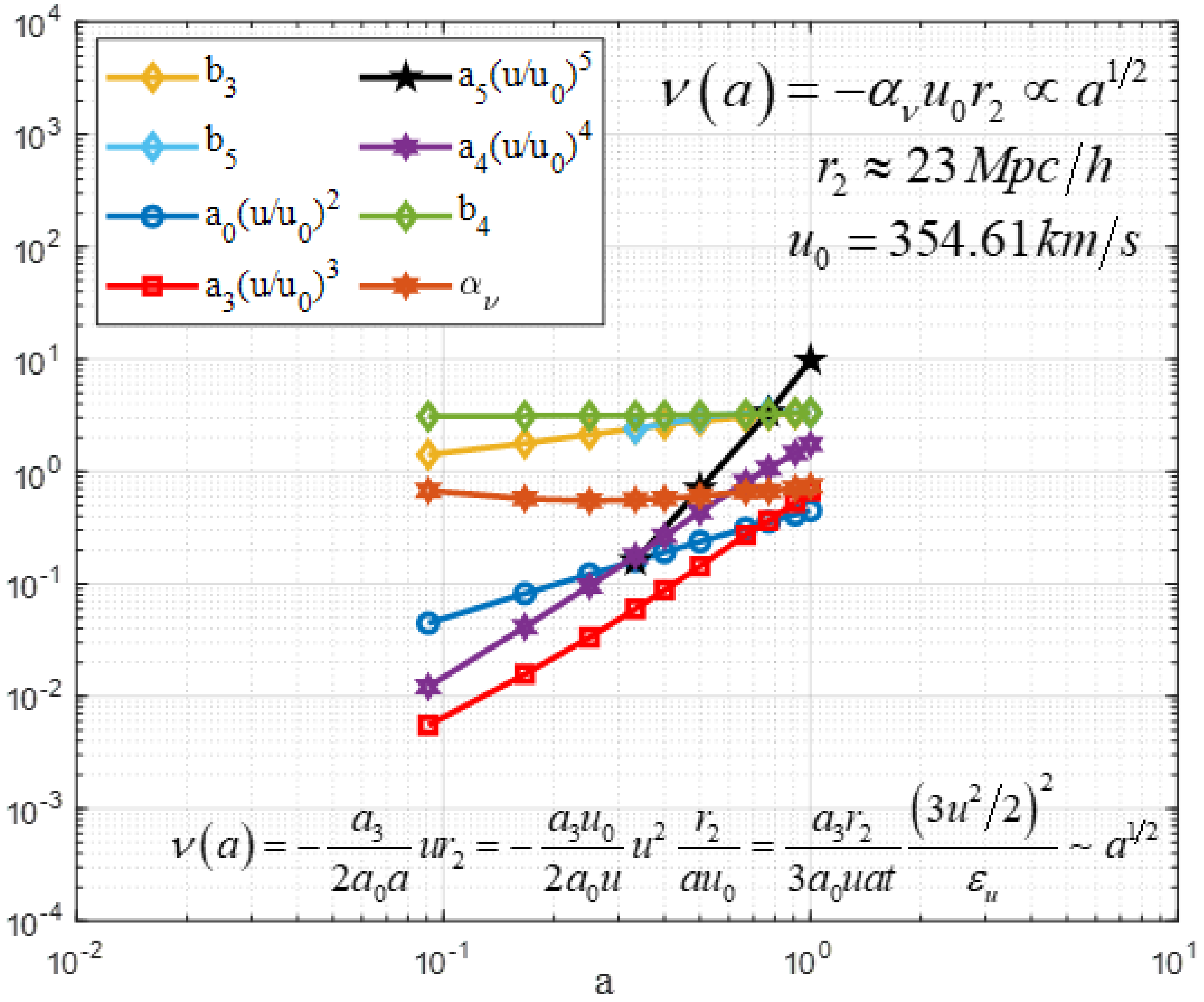
$$R_{(4,3)} = \left\langle u^2 \mathbf{u}\cdot\mathbf{u}^{'} \right\rangle = a_4 u^4 \exp\left(-\frac{r}{r_2}\right)\left(b_4 - \frac{r}{r_2}\right)$$

$$L_{(5,4)} = \left\langle u^4 u_L^{'} \right\rangle = a_5 u^5 \exp\left(-\frac{r}{r_2}\right)\left(\frac{r}{r_2} - b_5\right)$$

Generalize to any order correlation functions:

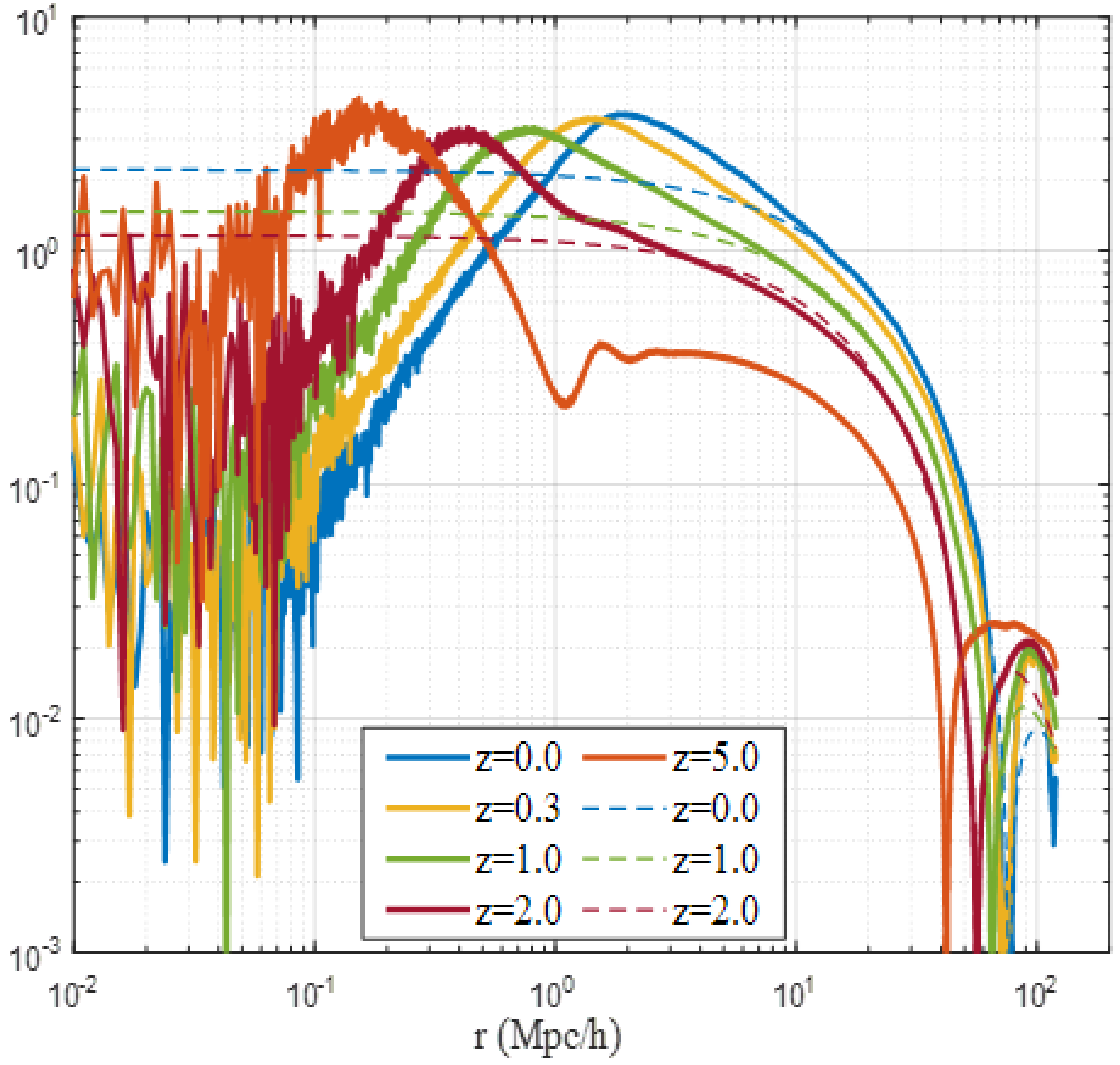
$$L_{(q+1,q)} = \left\langle u^q u_L^{'} \right\rangle \propto u^q \left\langle u_L^{'} \right\rangle \propto \left(\nu H a^2\right)^{q/2} L_{(1,0)} \propto a^{(q+3)/2}$$

$$R_{(q,q-1)} = \left\langle u^{q-2} \mathbf{u}\cdot\mathbf{u}^{'} \right\rangle \propto u^{q-2} \left\langle \mathbf{u}\cdot\mathbf{u}^{'} \right\rangle \propto \left(\nu H a^2\right)^{(q-2)/2} R_{(2,1)} \propto a^{q/2}$$

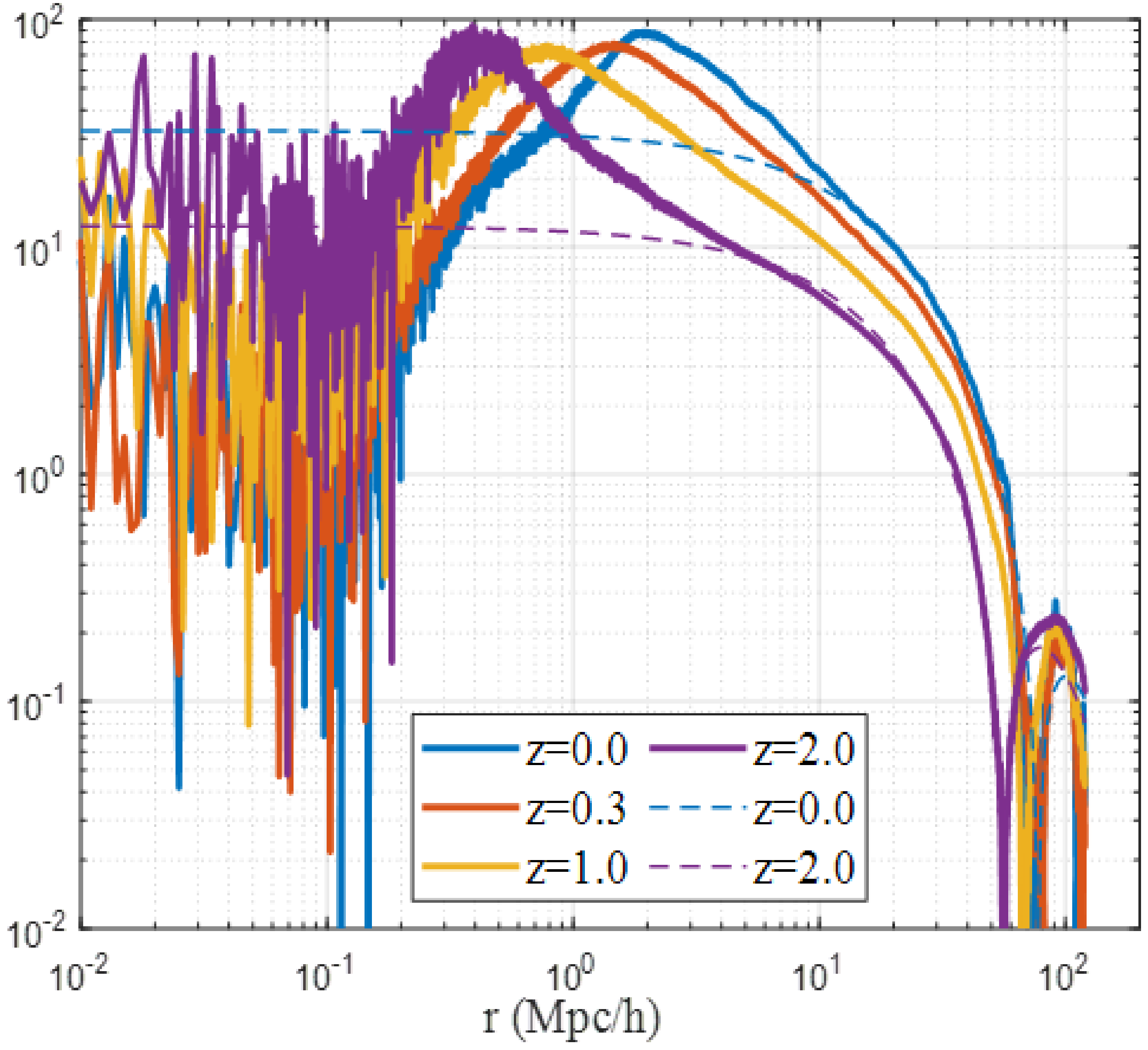


V(a) is artificial viscosity

Modeling high order correlation functions on large scale



Two-point third order velocity correlation $L_{(3,2)}$



Two-point fifth order velocity correlation $L_{(5,4)}$

Dynamic relations from dynamics on large scale

$$\langle\theta\rangle = \langle\nabla\cdot\mathbf{u}\rangle = \frac{1}{2r^2}\left(r^2\langle\Delta u_L\rangle\right)_{,r}$$ ⬅ Kinematic relation

From pair conservation equation:

$$\langle\Delta u_L\rangle \approx -\frac{2}{3}Har\bar{\xi}(r,a) = -\frac{2Ha}{r^2}\int_0^r \xi(y)\,y^2dy$$

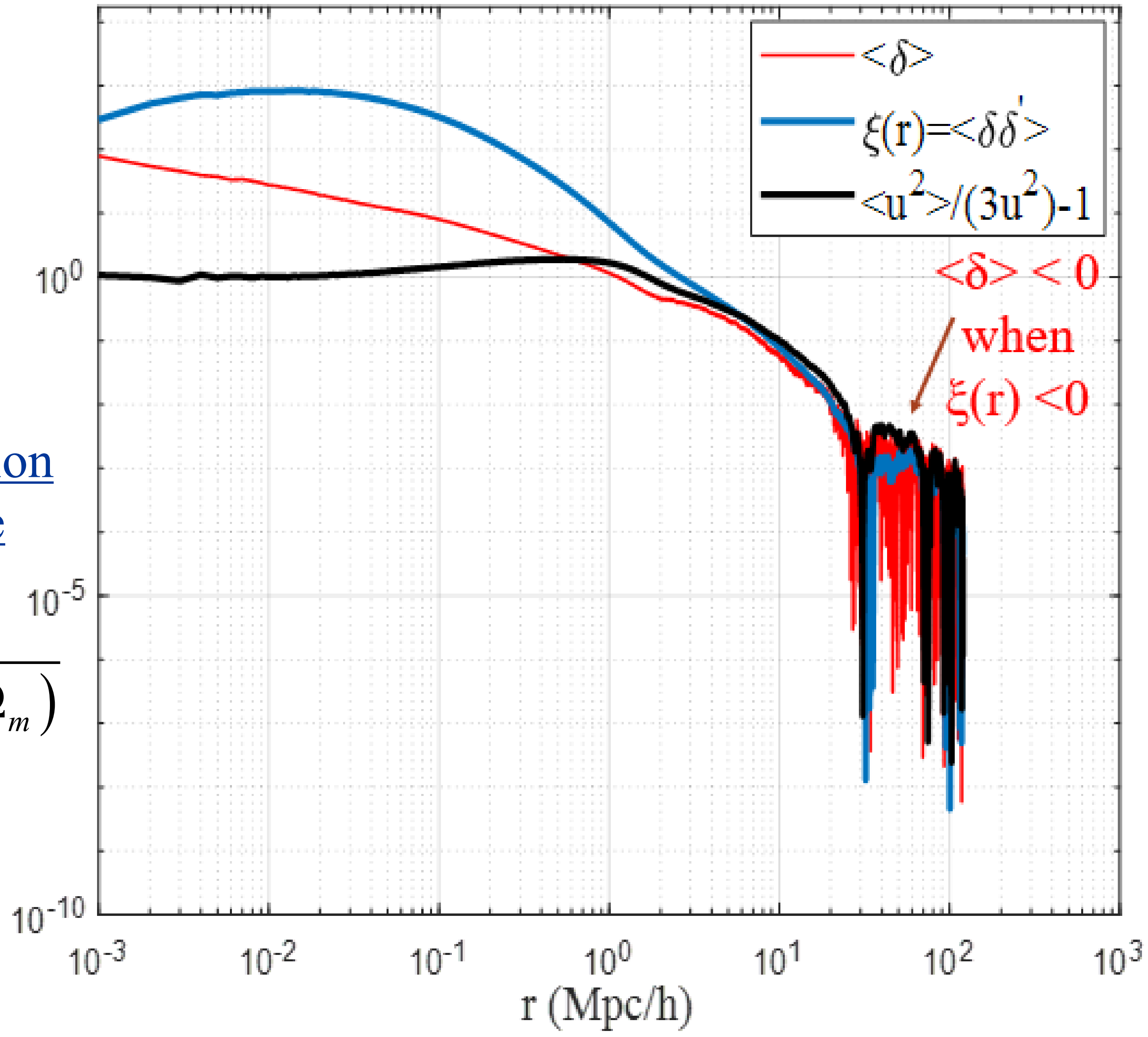
$$\boxed{\begin{aligned}\langle\theta\rangle &= \langle\nabla\cdot\mathbf{u}\rangle = -Ha\xi(r)\\ \langle\theta\rangle &= \langle\nabla\cdot\mathbf{u}\rangle = -aHf(\Omega_m)\langle\delta\rangle\end{aligned}}$$

Dynamic equation on large scale

$$\delta \approx \eta = -\frac{\nabla\cdot\mathbf{u}}{aHf(\Omega_m)}$$

$$f(\Omega_m)\langle\delta\rangle = f(\Omega_m)\langle\delta+\delta'\rangle/2 = \xi(r) = \langle\delta\delta'\rangle$$

On large scale, mean density at two locations is proportional to density correlation on the same scale



Dynamic relations from dynamics on large scale

Use dynamic equations at locations x and x':

$$\frac{\partial v_j}{\partial t}+\frac{1}{2a}\frac{\partial\left(v_i v_i\right)}{\partial x_j}=cv_j+\nu\nabla^2 v_j \quad \times\hat{r}_j$$

$$-\quad \frac{\partial v_i^{'}}{\partial t}+\frac{1}{2a}\frac{\partial\left(v_j^{'} v_j^{'}\right)}{\partial x_i^{'}}=cv_i^{'}+\nu\nabla^{'2} v_i^{'} \quad \times\hat{r}_i$$

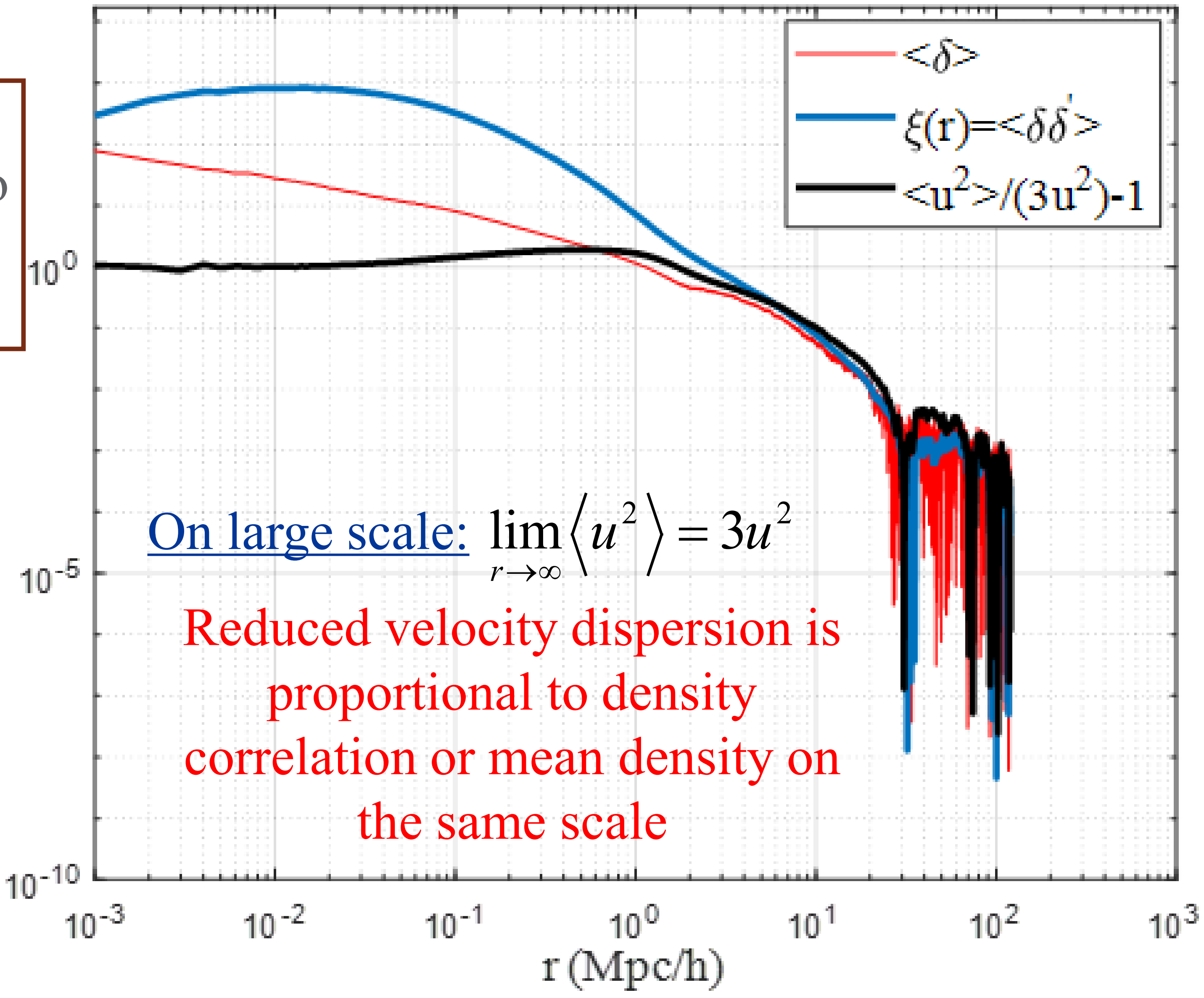
Unit vector between two particles $\hat{\mathbf{r}}=\mathbf{r}/r$

$$=\left\langle \hat{r}_i\frac{\partial v_i^{'}}{\partial t}-\hat{r}_j\frac{\partial v_j}{\partial t}\right\rangle+\frac{1}{a}\frac{\partial\left\langle u^2\right\rangle}{\partial r}=c\left\langle \Delta u_L\right\rangle+2\nu\frac{\partial\left\langle\theta\right\rangle}{\partial r}$$

$$\boxed{\frac{\partial \mathbf{v}}{\partial t}=c(a)\mathbf{v} \Rightarrow \begin{array}{l}\hat{r}_i\frac{\partial v_i}{\partial t}=c(a)\hat{r}_i v_i=c(a)u_L \\ \hat{r}_i\frac{\partial v_i^{'}}{\partial t}=c(a)\hat{r}_i v_i^{'}=c(a)u_L^{'}\end{array}}$$

Use $\left\langle\theta\right\rangle=\left\langle\nabla\cdot\mathbf{u}\right\rangle=-Ha\xi(r)$ and $f\left(\Omega_m\right)\left\langle\delta\right\rangle=\xi(r)$

$$\frac{\partial\left\langle u^2\right\rangle}{\partial r}=2\nu a\frac{\partial\left\langle\theta\right\rangle}{\partial r}=-2\nu a^2 Hf\left(\Omega_m\right)\frac{\partial\left\langle\delta\right\rangle}{\partial r}=-2\nu Ha^2\frac{\partial\xi(r)}{\partial r} \Rightarrow \frac{\left\langle u^2\right\rangle}{3u^2}-1=-\frac{2\nu Ha^2\xi(r)}{3u^2}=-\frac{2\nu Ha^2}{3u^2}f\left(\Omega_m\right)\left\langle\delta\right\rangle$$



On large scale: $\lim_{r\to\infty}\left\langle u^2\right\rangle=3u^2$

Reduced velocity dispersion is proportional to density correlation or mean density on the same scale

Divergence of velocity on all scales

Kinematic relation (good for all scales):

$$\langle\theta\rangle = \langle\nabla\cdot\mathbf{u}\rangle = \frac{1}{2r^2}\left(r^2\langle\Delta u_L\rangle\right)_{,r}$$

From pair conservation equation:
(for large scale)

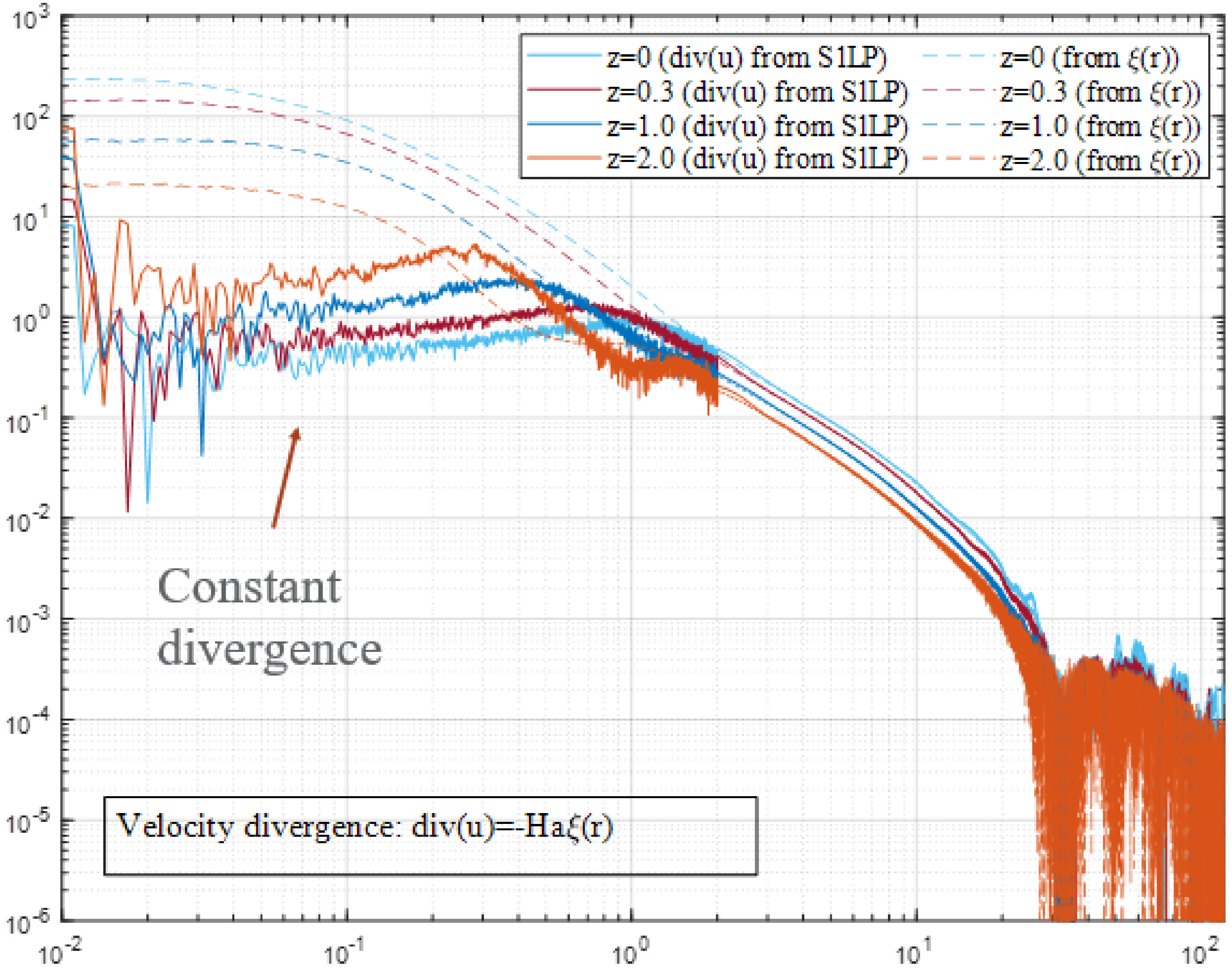
$$\langle\Delta u_L\rangle = -\frac{2Ha}{r^2}\int_0^r \xi(y)\,y^2\,dy$$

On large scale:

$$\langle\theta\rangle = \langle\nabla\cdot\mathbf{u}\rangle = -Ha\,\xi(r)$$

Dynamic equation on large scale

$$\delta = -\frac{\nabla\cdot\mathbf{u}}{aHf(\Omega_m)} = -\frac{\theta}{aHf(\Omega_m)}$$



Velocity divergence on different scales
(normalized by Ha)

Deriving exponential velocity correlation functions on large scale

- The exponential function was proposed for second order transverse velocity correlation T_2 on large scale.
- This is not a coincidence and must be deeply rooted in the dynamics and kinematics on large scale.

Velocity dispersion function for kinetic energy contained in all scales above r:

$$\sigma_u^2(r) = \frac{1}{3}\int_{-\infty}^{\infty} E_u(k) W(kr)^2 \, dk$$

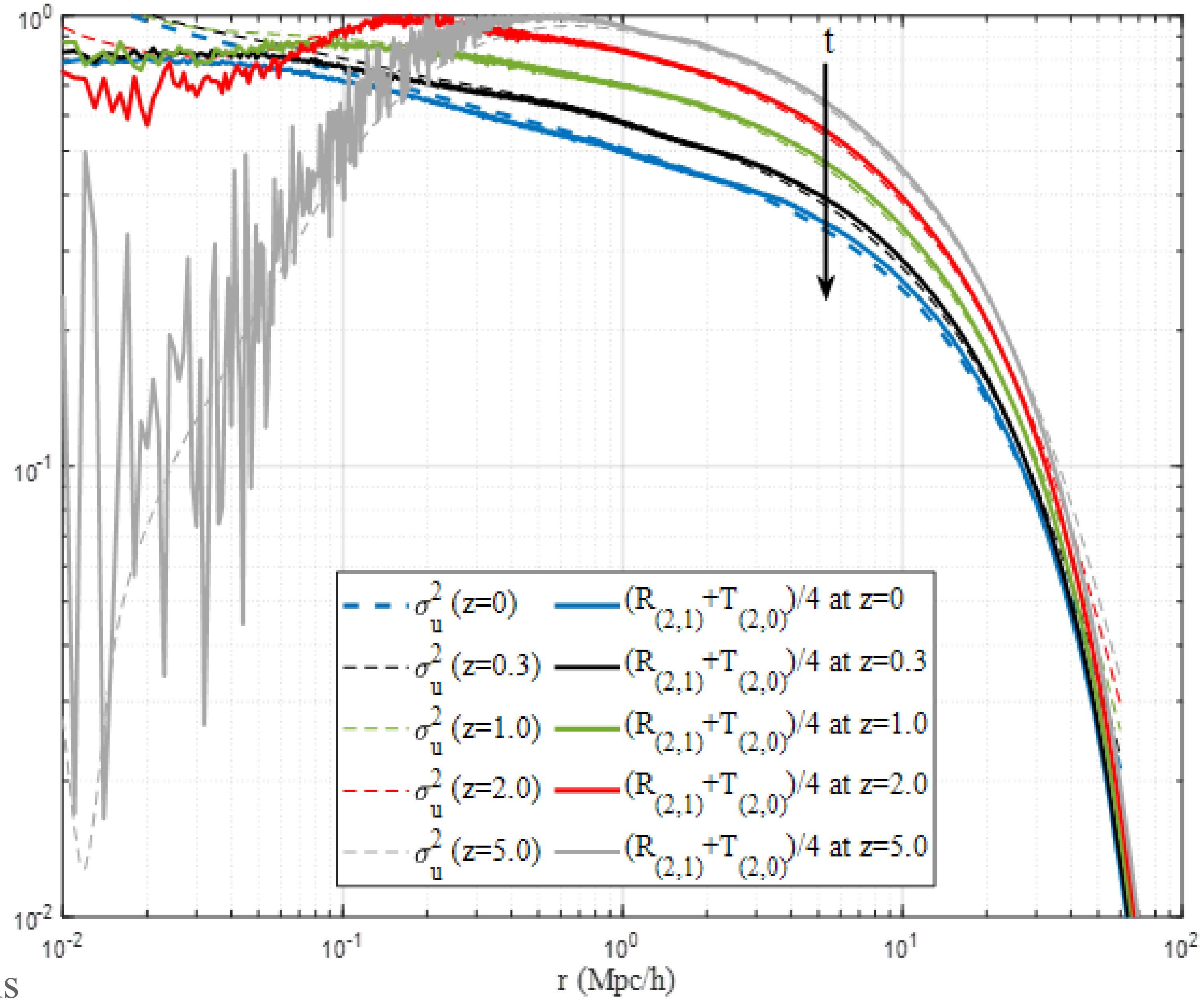
$$W(x) = \frac{3}{x^3}\left[\sin(x) - x\cos(x)\right] = 3\frac{j_1(x)}{x}$$ Window function

On large scale, velocity dispersion function can be approximated by:

$$\sigma_u^2(r) \approx \frac{1}{4}\left[R_{(2,1)}(r) + T_{(2,0)}(r)\right]$$

Relate to velocity correlation functions (Equipartition)

3 translational 1 rotational



Deriving exponential velocity correlation functions on large scale

On large scale velocity dispersion function can be approximated as,

$$\sigma_u^2(r) \approx \frac{1}{4}\left[R_{(2,1)}(r) + T_{(2,0)}(r)\right]$$

Relate to velocity correlation functions (Equipartition)

On large scale, the rate of energy cascade (m^2/s^3):

$$\Pi_u \propto \frac{\sigma_u^2(r)}{(ar)/u}$$

Kinetic energy in scales above r

Turnaround time for energy cascade

$$\Pi_u \propto \frac{\langle u^3 \rangle}{ar} \propto \frac{L_{(3,2)}(r)}{ar}$$

$$\Downarrow$$

$$L_{(3,2)}(r) \propto u\sigma_u^2(r)$$

From dynamic relation on large scale:

$$L_{(3,2)}(r) = -2av\frac{\partial R_{(2,1)}}{\partial r}$$

$$\Downarrow$$

$$\frac{8\nu a}{\alpha_r u}\frac{\partial R_{(2,1)}}{\partial r} = \left[R_{(2,1)}(r) + T_{(2,0)}(r)\right]$$

From kinematic relation on large scale for irrotational flow:

$$R_{(2,1)} = \frac{1}{r^2}\left(r^3 T_{(2,0)}\right)_{,r}$$

$$\Downarrow$$

Exponential second order transverse correlation function:

$$T_{(2,0)} = Const \cdot \exp\left(-\frac{r}{r_2}\right) \quad \text{with} \quad r_2 = -\frac{8\nu a}{\alpha_r u}$$

Deriving power-law velocity correlation functions on small scale

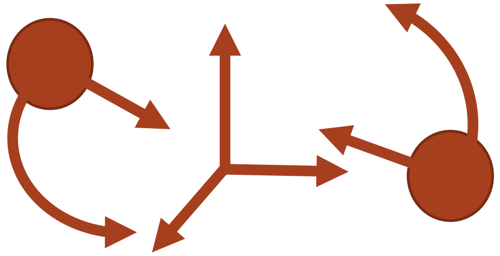
- Similar idea can be applied to determine the power-law exponent of correlation functions on small scale
- On small scale, velocity dispersion function can be approximated as

$$\sigma_u^2(r) \approx \frac{1}{5}\left[R_{(2,1)}(r) + T_{(2,0)}(r) + L_{(2,0)}(r)\right]$$

3 translational

1 internal rotational (two-body is planar)

1 internal longitudinal relative motion



$$S_2^l = 2u^2 (r/r_1)^n$$

Power-law that can be related to virial theorem

From kinematic relations on small scale:

$$L_2(r) = u^2 - \frac{S_2^l}{2} = u^2\left[1 - \left(\frac{r}{r_1}\right)^n\right]$$

$$T_2 = \frac{1}{2r}\left(r^2 L_2\right)_{,r} = u^2\left[1 - \frac{2+n}{2}\left(\frac{r}{r_1}\right)^n\right]$$

$$R_2 = \frac{1}{r^2}\left(r^3 L_2\right)_{,r} = u^2\left[3 - (3+n)\left(\frac{r}{r_1}\right)^n\right]$$

See slides

$$\sigma_d^2(r) = u^2 - \sigma_u^2(r) \Rightarrow$$

$$\boxed{\sigma_d^2(r) = \left(1 + \frac{3}{10}n\right)u^2\left(\frac{r}{r_1}\right)^n \qquad \sigma_d^2(r) = \frac{24 \cdot 2^n}{(4+n)(6+n)}u^2\left(\frac{r}{r_1}\right)^n}$$

See slides

n =0.27 ≈ ¼, the one-forth law on small scale

Dynamic relations from dynamics on small scale

- Self-closed equations for velocity evolution on small scale seems not exist.
- we will first formulate the self-close equations for velocity on small scale.
- These equations are subsequently applied to derive the dynamic relations on small scale.

Jeans equation (not self-closed):

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v}+H\mathbf{v}=-\frac{1}{a}\frac{\nabla\cdot\mathbf{p}}{\rho}-\frac{1}{a}\nabla\phi$$

$$\mathbf{p}=\rho\boldsymbol{\sigma}^2$$

Stress tensor

Velocity dispersion tensor

- γ=1/2 for small scale dynamic equation.
- γ=1 for large scale dynamic equation.

Decompose total velocity into halo velocity and velocity in halos

$$\mathbf{v}(\mathbf{x},t)=\mathbf{v}_h(\mathbf{x}_h,t)+\mathbf{v}_v(\mathbf{r},t)$$

Decompose velocity in halos into radial and azimuthal flow

$$\mathbf{v}_v=\mathbf{v}_r+\mathbf{v}_\varphi$$

Polar flow is neglected

Self-closed description of mean flow (derivation skipped):

$$\nabla\cdot\mathbf{v}=\theta(t)$$ Four equations and four unknowns

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v}+H\mathbf{v}=-\frac{1}{a}\nabla\phi^*+\gamma\frac{1}{a}\underbrace{(\nabla\times\mathbf{v})\times\mathbf{v}}_{1}$$

Centripetal acceleration, significant on small scale

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}(1-\gamma)\mathbf{v}\cdot\nabla\mathbf{v}+\frac{\gamma}{2a}\nabla(\mathbf{v}\cdot\mathbf{v})+H\mathbf{v}=-\frac{1}{a}\nabla\phi^*$$

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}(1-\gamma)\nabla\cdot(\mathbf{v}\otimes\mathbf{v})+\frac{\gamma}{2a}\nabla(\mathbf{v}\cdot\mathbf{v})+\left[H-\frac{1}{a}(1-\gamma)\theta\right]\mathbf{v}=-\frac{1}{a}\nabla\phi^*$$

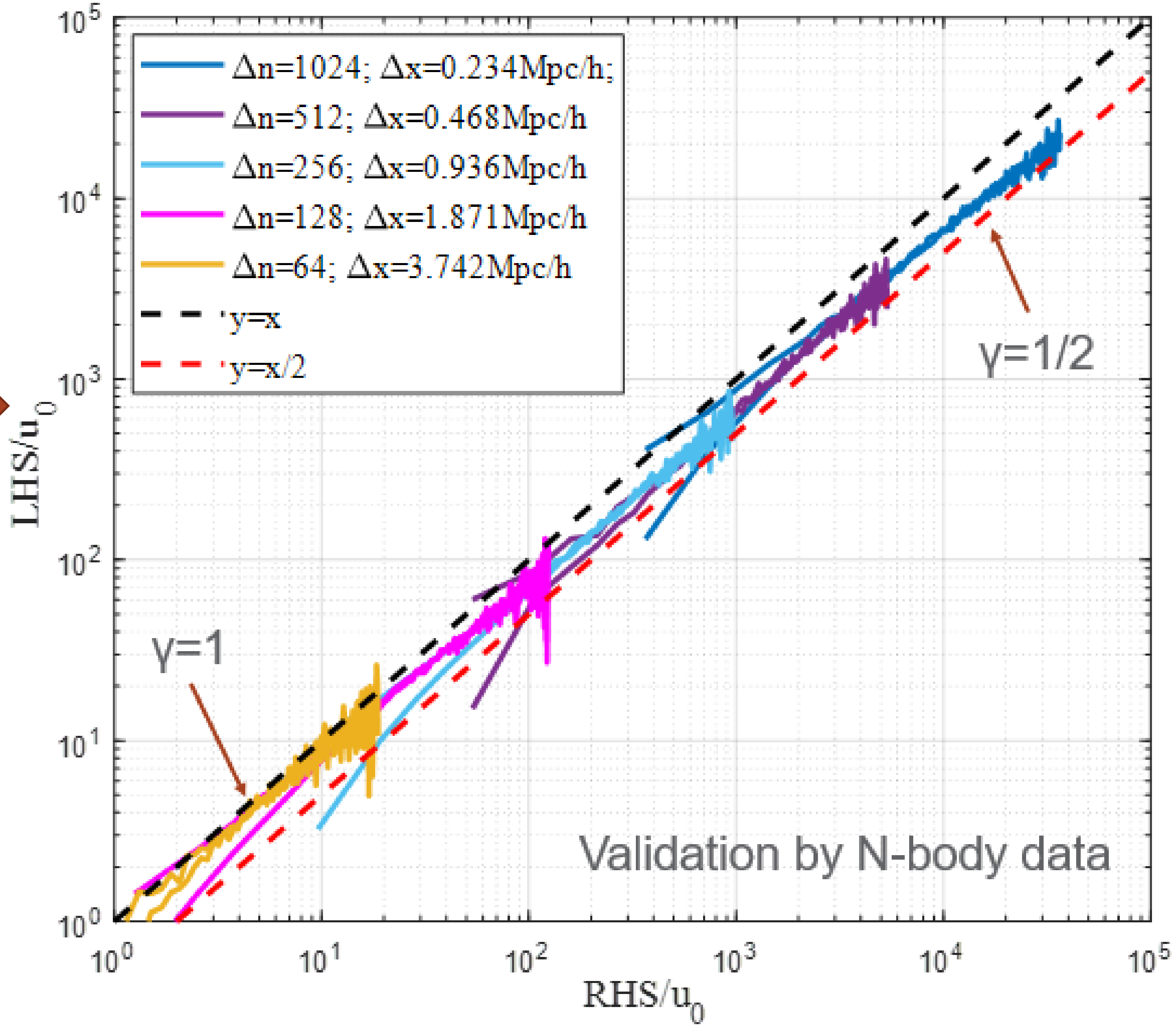
Self-closed description of dynamics

Taking curl on both sides:

$$\nabla\times\quad \frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v}+H\mathbf{v}=-\frac{1}{a}\nabla\phi^{*}+\gamma\frac{1}{a}\underbrace{(\nabla\times\mathbf{v})\times\mathbf{v}}_{1}$$

Equation for vorticity: $\boldsymbol{\omega}=\nabla\times\mathbf{v}$

$$\underbrace{\frac{\partial\boldsymbol{\omega}}{\partial t}+\frac{1}{a}\nabla\times(\mathbf{v}\cdot\nabla\mathbf{v})+H\boldsymbol{\omega}}_{LHS}=\underbrace{\gamma\frac{1}{a}\nabla\times[\boldsymbol{\omega}\times\mathbf{v}]}_{RHS}$$



- On large scale (large grid size Δx), γ≈1
- On small scale (small grid size Δx), γ≈1/2.
- There is a transition between the two regimes.

Averaged dynamic equations for velocity and the origin of effective viscosity

With the self-closed description of velocity, we can derive the effective equations for mean flow

Similar to Reynolds decomposition, decompose velocity and potential into mean and fluctuation in time,

$$\mathbf{v}=\overline{\mathbf{v}}+\mathbf{v}' \quad \phi^*=\overline{\phi^*}+\phi^{*'}$$

Averaging is essentially a filtering process with a cutoff resolution to separate variables into resolved and unresolved parts

Substitute into the self-closed description:

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}(1-\gamma)\mathbf{v}\cdot\nabla\mathbf{v}+\frac{\gamma}{2a}\nabla(\mathbf{v}\cdot\mathbf{v})+H\mathbf{v}=-\frac{1}{a}\nabla\phi^*$$

$$\frac{\partial \overline{\mathbf{v}}}{\partial t}+\frac{1}{a}(1-\gamma)\overline{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}}+\frac{\gamma}{2a}\nabla(\overline{\mathbf{v}}\cdot\overline{\mathbf{v}})+H\overline{\mathbf{v}}=-\frac{1}{a}\nabla\overline{\phi^*}-\left(\frac{1-\gamma}{a}\underbrace{\overline{\mathbf{v}'\cdot\nabla\mathbf{v}'}}_{1}+\frac{\gamma}{2a}\underbrace{\overline{\nabla(\mathbf{v}'\cdot\mathbf{v}')}}_{2}\right)$$

$$\nabla\overline{\phi^*}=-3Ha\overline{\mathbf{v}}/2 \quad \text{and} \quad \gamma=1$$

$$\frac{\partial \overline{\mathbf{v}}}{\partial t}+\frac{1}{2a}\nabla(\overline{\mathbf{v}}\cdot\overline{\mathbf{v}})=\frac{1}{2}Ha\overline{\mathbf{v}}-\frac{1}{2a}\nabla\left(\overline{\mathbf{v}'\cdot\mathbf{v}'}\right)$$

Compare to dynamic equation on large sale:

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{2a}\nabla(\mathbf{v}\cdot\mathbf{v})=c(a)\mathbf{v}+\nu(a)\nabla^2\mathbf{v}$$

$$-\frac{1}{2a}\nabla\left(\overline{\mathbf{v}'\cdot\mathbf{v}'}\right)=\nu\nabla^2\overline{\mathbf{v}}=\nu\nabla(\nabla\cdot\overline{\mathbf{v}}) \quad \text{Subgrid model}$$

Force as the gradient of kinetic energy in unresolved fluctuation

Force from Newtonian law of viscosity for mean flow

Divergence proportional to overdensity δ

The artificial viscosity on large scale origins from the unresolved velocity fluctuations

Use $\overline{\delta}=-\frac{\nabla\cdot\overline{\mathbf{v}}}{aHf(\Omega_m)}$ and integrate both sides of subgrid model

$$\overline{\mathbf{v}'^2}=F(t)+2\nu a^2Hf(\Omega_m)\overline{\delta}$$

The larger mean density (higher resolution), the smaller unresolved velocity fluctuations

Dynamic evolution of vorticity, enstrophy, and energy

Taking curl on both sides of self-closed description:

$$\nabla\times\quad \frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}(1-\gamma)\mathbf{v}\cdot\nabla\mathbf{v}+\frac{\gamma}{2a}\nabla(\mathbf{v}\cdot\mathbf{v})+H\mathbf{v}=-\frac{1}{a}\nabla\phi^*$$

Equation for vorticity: $\boldsymbol{\omega}=\nabla\times\mathbf{v}$

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+H\boldsymbol{\omega}=\frac{1}{a}(\gamma-1)\nabla\times(\mathbf{v}\cdot\nabla\mathbf{v})$$

Dynamic evolution of vorticity:

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\frac{1-\gamma}{a}\underbrace{\mathbf{v}\cdot\nabla\boldsymbol{\omega}}_{1}+\left[1+(1-\gamma)\frac{\theta}{Ha}\right]\underbrace{H\boldsymbol{\omega}}_{2}=\frac{1-\gamma}{a}\underbrace{\boldsymbol{\omega}\cdot\nabla\mathbf{v}}_{3}$$

1: Transport of vorticity

2: Destroy of vorticity on large scale

3: Generation of vorticity on small scale

Dynamic evolution of enstrophy:

$$\frac{\partial\boldsymbol{\omega}^2/2}{\partial t}+\frac{1-\gamma}{a}\underbrace{\mathbf{v}\cdot\nabla\frac{\boldsymbol{\omega}^2}{2}}_{1}+\left[1+(1-\gamma)\frac{\theta}{Ha}\right]\underbrace{H\boldsymbol{\omega}^2}_{2}=\frac{1-\gamma}{a}\underbrace{\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla\mathbf{v})}_{3}$$

Taking scalar product on both sides:

$$\mathbf{v}\cdot\quad \frac{\partial\mathbf{v}}{\partial t}+\frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v}+H\mathbf{v}=-\frac{1}{a}\nabla\phi^*+\gamma\frac{1}{a}\underbrace{(\nabla\times\mathbf{v})\times\mathbf{v}}_{1}$$

$$\frac{\partial\mathbf{v}^2/2}{\partial t}=-\frac{1}{a}\nabla\cdot\left[\left(\frac{1}{2}\mathbf{v}^2+\phi^*\right)\mathbf{v}\right]-H\mathbf{v}^2+\frac{1}{a}\left(\frac{1}{2}\mathbf{v}^2+\phi^*\right)\nabla\cdot\mathbf{v}$$

Specific kinetic energy:

$$K=\int_V\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\,dV$$

Total energy:

$$E=\frac{1}{2}\mathbf{v}^2+\phi^*$$

Virial relation:

$$\int_V\left(2\mathbf{v}^2+\beta\phi^*\right)dV=0$$

Dynamic evolution of energy E at different location:

$$\nabla^2E+Ha\theta\left(1+\frac{\partial\ln\theta}{\partial\ln a}\right)=(1-\gamma)\left(\left[\mathbf{v}\cdot\left(\nabla^2\mathbf{v}-\nabla\theta\right)+\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right]\right)$$

Decay on large scale

Velocity gradient

Rotational contribution

Dynamic relations from dynamics on small scale

Self-closed dynamic equations at two locations x and x':

$$\frac{\partial v_i}{\partial t}+\frac{1-\gamma}{a}\frac{\partial\left(v_i v_k\right)}{\partial x_k}+\frac{\gamma}{2a}\frac{\partial\left(v_k v_k\right)}{\partial x_i}+\left[1-\frac{\left(1-\gamma\right)}{aH}\theta\right]Hv_i=-\frac{1}{a}\frac{\partial\phi^*}{\partial x_i}\qquad\times v_j'$$

$$+\quad\frac{\partial v_j'}{\partial t}+\frac{1-\gamma}{a}\frac{\partial\left(v_j' v_k'\right)}{\partial x_k'}+\frac{\gamma}{2a}\frac{\partial\left(v_k' v_k'\right)}{\partial x_j'}+\left[1-\frac{\left(1-\gamma\right)}{aH}\theta\right]Hv_j'=-\frac{1}{a}\frac{\partial\phi^{*'}}{\partial x_j'}\qquad\times v_i$$

With self-closed dynamic equations on small scale, we are ready to covert it into dynamic relations. Same approach was applied for irrotational flow on large scale.

$$\frac{\partial Q_{ij}}{\partial t}+2\left[1-\frac{\left(1-\gamma\right)}{aH}\theta\right]HQ_{ij}=\frac{2-2\gamma}{a}\frac{\partial Q_{ikj}}{\partial r_k}+\frac{\gamma}{a}\frac{\partial Q_{kkj}}{\partial r_i}-\frac{1}{a}\left[\frac{\partial\left\langle\phi^* v_j'\right\rangle}{\partial x_i}+\frac{\partial\left\langle\phi^{*'} v_i\right\rangle}{\partial x_j'}\right]\qquad\times\delta_{ij}$$

$$\frac{\partial R_{(2,1)}}{\partial t}+2\left[1-\frac{\left(1-\gamma\right)}{aH}\theta\right]HR_{(2,1)}=\frac{1}{ar^2}\left[\frac{\partial}{\partial r}\left(r^2\left[\left(2-2\gamma\right)R_{(3,1)}+\gamma L_{(3,2)}\right]\right)\right]+\frac{2}{a}\theta\left\langle\phi^*\right\rangle$$

Dynamic relations between second and third order correlations on small scale

$$\frac{-\left\langle\phi^*\right\rangle}{u^2}=\frac{\left\langle u^2\right\rangle}{\beta^* u^2}=\underbrace{\frac{5}{u^2 r^3}\int_0^r R_{(2,1)}\left(y\right)y^2 dy}_{1}-\underbrace{\frac{1}{Haru^2}\left(R_{(3,1)}+\frac{1}{2}L_{(3,2)}\right)}_{2}$$

$$\left(R_{(3,1)}+\frac{1}{2}L_{(3,2)}\right)=-Hau^2 r=\left\langle\Delta u_L\right\rangle u^2=\frac{4}{9}\varepsilon_u ar$$

Dynamic relations from dynamics on small scale

Dynamic relations:

$$\left(R_{(3,1)} + \frac{1}{2}L_{(3,2)}\right) = -Hau^2 r = \langle \Delta u_L \rangle u^2 = \frac{4}{9}\varepsilon_u ar$$

GSCH:

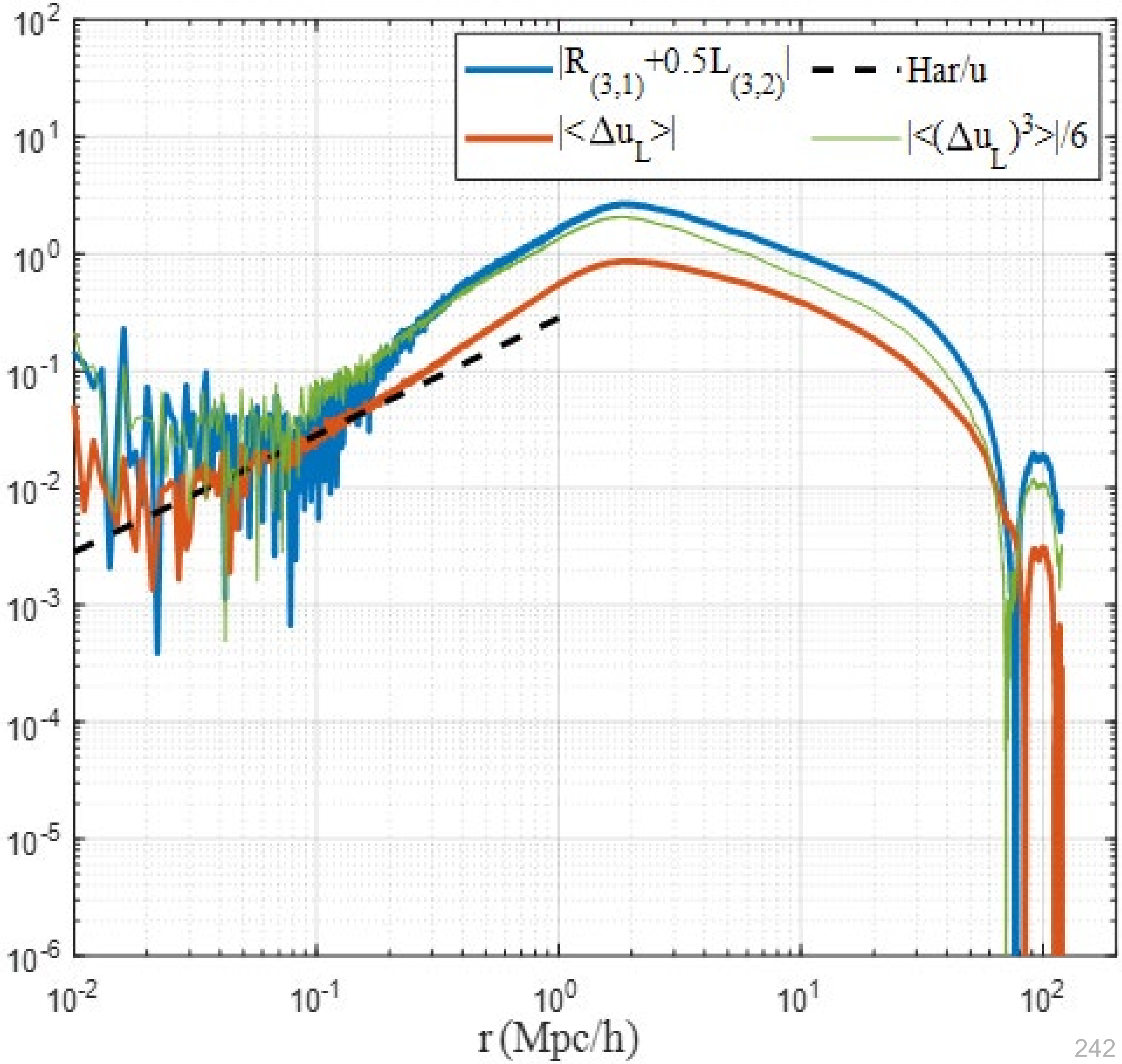
$$\left\langle (\Delta u_L)^3 \right\rangle = 3\left\langle (\Delta u_L)^2 \right\rangle \langle \Delta u_L \rangle \approx 6u^2 \langle \Delta u_L \rangle$$

$$\left(R_{(3,1)} + \frac{1}{2}L_{(3,2)}\right) = \frac{1}{6}\left\langle (\Delta u_L)^3 \right\rangle$$

$$\left\langle (\Delta u_L)^3 \right\rangle = \frac{8}{3}\varepsilon_u ar \quad \Longleftrightarrow \quad \left\langle (\Delta u_L)^3 \right\rangle = -\frac{4}{5}\varepsilon_u r$$

For comparison, the four-fifths law for incompressible flow

$$\varepsilon_u = \frac{3}{8}\frac{\left\langle (\Delta u_L)^3 \right\rangle}{ar}$$



Summary and keywords

Third order velocity correlation tensor	Vorticity, Energy and Enstrophy	Self-closed velocity equation
Effective viscosity	Kinematic relations	Dynamic relations

- Analogy between dark matter flow and homogeneous isotropic turbulence is established for development of statistical theory in terms of correlation, structure, dispersion, and spectrum functions;

- General kinematic relations for two-point velocity statistics are developed on small and large scales respectively;

- On large scale, the redshift dependence of qth order velocity correlations follows $\sim a^{(q+2)/2}$ for odd q and $\sim a^{q/2}$ for even q; The overdensity is proportional to density correlation on the same scale, i.e. $\langle\delta\rangle=\langle\delta\delta'\rangle$; (Negative) Effective viscosity in adhesion model originates from velocity fluctuations.

- On small scale, self-closed description for velocity is developed such that the dynamic relation can be obtained, which can be validated by N-body simulation.

Applications of dark matter flow

Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow

Xu Z., 2022, arXiv:2202.07240v1 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2202.07240

Introduction

- The existence of dark matter (DM) is supported by numerous astronomical observations:
  - Rotation curves of spiral galaxies
  - Motion of galaxies in galaxy clusters
  - Gravitational lensing
  - Bullet clusters
  - CMB ……
- Though the nature of dark matter is still unclear, dark matter is believed to be cold (non-relativistic), collisionless, dissipationless, non-baryonic, and barely interacting with baryonic matter except through gravity.
- Dark matter must be sufficiently smooth on large scales with a fluid-like behavior that is best described by self-gravitating collisionless flow dynamics (SG-CFD).
- It is often assumed to be a thermal relic, weakly interacting massive particles (WIMPs)
- However, no conclusive signals have been detected in searches for thermal WIMPs.
  - Direct detection by underground experiments
    - XENON
    - DarkSide
    - LUX, SuperCDM
  - Indirect Astronomical observations like high energy cosmic rays
    - Pierre Auger Observatory
  - Production by the accelerator such as LHC

The null results from the detection of standard WIMP particles suggest new perspectives maybe needed.

A classical "top-down" example in physics

What is the typical speed of electron?

At the scale of electron, we have three fundamental constants

Vacuum permittivity	$\varepsilon_0 \sim s^4 \cdot A^2 \cdot kg^{-1} \cdot m^{-3}$	Required by Coulomb force
Elementary charge	$e \sim A \cdot s$	
Planck constant	$\hbar \sim m^2 \cdot kg \cdot s^{-1}$	Required by quantum effect

Even if the detail of physics is unknown, we can use simple dimensional analysis to predict the electron speed:

Electron speed: $v_e \propto e^2 / \varepsilon_0 \hbar \sim m \cdot s^{-1}$

Goal: can we apply similar method (by identifying key constants) to find dark matter particle properties ??

If we know the physics:

$$m_e v_e r_e = \hbar$$ Heisenberg's uncertainty principle

$$\frac{e^2}{4\pi\varepsilon_0 r_e} = m_e v_e^2$$ Virial theorem

Potential energy — Kinetic energy

More accurate electron speed: $v_e = \dfrac{e^2}{4\pi\varepsilon_0\hbar}$

Sommerfeld's interpretation of the fine structure constant:

$$\alpha = \frac{v_e}{c} = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}$$

What we need for predicting DM particle mass?

What is the mass of dark matter particles?

At the scale of DM particle, Assumptions:

- Only gravity is present without any other known interactions involved;
- DM particles still exhibit the wave-particle duality on the quantum level;

Then we have at least two fundamental constants:

Gravitational constant	$G \sim m^3 \cdot s^{-2} \cdot kg^{-1}$ ←	Required by Newtonian gravity
Planck constant	$\hbar \sim m^2 \cdot kg \cdot s^{-1}$ ←	Required by quantum effect

Dimensional analysis points out:

- No matter how you combine two constants, you cannot get mass;
- These two constants are not sufficient to solve problem;

Then what is the other constant besides these two?

This additional constant might come from the properties of dark matter flow.

Energy cascade in hydrodynamic turbulence

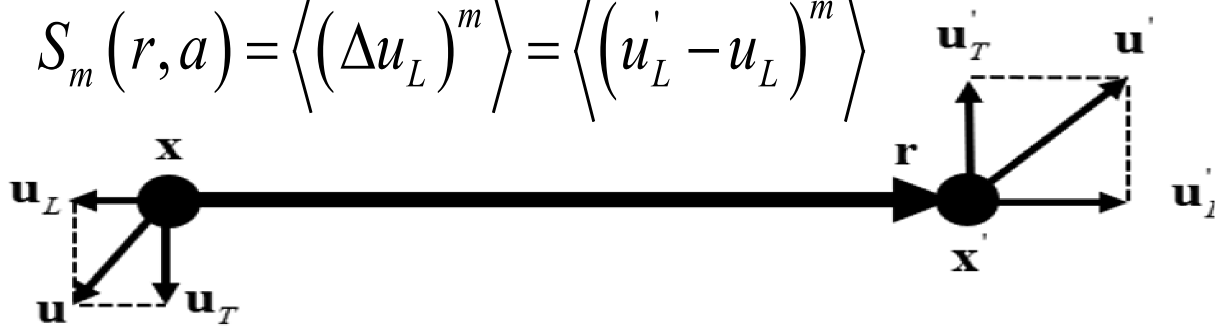
- There exist an inertial range with a scale-independent rate of energy cascade (ε does not depend on eddy size *l*) for eddy size η< *l* <L. η is a dissipative scale determined by viscosity ν and ε.

- In inertial range, inertial force is dominant over viscous force. A general scaling for velocity structure functions $S_m(r)$ for pairwise velocity Δu_L can be identified:

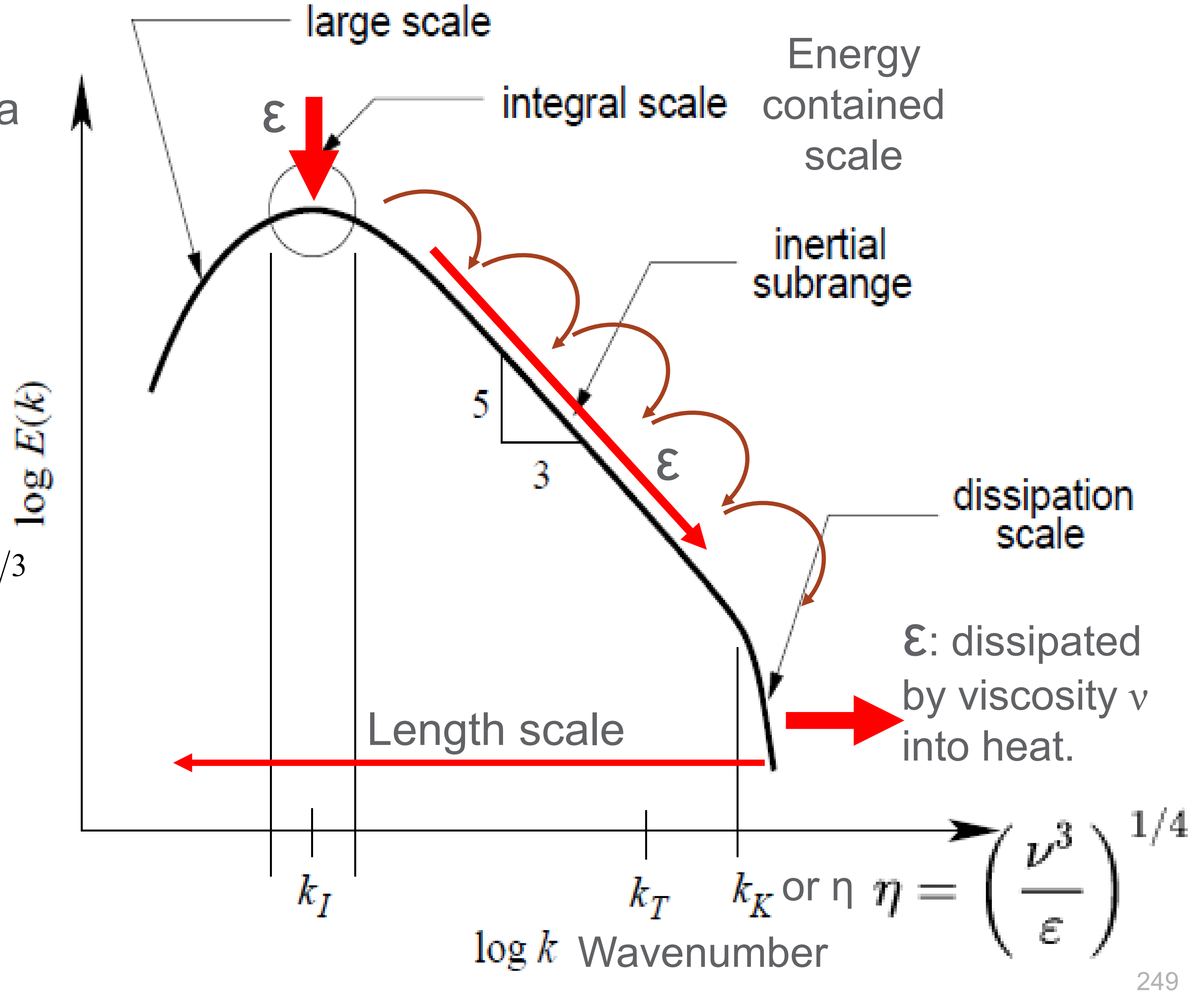
$$S_m(r) \propto (\varepsilon_u)^{m/3} r^{m/3} \xrightarrow{m=2} S_2 \propto (-\varepsilon_u)^{2/3} r^{2/3}$$

Two-thirds law

$$S_m(r,a) = \left\langle (\Delta u_L)^m \right\rangle = \left\langle (u_L' - u_L)^m \right\rangle$$



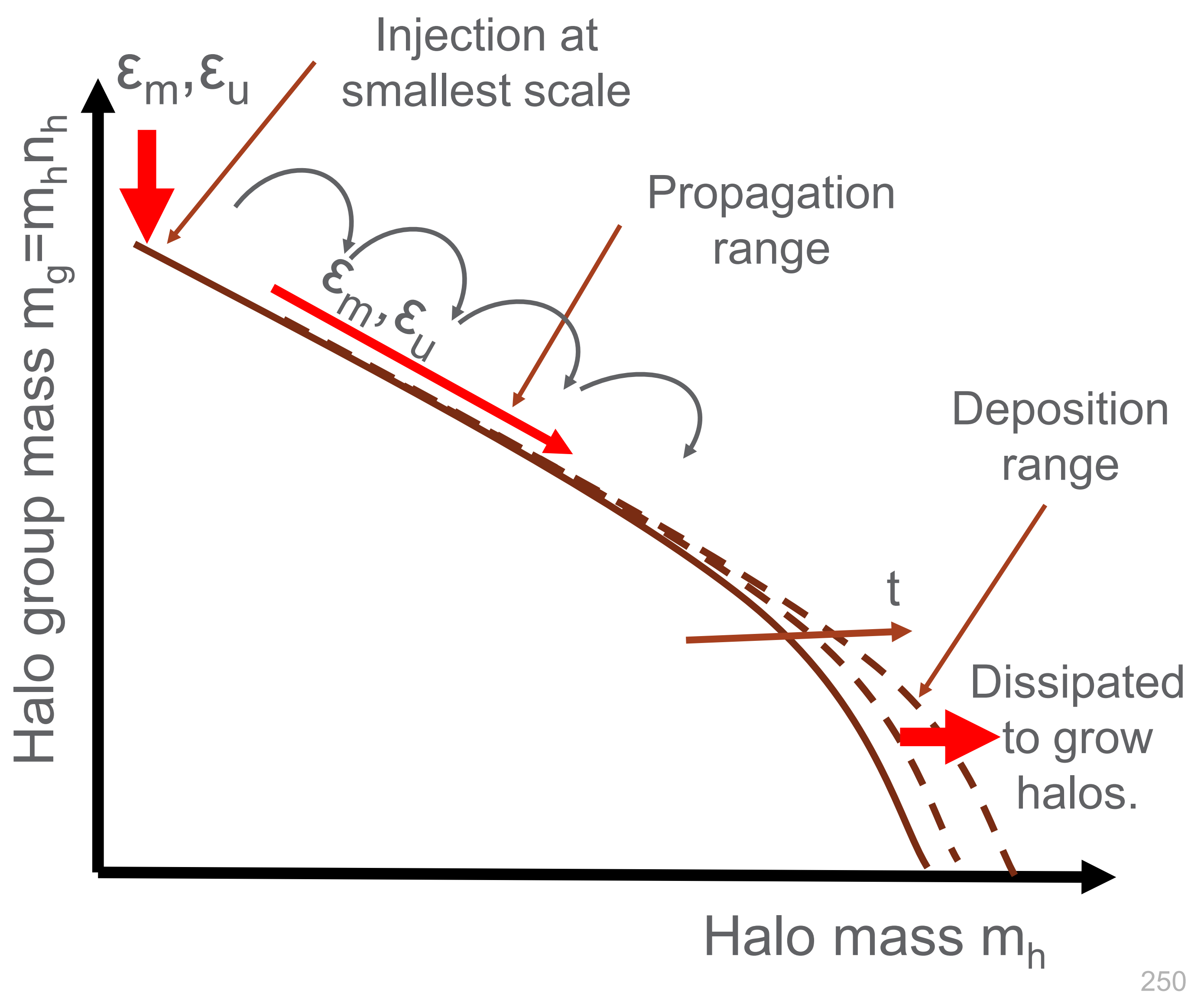
Big whirls have little whirls, That feed on their velocity;
And little whirls have lesser whirls, And so on to viscosity.



Mass/Energy cascade in dark matter flow (SG-CFD)

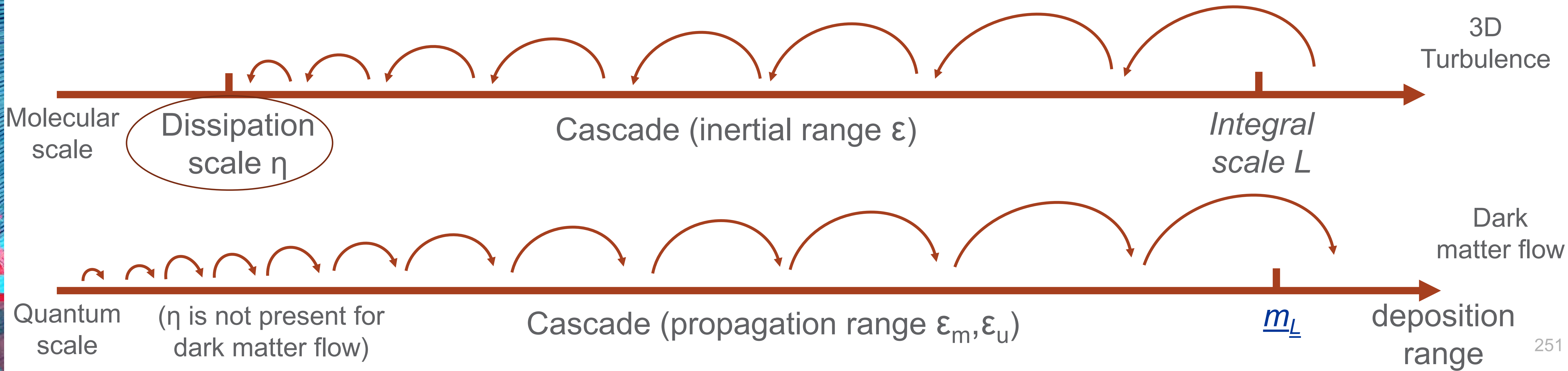
- Collisionless nature and long-range interaction.
- Long-range gravity requires a broad spectrum of halos to be formed to maximize system entropy.
- A continuous cascade of mass/energy from smaller to larger mass scales with a scale-independent rate of mass transfer ε_m and ε_u in a certain range of mass scales (propagation range).
- The mass/energy cascade is an intermediate statistically steady state for non-equilibrium systems to continuously maximize system entropy.
- The maximum entropy distribution of dark matter flow.

Little halos have big halos, That feed on their mass;
And big halos have greater halos, And so on to growth.



Mass/Energy cascade in dark matter flow (SG-CFD)

- Collisionless, no dissipation range in SG-CFD.
- The smallest length scale of inertial range is not limited by viscosity.
- This enable us to extend the scale-independent ε_u down to the smallest scale, where quantum effects become important
- Dark matter flow exhibits scale-dependent flow behaviors for peculiar velocity, i.e. a constant divergence flow on small scales and an irrotational flow on large scales.
- The constant divergence flow shares the same even order kinematic relations with those of incompressible (divergence free) flow. This hints to similar scaling laws holds for dark matter.



Constant (time and scale independent) rate of energy cascade

Power-law time evolution for energy in terms of rate of energy cascade ε_u:

$$K_p = -\varepsilon_{\mathbf{u}} t$$

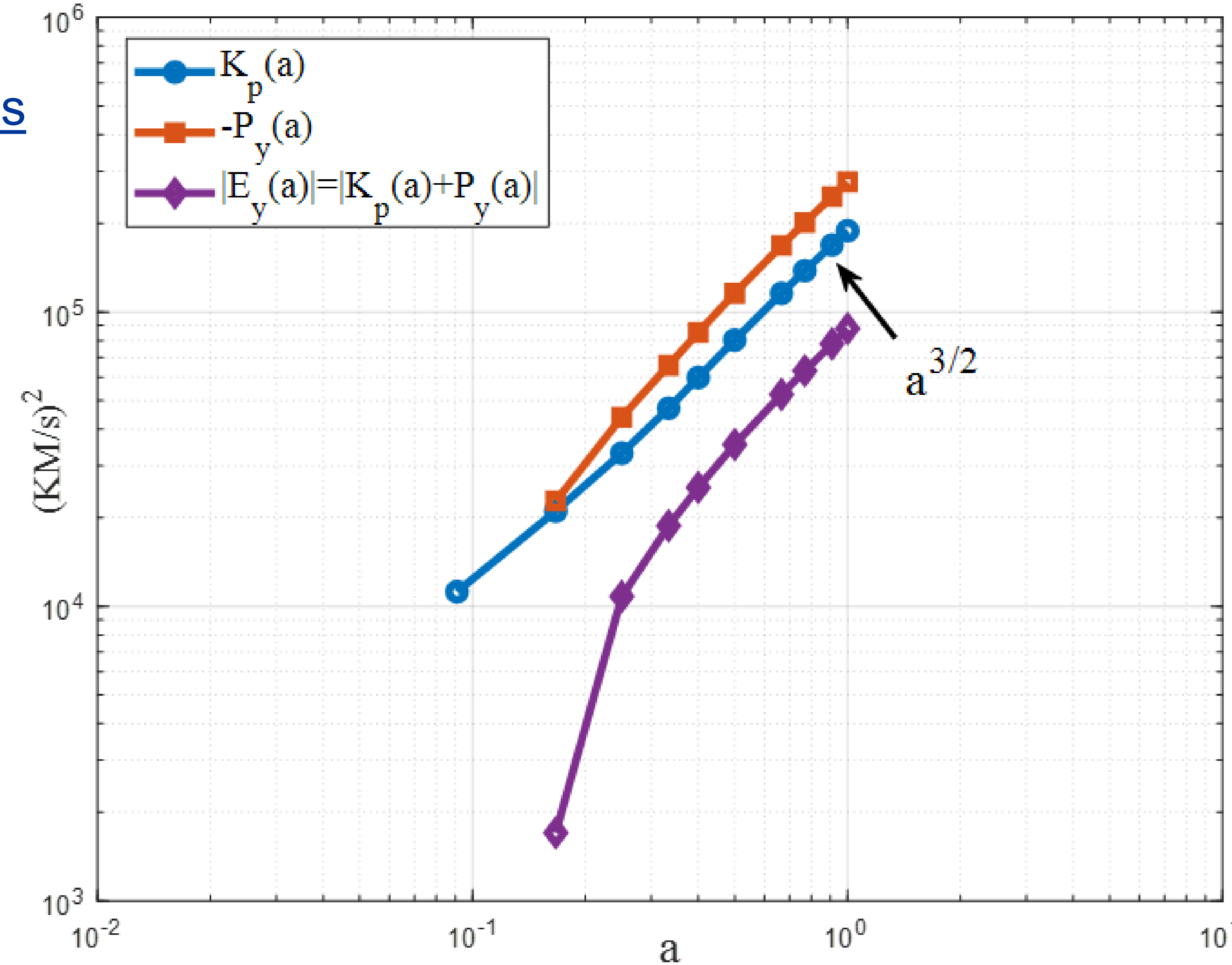
Power-law for Peculiar kinetic energy

$$P_y = \frac{7}{5}\varepsilon_{\mathbf{u}} t$$

Power-law for potential energy

$$\varepsilon_u = -\frac{K_p}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} \approx -4.6\times10^{-7}\,\frac{m^2}{s^3}$$

Also see detail analysis for inverse kinetic energy cascade.



The time variation of specific kinetic and potential energies from *N*-body simulation.

The two-thirds law on small scales

Odd order moment (generalized stable clustering hypothesis):

$$S_{2n+1}^{lp}(r) = (2n+1)\, S_1^{lp}(r)\, S_{2n}^{lp}(r) \propto r^1$$

Even order (two-thirds law):

$$S_{2n}^{lp}(r) - 2^n u^{2n} K_{2n}(\Delta u_L, 0) = \beta_{2n}^*\left(r/r_s\right)^{2/3} \propto r^{2/3}$$

Second order (two-thirds law):

$$S_2^{lp}(r) - 2u^2 = S_{2r}^{lp} = \beta_2^*\left(r/r_s\right)^{2/3} \propto r^{2/3}$$

Introduce a velocity scale:

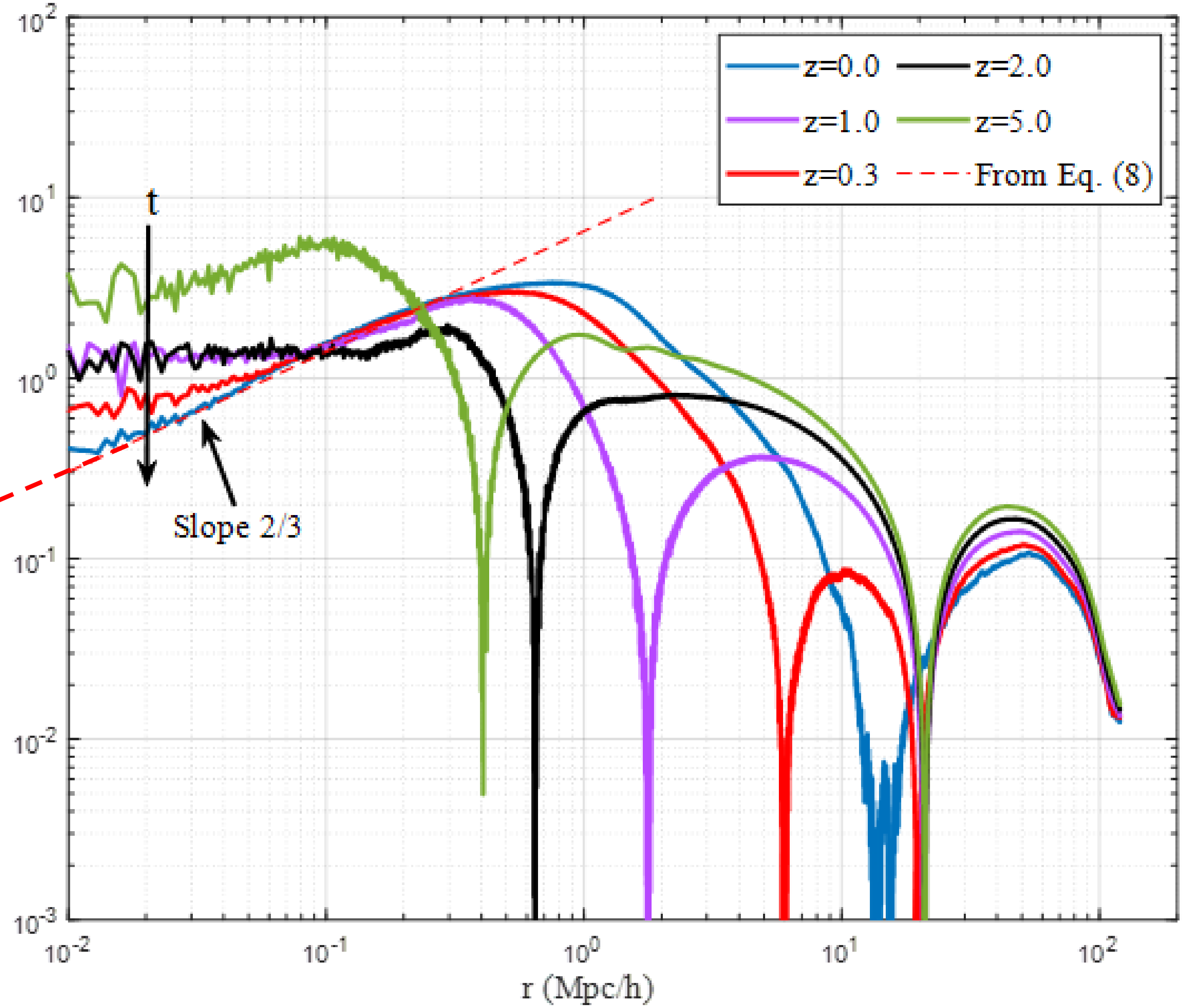
$$v_l^2 = S_{2r}^{lp}(r) \Big/ \left(2^{2/3}\beta_2^* a^{3/2}\right)$$

$$(-\varepsilon_u) = \frac{2v_l^2}{r} v_l = \frac{2v_l^2}{r/v_l}$$

Acceleration

Turnaround time

Extend all the way to the smallest scale



Variation of normalized reduced longitudinal structure function and two-thirds law

Postulating dark matter particle mass and properties

At the smallest scale, we have three fundamental constants:

Gravitational constant	$G = 6.67\times10^{-11}\, m^3/(kg\cdot s^2)$	Required by Newtonian gravity
Rate of energy cascade	$\varepsilon_u = -4.6\times10^{-7}\, m^2/s^3$	Required by dark matter flow
Planck constant	$\hbar = 1.05\times10^{-34}\, kg\cdot m^2/s$	Required by quantum effect

If we know the physics:

$m_X v_X \cdot l_X/2 = \hbar$	Heisenberg's uncertainty principle
$a_X \cdot v_X = -\varepsilon_u$	Energy cascade (two-thirds law)
$Gm_X/l_X^2 = a_X$	Acceleration
$Gm_X/l_X = 2v_X^2$	Virial theorem

Even if the detail of physics is unknown, we can use simple dimensional analysis to predict :

Mass scale: $m_X \propto \left(-\varepsilon_u \hbar^5/G^4\right)^{\frac{1}{9}}$

Length scale: $l_X \propto \left(-G\hbar/\varepsilon_u\right)^{\frac{1}{3}}$

Time scale: $t_X \propto \left(G^2\hbar^2/\varepsilon_u^5\right)^{\frac{1}{9}}$

$$m_X = \left(-256\varepsilon_u \hbar^5/G^4\right)^{\frac{1}{9}} = 1.62\times10^{-15}\,kg = 0.90\times10^{12}\,GeV$$

$$l_X = \left(-2G\hbar/\varepsilon_u\right)^{\frac{1}{3}} = 3.12\times10^{-13}\,m$$

$$t_X = l_X/v_X = \left(-32G^2\hbar^2/\varepsilon_u^5\right)^{\frac{1}{9}} = 7.51\times10^{-7}\,s$$

Velocity scale: $v_X = \left(\varepsilon_u^2 \hbar G/4\right)^{\frac{1}{9}} = 4.16\times10^{-7}\,m/s$

Acceleration scale: $a_X = \left(-4\varepsilon_u^7/(\hbar G)\right)^{\frac{1}{9}} = 1.11\,m/s^2$

Postulating dark matter particle mass and properties

Density scale: $m_X / l_X^3 \approx 5.33\times10^{22}\, kg/m^3$ ⟷ Nuclear density: $10^{17}\, kg/m^3$

Power scale (Joule/s):

$$\mu_X = m_X a_X \cdot v_X = F_X \cdot v_X = -m_X \varepsilon_u = \left(-\frac{256\varepsilon_u^{10}\hbar^5}{G^4}\right)^{\frac{1}{9}} = 7.44\times10^{-22}\, kg\cdot m^2/s^3 = 0.0046\, eV/s$$

Energy scale: $\mu_X t_X / 4 = \hbar / t_X = \frac{1}{2} m_X v_X^2 = 0.87\times10^{-9}\, eV$

Rydberg energy of 13.6 eV for the ionization energy of the hydrogen atom

Particle lifetime: $\tau_X = \frac{m_X c^2}{\mu_X} = -\frac{c^2}{\varepsilon_u} = 2\times10^{23}\, s = 6.2\times10^{15}\, yr \approx \frac{\hbar e^{1/\alpha_X}}{m_X c^2}$

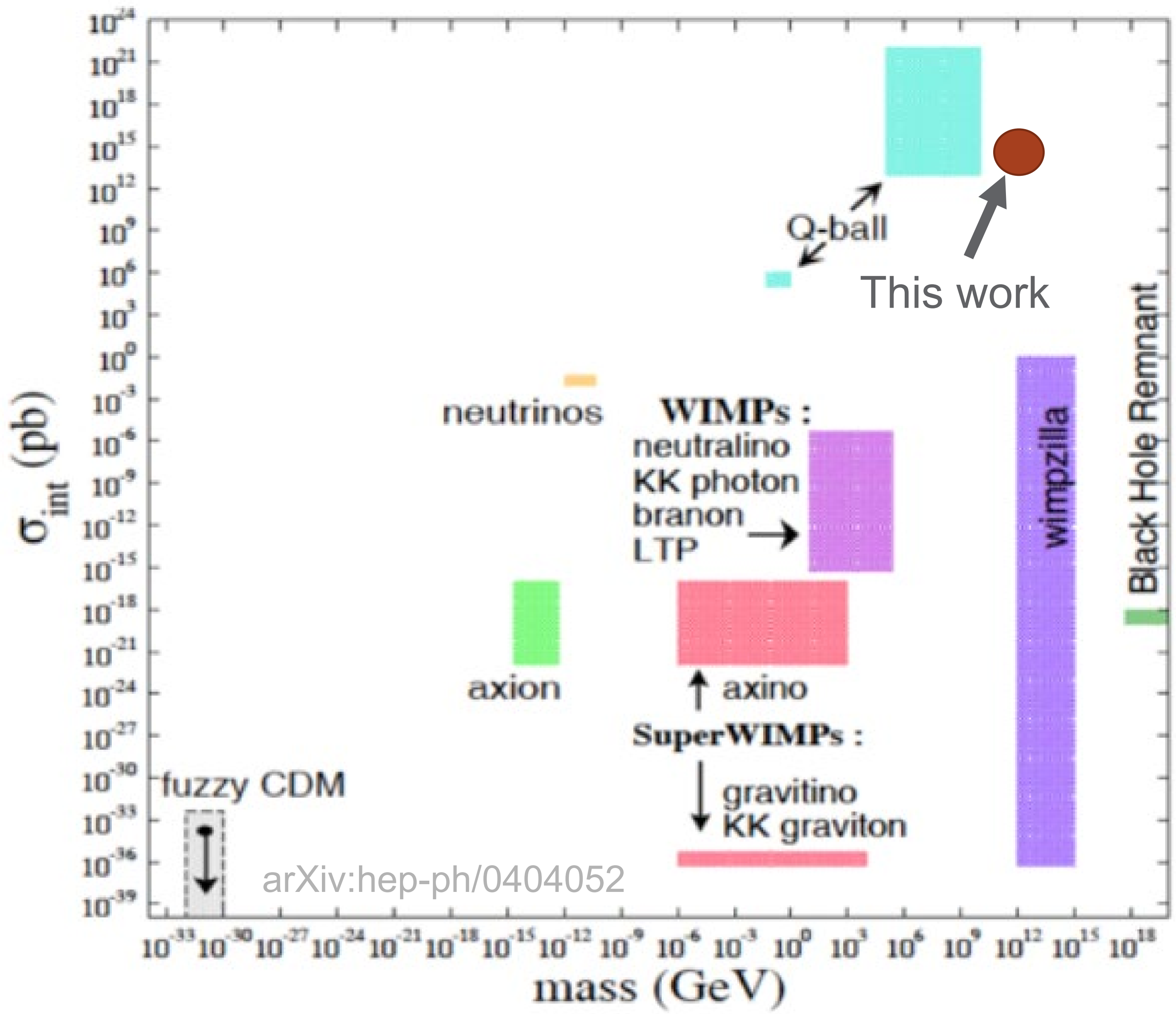
If instantons are responsible for the decay [1]:

$$\tau_X = \frac{\hbar e^{1/\alpha_X}}{m_X c^2} = 6.2\times10^{15}\, yr$$

➡ Fine structure constant: $\alpha_X \approx \frac{1}{136.85}$

[1] Anchordoqui, L.A., et al.,Astroparticle Physics, 2021. 132.

Where is our prediction?



Summary and keywords

Dark matter flow	Mass/energy cascade	Dark matter particle mass
Two-thirds law	Rate of energy cascade	Fine structure constant

- Establish connections between dark matter flow and hydrodynamic turbulence.
- Review direct energy cascade from large to small scales in hydrodynamic turbulence with the smallest length scale η determined by viscosity and the rate of cascade ε.
- Review the inverse energy cascade in dark matter flow from small to large mass scales with a constant rate of energy cascade.
- Two-thirds law for pairwise velocity dispersion on small scale r.
- The collisionless nature of dark matter flow enables us to extend constant rate of cascade and two-thirds law down to the smallest scale where quantum effects are dominant.
- Suggests a heavy dark matter scenario by combining rate of energy cascade, Planck constant, and gravitational constant to predict dark matter particles with a mass ~0.9x1012 GeV and a size ~3x10-13 m.

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The origin of MOND acceleration from mass and energy cascade in dark matter flow

Xu Z., 2022, arXiv:2203.05606v1 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2203.05606

Introduction

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  - Flat rotation curves of spiral galaxies
  - Motion of galaxies in galaxy clusters
  - Gravitational lensing
  - Bullet clusters, CMB ……

- Though the nature of dark matter is still unclear, dark matter is believed to be cold (non-relativistic), collisionless, dissipationless, non-baryonic, barely interacting with baryonic matter except through gravity, and sufficiently smooth with a fluid-like behavior.

- However, no conclusive signals have been detected in searches for dark matter particles.

- Alternative theory of dark matter: Modified Newtonian Dynamics (MOND)

- Empirical Tully and Fisher relation:

$$v_f \propto M^{1/4} \Leftarrow \text{ observed baryonic mass}$$

- MOND (Milgrom) is a popular empirical model to reproduce flat rotation curve without invoking dark matter hypothesis.

$a_0 \approx 1.2\times 10^{-10}\, m/s^2$ Critical MOND acceleration

$F = ma \qquad a \gg a_0$ Newtonian

$F = m a^2/a_0 \propto a^2 \qquad a \ll a_0$ Deep MOND

$$\frac{GMm}{r^2} = m\frac{\left(v_f^2/r\right)^2}{a_0} \Rightarrow v_f = \left(GMa_0\right)^{1/4}$$

- What is the origin of MOND acceleration?
- What is the origin of deep "MOND" behavior?
- Could MOND be an intrinsic property of dark matter flow?
- Instead of falsifying, MOND supports the existence of dark matter?

Hydrodynamic turbulence vs. dark matter flow

Key attributes of hydrodynamic turbulence:

- Disorganized, chaotic, random;
- Nonrepeatability (sensitivity to initial cond.);
- Multiscale in length and time scales;
- Intermittency in space and time;

- Dissipative and collisional
- No long-range interaction
- Velocity fluctuation
- Vortex as fundamental building block
- Maximum entropy distribution (Gaussian)
- Incompressible on all scales
  - Divergence-free $\nabla \cdot \mathbf{v} = 0$
  - Constant density
- Energy cascade from large to small length scales

Key attributes of dark matter flow:

- Disorganized, chaotic, random;
- Nonrepeatability;
- Multiscale in mass/length/time scales;
- Intermittency in space and time;

- Dissipationless and collisionless
- Long-range gravity
- Velocity & acceleration fluctuation
- Halos as fundamental building block
- Maximum entropy distribution? (the X dist.)
- Flow behavior is scale-dependent
  - Small scale: constant divergence $\nabla \cdot \mathbf{v} = \theta$
  - Large scale: irrotational (curl-free) $\nabla \times \mathbf{v} = 0$
- Mass/energy cascade from small to large mass scales

MOND acceleration

Deep MOND

Energy cascade in hydrodynamic turbulence

- There exist an inertial range with a scale-independent rate of energy cascade (ε does not depend on eddy size l) for eddy size $\eta < l < L$. η is a dissipative scale determined by viscosity ν and ε.

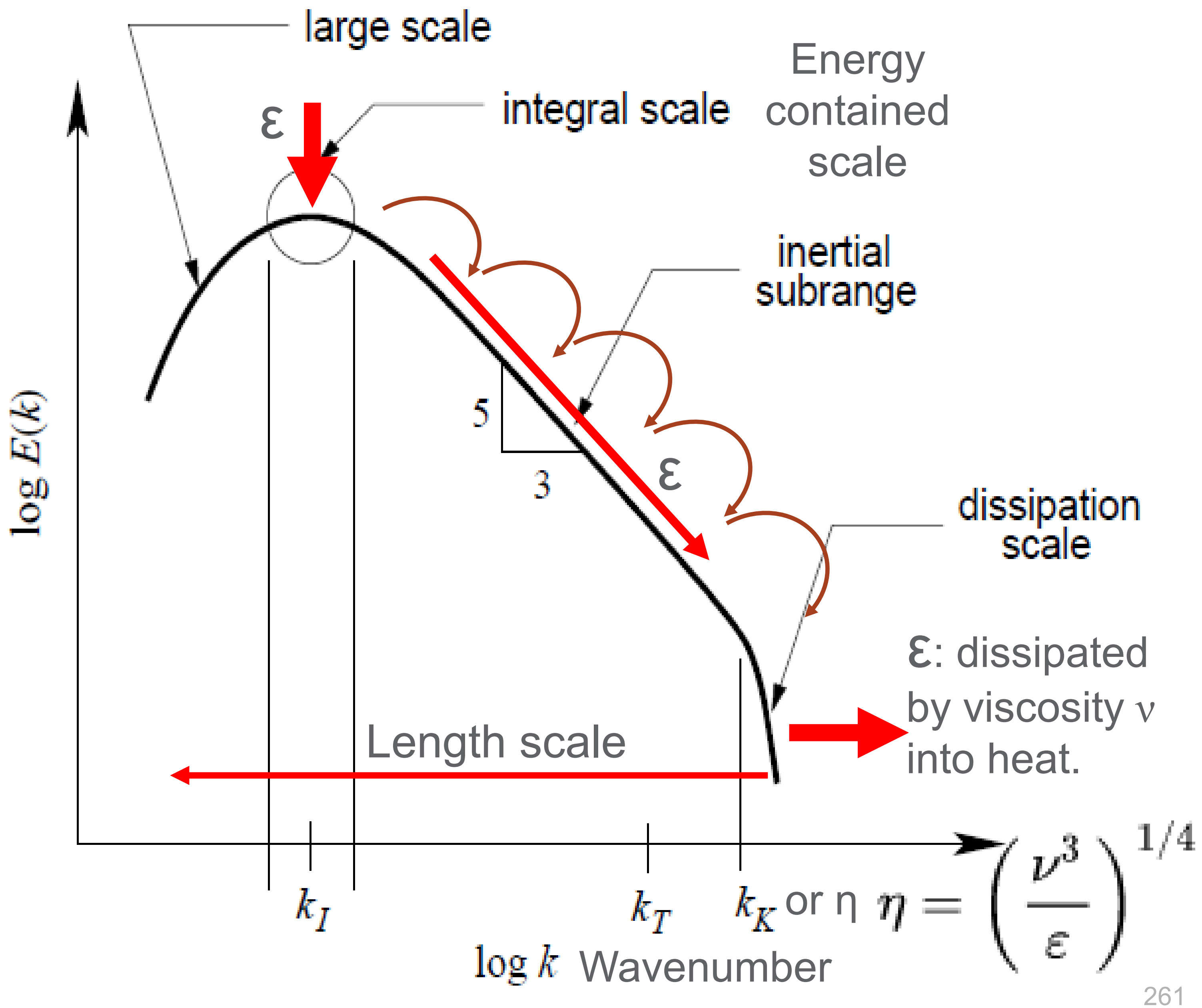
- In this range, inertial force is dominant over viscous force. For eddies with a characteristic velocity u and size l , the lifetime (turnaround time) of eddy is l/u. The rate ε can be computed as the kinetic energy passed per eddy lifetime.

$$\varepsilon \approx \frac{u^2}{(l/u)} \approx \frac{u^2}{l}u \;\Rightarrow\; u^3 \propto l$$

turnaround time

acceleration

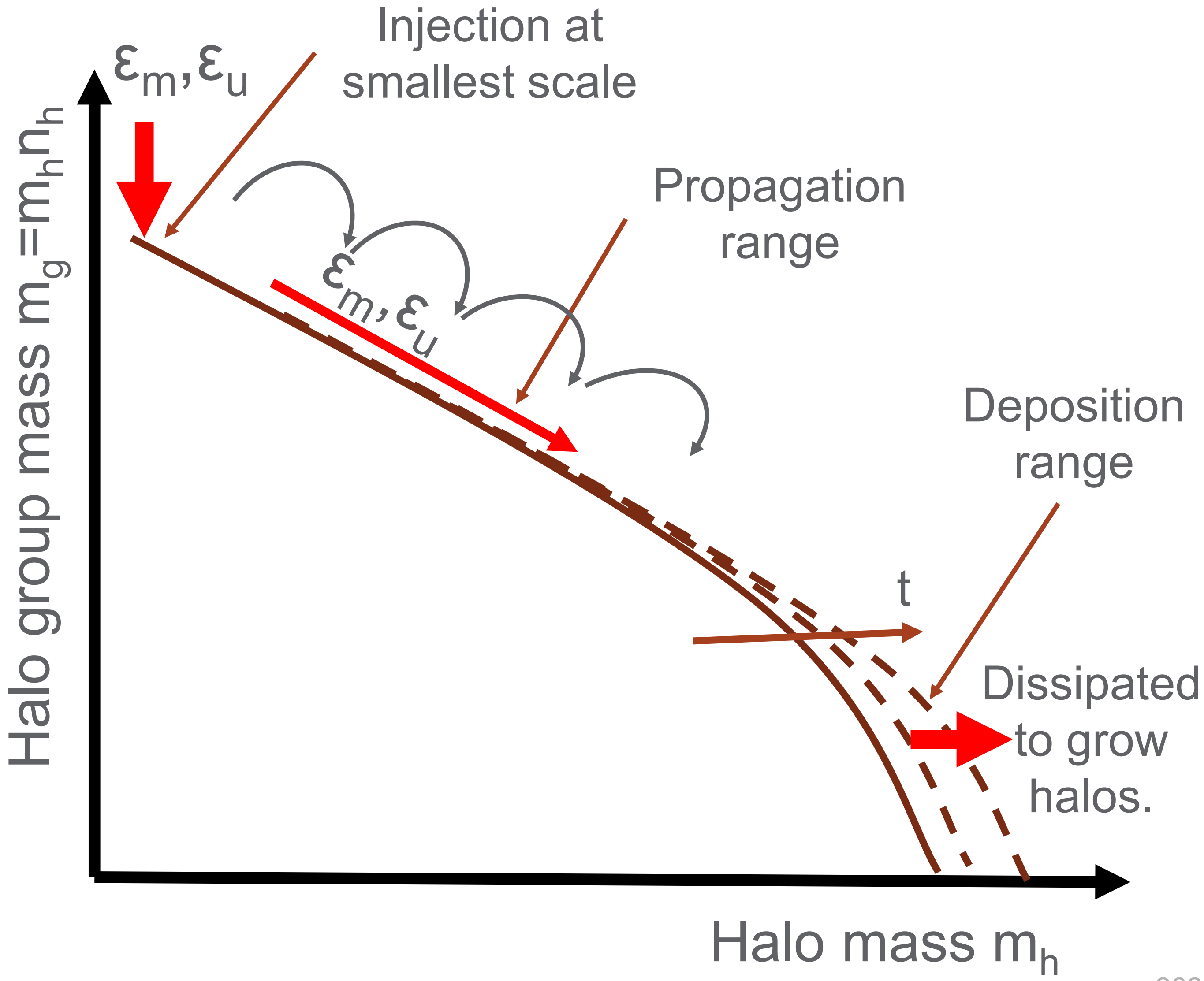
Big whirls have little whirls, That feed on their velocity;
And little whirls have lesser whirls, And so on to viscosity.



Mass/Energy cascade in dark matter flow (SG-CFD)

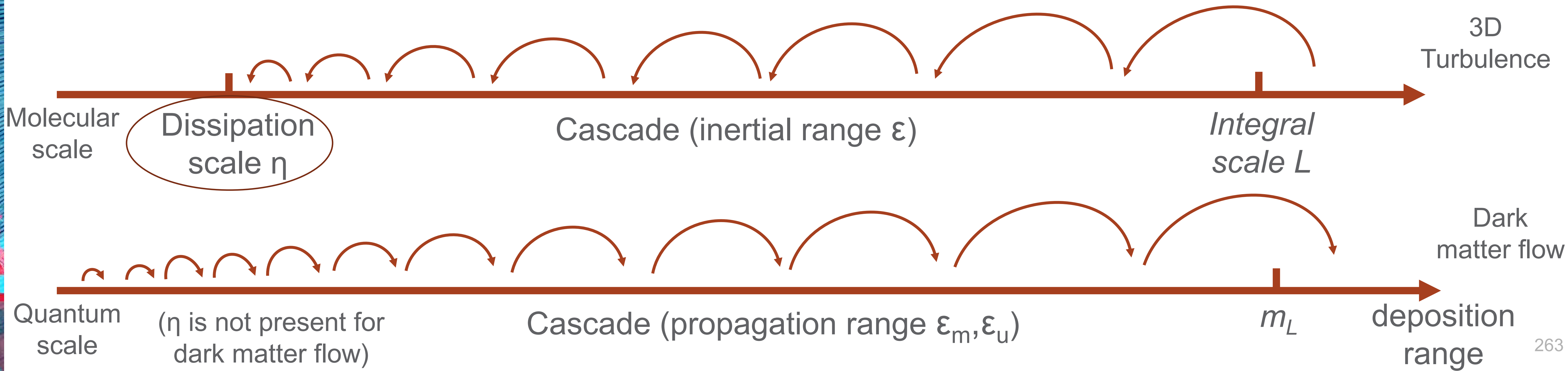
- Collisionless nature and long-range interaction.
- Long-range gravity requires a broad spectrum of halos to be formed to maximize system entropy. No halo structure for short-range forces.
- A continuous cascade of mass/energy from smaller to larger mass scales with a scale-independent rate of mass transfer ε_m and ε_u in a certain range of mass scales (propagation range).
- The mass/energy cascade is an intermediate statistically steady state for non-equilibrium systems to continuously maximize system entropy.
- The maximum entropy distribution of dark matter flow (the X distribution).

Little halos have big halos, That feed on their mass;
And big halos have greater halos, And so on to growth.



Mass/Energy cascade in dark matter flow (SG-CFD)

- Collisionless, no dissipation range in SG-CFD.
- The smallest length scale of inertial range is not limited by viscosity.
- This enable us to extend the scale-independent ε_u down to the smallest scale, where quantum effects become important
- Dark matter flow exhibits scale-dependent flow behaviors for peculiar velocity, i.e. a constant divergence flow on small scales and an irrotational flow on large scales.
- The constant divergence flow shares the same even order kinematic relations with those of incompressible (divergence free) flow. This hints to similar scaling laws holds for dark matter.



Constant (time and scale independent) rate of energy cascade

[Power-law time evolution for energy in terms of rate of energy cascade]ε_u:

$$K_p = -\varepsilon_{\mathbf{u}} t$$

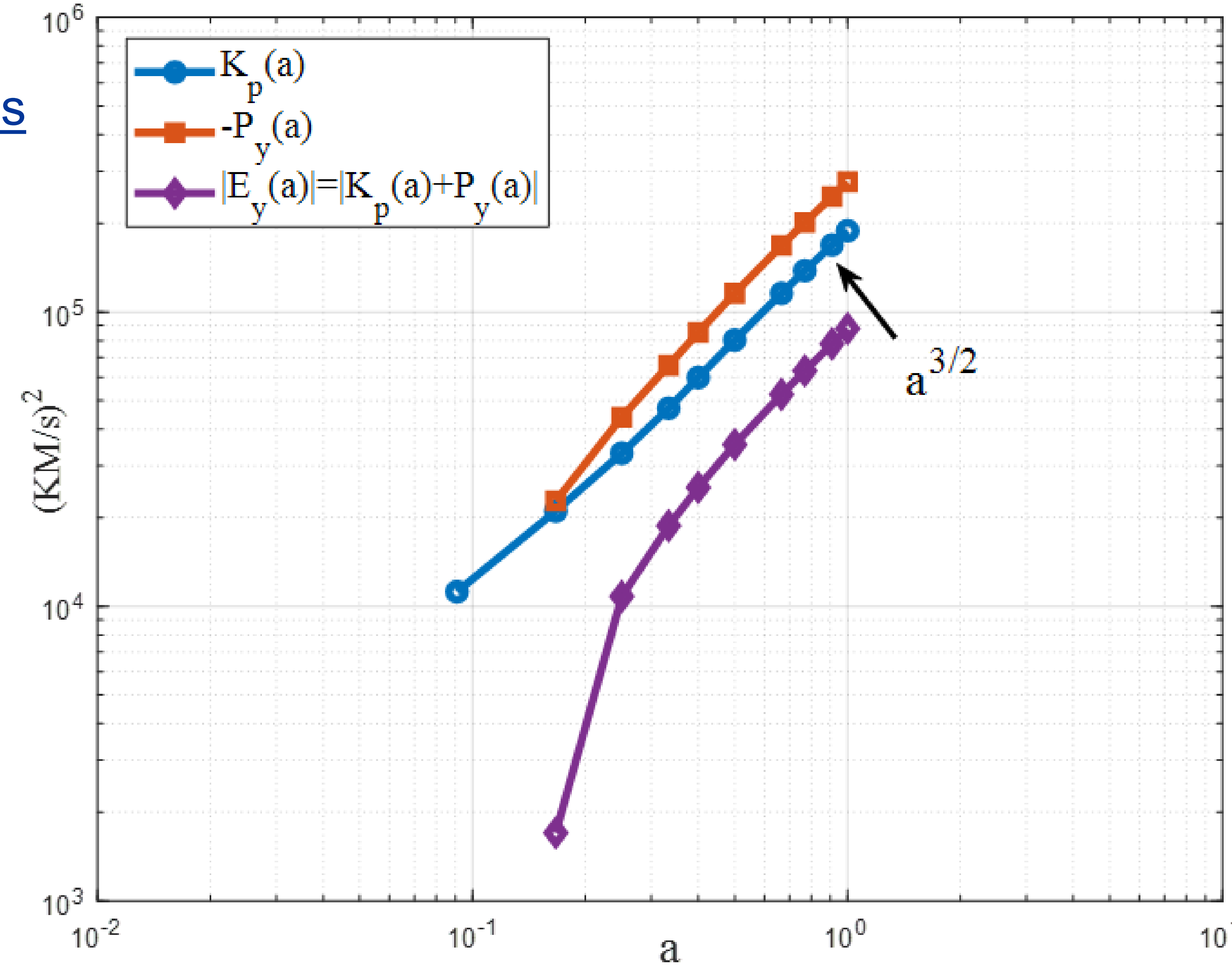
Power-law for Peculiar kinetic energy

$$P_y = \frac{7}{5}\varepsilon_{\mathbf{u}} t$$

Power-law for potential energy

$$\varepsilon_u = -\frac{K_p}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} \approx -4.6\times10^{-7}\frac{m^2}{s^3}$$

[Also see detail analysis for inverse kinetic energy cascade.]



The time variation of specific kinetic and potential energies from *N*-body simulation.

The maximum entropy distribution in dark matter flow

In dark matter flow, the maximum entropy distribution of velocity can be derived as the X distribution:

α: shape parameter;
v_0: velocity scale;

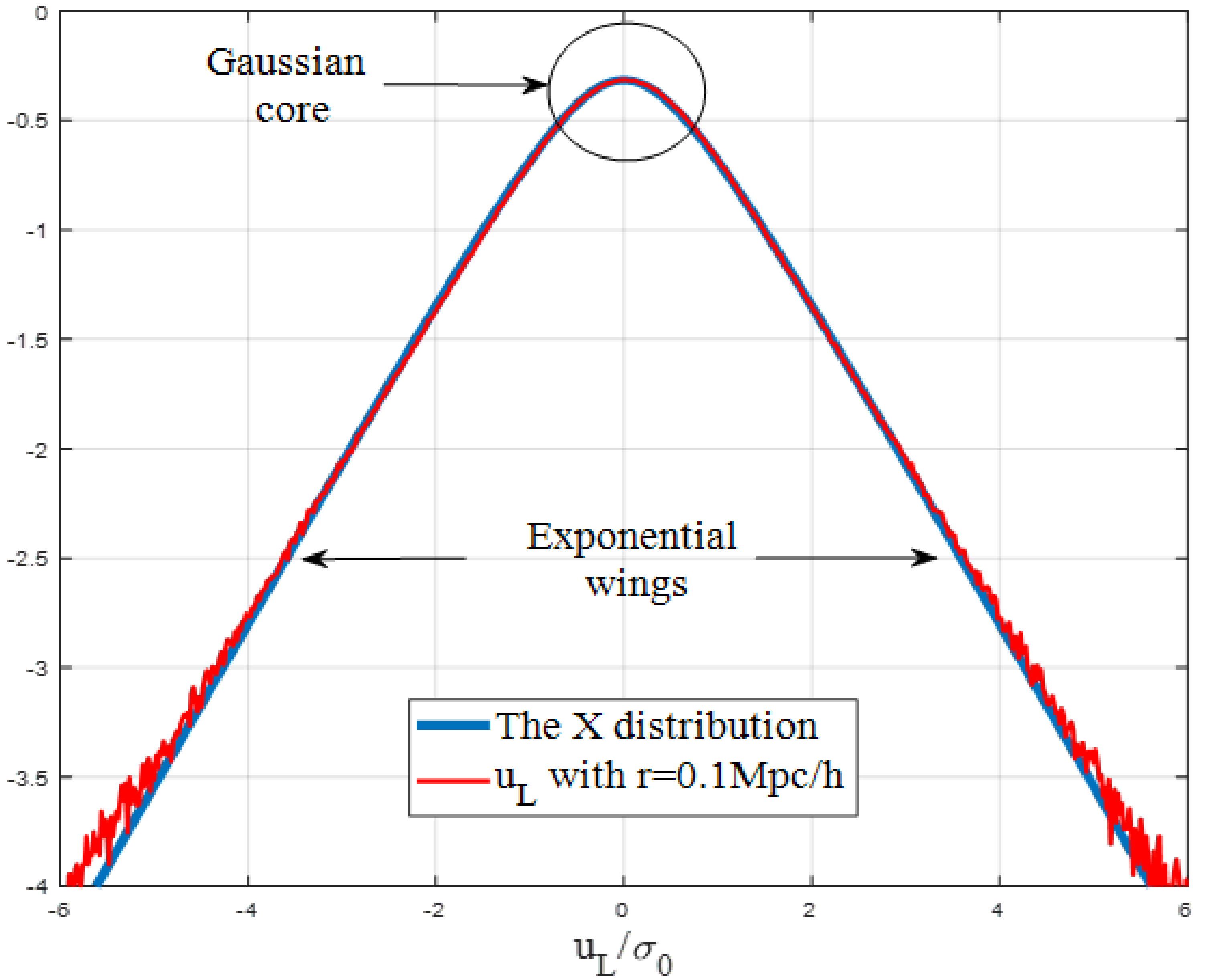
$$X(v)=\frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)}$$

The relation between particle energy and velocity can be obtained from X distribution:

Energy per particle with a speed of v:

$$\varepsilon(v)=-\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

$$\varepsilon(v)=\frac{3}{2}\left(1+\frac{2}{n}\right)v_0^2\sqrt{\alpha^2+\left(\frac{v}{v_0}\right)^2}$$



The *X* distribution with a unit variance compared with the velocity distribution from *N*-body simulation

Particle energy in dark matter flow

$$X(v)=\frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)} \qquad \varepsilon(v)=-\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

Particle energy: $$\varepsilon(v)=\frac{3}{2}\left(1+\frac{2}{n}\right)v_0^2\sqrt{\alpha^2+\left(\frac{v}{v_0}\right)^2}$$

Gaussian core for $|v| \ll v_0$

$$\varepsilon(v)\approx\frac{3}{2}\left(1+\frac{2}{n}\right)\left(\alpha v_0^2+\frac{v^2}{2\alpha}\right)\propto v^2$$

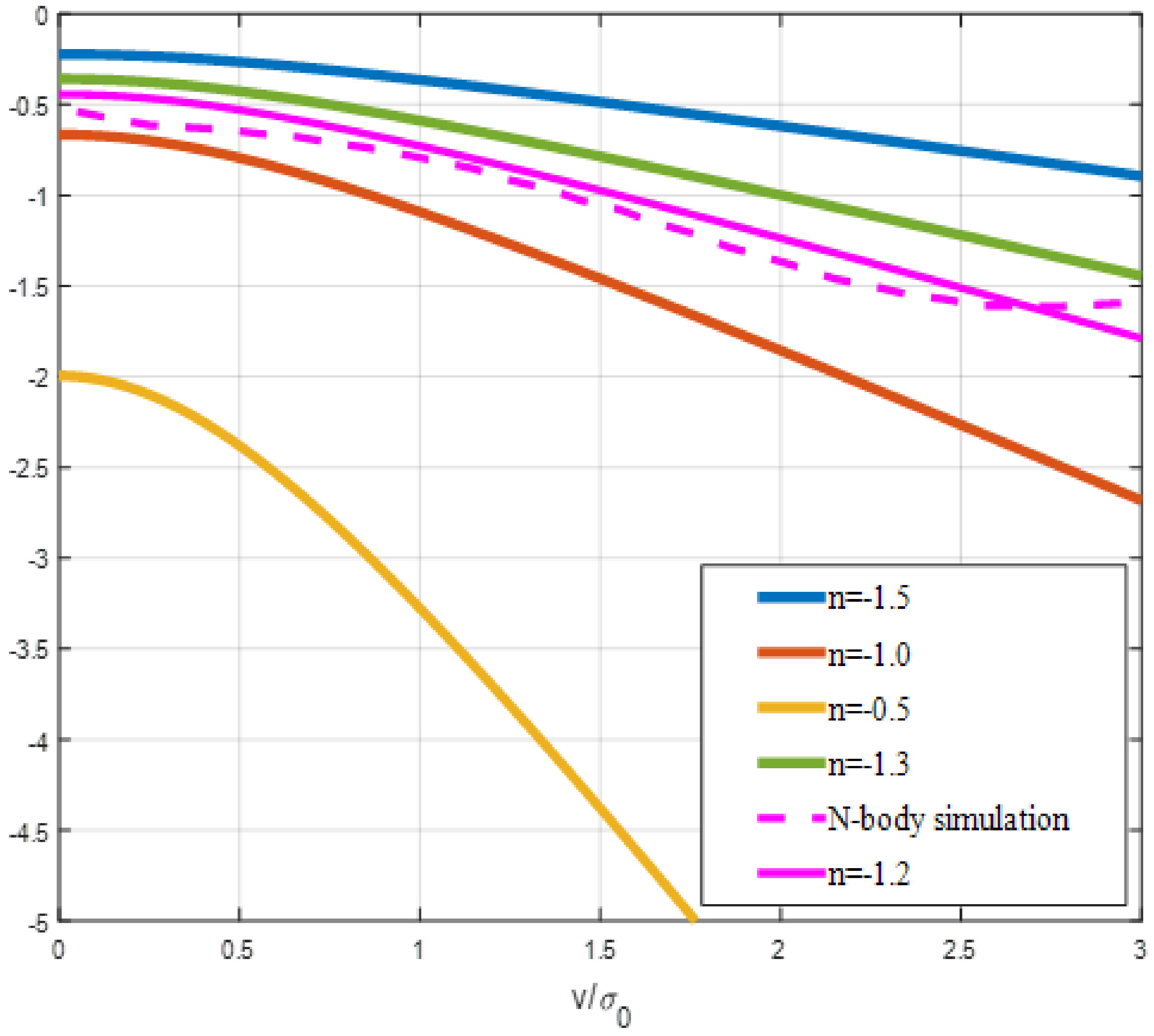
Inner halo, Newtonian behavior

Exponential wings for $|v| \gg v_0$

$$\varepsilon(v)\approx\frac{3}{2}\left(1+\frac{2}{n}\right)v_0 v\propto v$$

Outer region of halo, non-Newtonian behavior

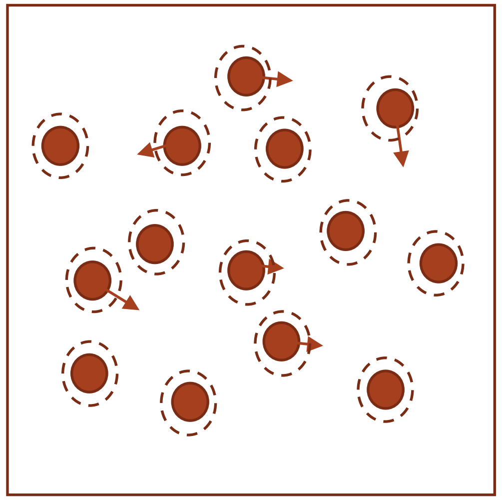
External field effects and MOND??



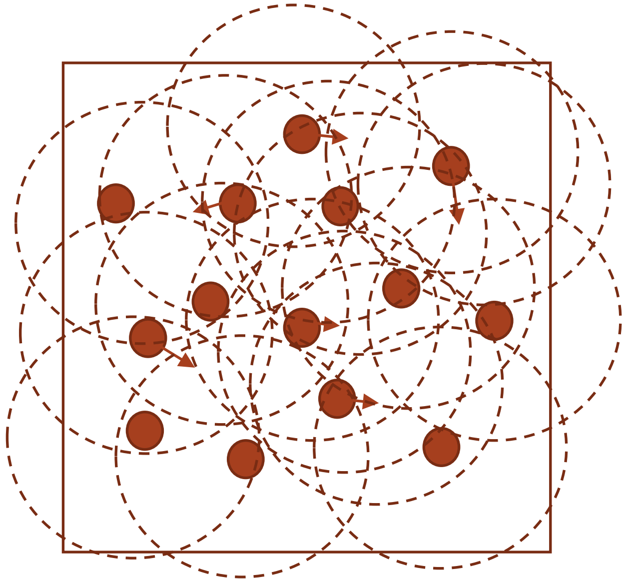
Comparison with N-body simulation

Acceleration fluctuation in dark matter flow

- In kinetic theory of gases, molecules undergo random elastic collisions with a short-range of interaction. Only velocity fluctuation and no fluctuation of acceleration.
- The long-range gravity in dark matter flow leads to fluctuations in acceleration, in addition to the fluctuation in velocity.
- This unique feature hints to the potential generalization of standard Brownian/Langevin dynamics to include acceleration fluctuation in dark matter flow.
- Critical MOND acceleration can be related to the fluctuation of acceleration.



Short range: molecule acceleration vanishes



Long range: nonvanishing and fluctuating acceleration

Acceleration distributions in dark matter flow

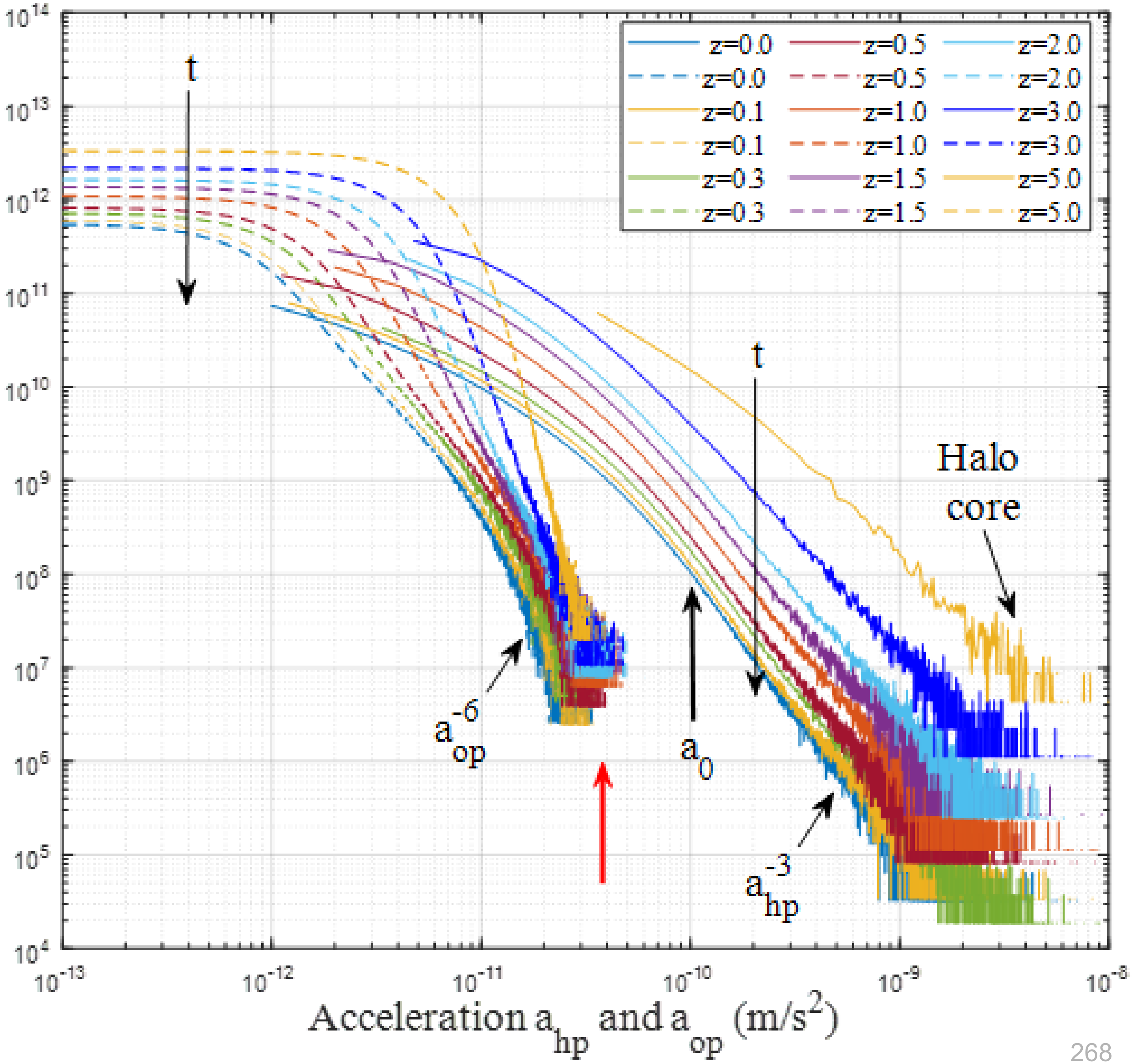
Fluctuation leads to distributions of acceleration

Proper acceleration for particle i:

$$\mathbf{a}_p = \frac{Gm_p}{a^2}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i - \mathbf{x}_j}{\left|\mathbf{x}_i - \mathbf{x}_j\right|^3}$$

Halo-based non-projection approach for acceleration distributions:

- Halo particle acceleration: a_{hp}
- Out-of-halo particles acceleration: a_{op} (Gaussian)
- Acceleration decreases with time
- A long tail $\sim a_{hp}^{-3}$ in halo core region
- MOND acceleration a_0 is right in the middle
- Analytical models of acceleration distribution? (future work)

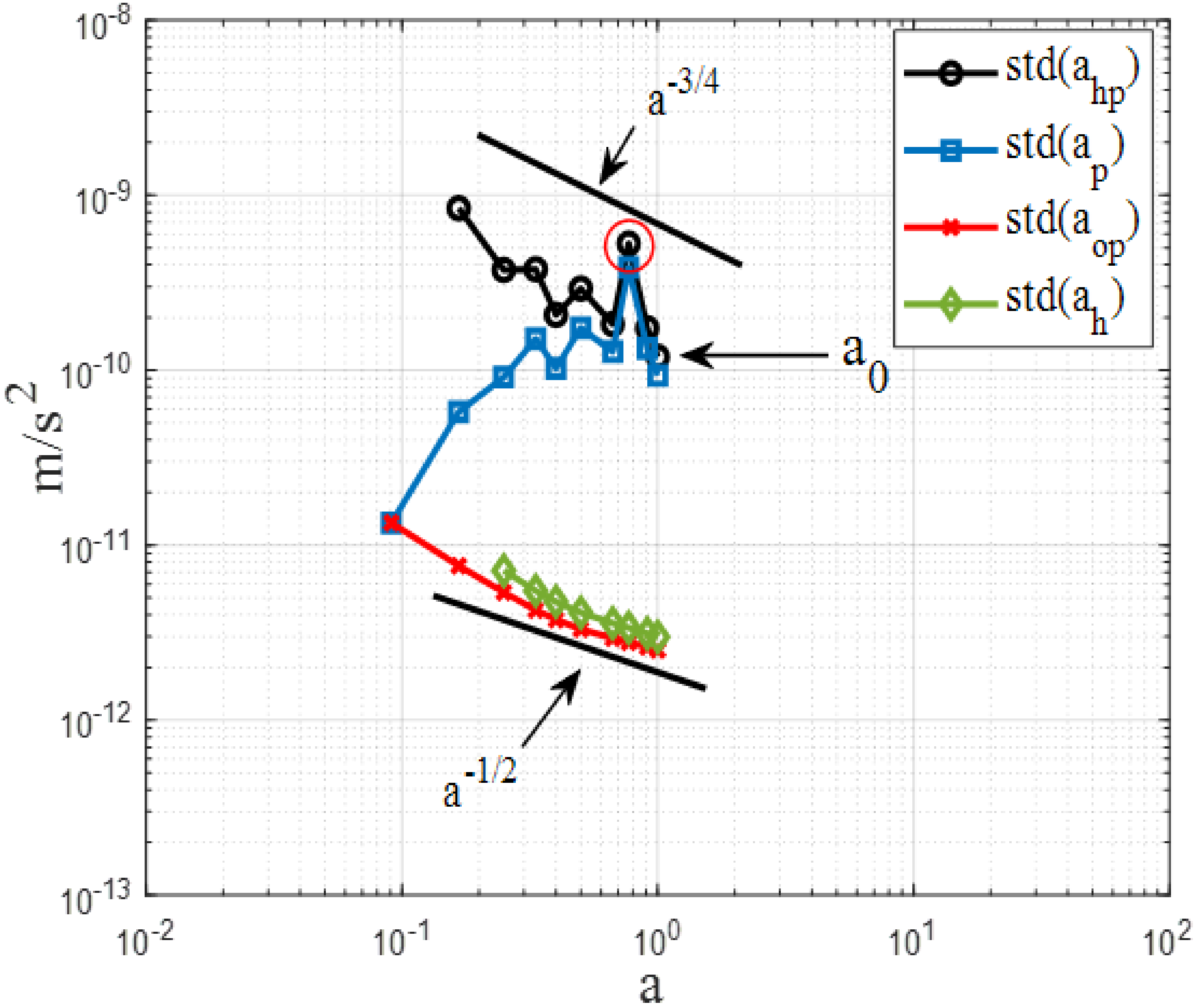


The variation of acceleration with redshift

Halo-based non-projection approach
Root-mean-square accelerations:

- Acceleration of all particles: a_p
- Halo particle acceleration: $a_{hp} \sim a^{-3/4}$
- Out-of-halo particles acceleration: $a_{op} \sim a^{-1/2}$
- Halo acceleration: $a_h \sim a^{-1/2}$

- All typical accelerations decrease with time
- The only exception a_{hp} at z=0.3 requires further confirmation
- Halos and out-of-halo particles have similar accelerations that are much smaller due to greater distance
- At z=0, the typical acceleration of halo particles matches the critical MOND acceleration



The variation of typical (root-mean-square) accelerations with scale factor a

Pacific Northwest NATIONAL LABORATORY

The variation of acceleration with halo size

Acceleration decomposition:
(similar to velocity decomposition)

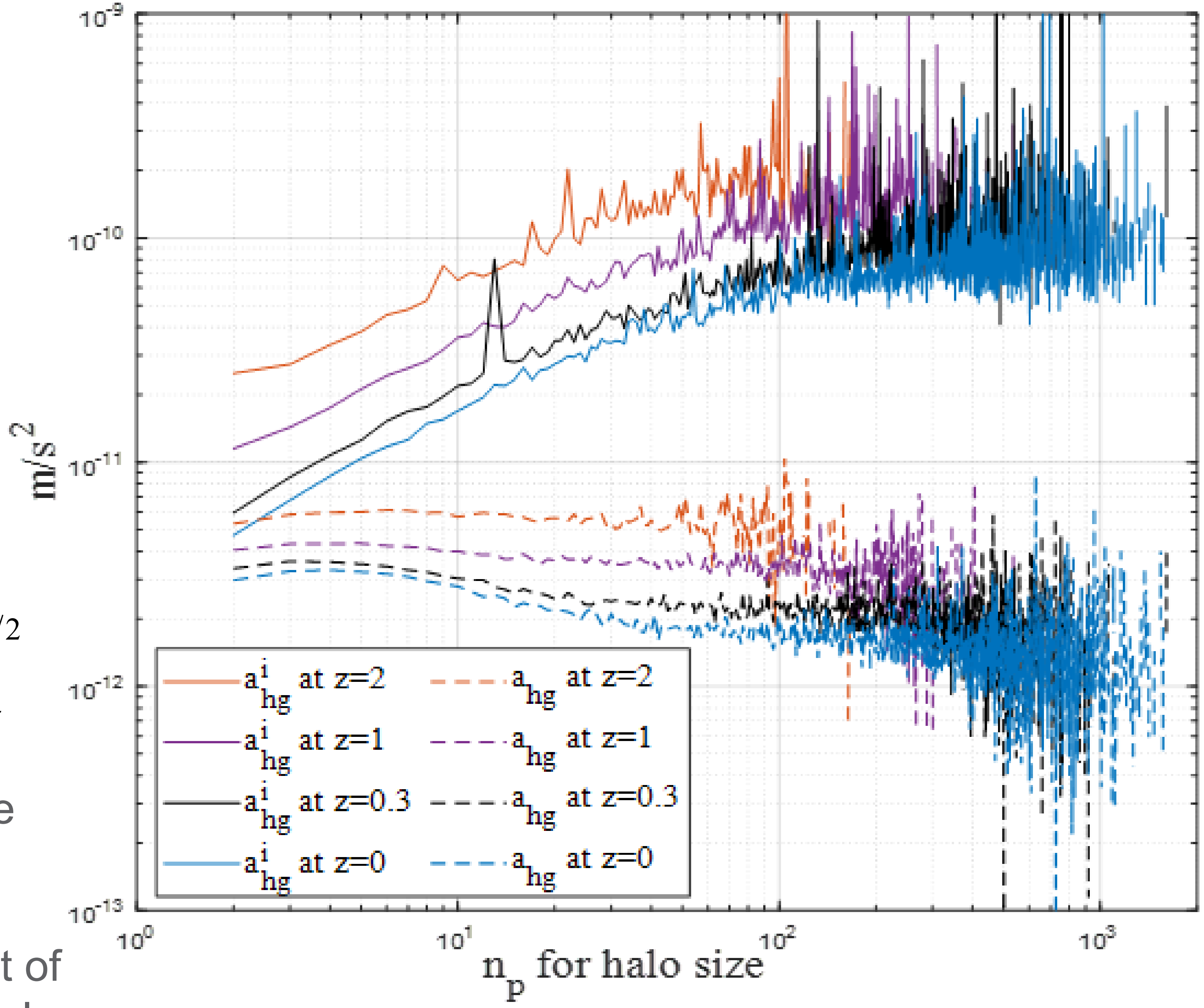
Intra-halo acceleration: $$\mathbf{a}_{hp}^{\mathrm{i}} = \mathbf{a}_{hp} - \left\langle \mathbf{a}_{hp} \right\rangle_h = \mathbf{a}_{hp} - \mathbf{a}_h$$

Halo acceleration (inter-halo): $$\mathbf{a}_h = \left\langle \mathbf{a}_{hp} \right\rangle_h = \frac{1}{n_p} \sum_{k=1}^{n_p} \mathbf{a}_{hp}$$

Group average intra-halo | Inter-halo acceleration

$$a_h^{\mathrm{i}} = \left\langle \left| \mathbf{a}_{hp}^{\mathrm{i}} \right|^2 \right\rangle_h^{1/2} \qquad a_{hg}^{\mathrm{i}} = \left\langle a_h^{\mathrm{i}} \right\rangle_g \qquad a_{hg} = \left\langle \left| \mathbf{a}_h \right|^2 \right\rangle_g^{1/2}$$

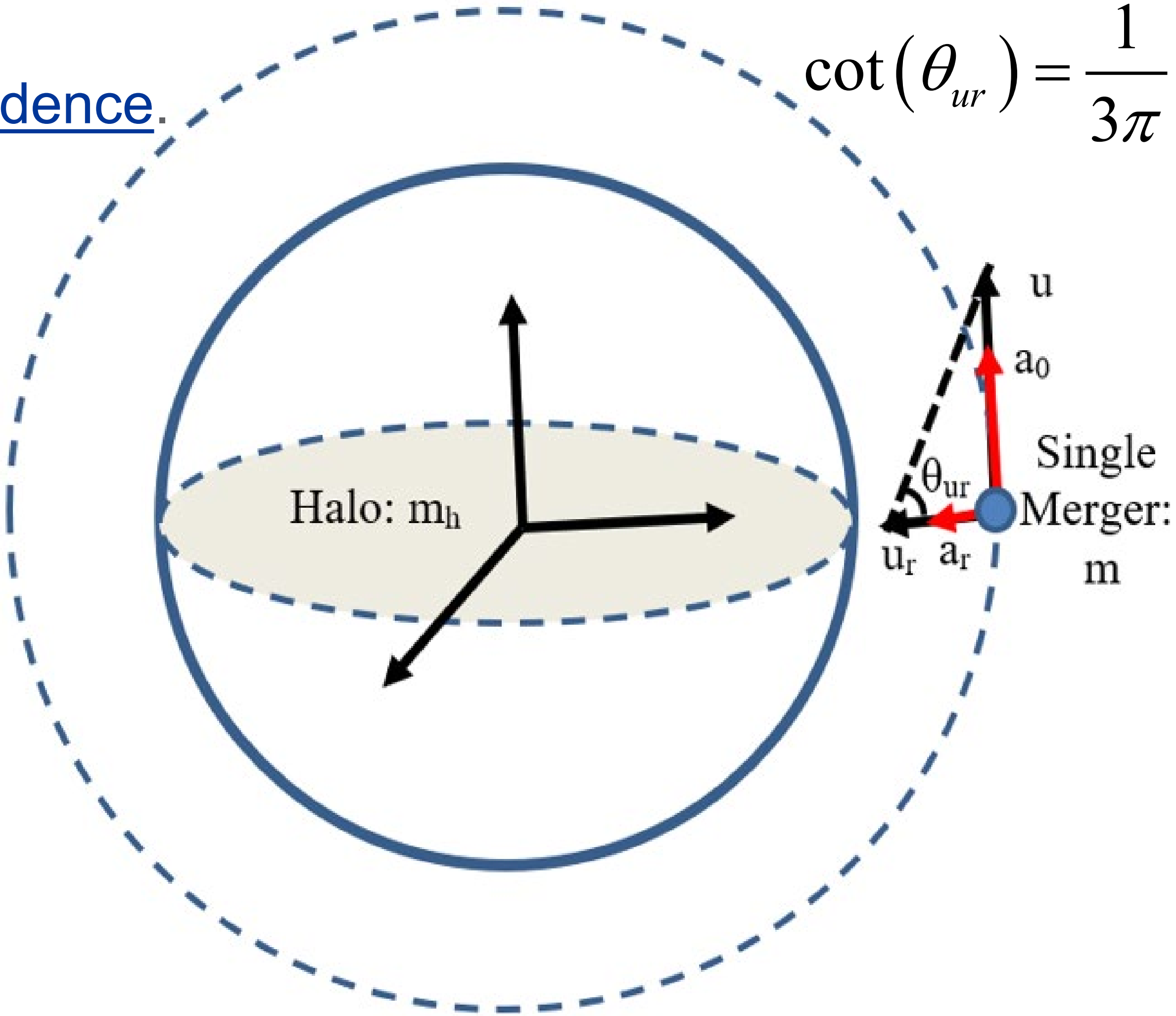
- Acceleration in halos increases with halo size and reaches about for large halos.
- Acceleration of halos is relatively independent of halo size, much smaller than acceleration in halos.



The original of MOND acceleration

Assume a_0 is the typical acceleration scale of fluctuation,
u is the typical velocity scale of fluctuation, θ_{ur} is the angle of incidence.

$$\cot(\theta_{ur}) = \frac{1}{3\pi}$$



The rate of energy cascade in terms of a_0, u and θ_{ur} :

$$\varepsilon_u = -a_r u_r = -a_0(a)\cot(\theta_{ur})u(a)\cot(\theta_{ur})$$

$$a_0(a) = -\frac{\Delta_c}{2}\cdot\frac{\varepsilon_u}{u} = -(3\pi)^2\frac{\varepsilon_u}{u} = \frac{81}{4}\pi^2 H_0\frac{u_0^2}{u} \propto a^{-3/4}$$

The rate of energy cascade:

$$\varepsilon_u \approx -\frac{3}{2}\frac{u^2}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} = -\frac{9}{4}H_0 u_0^2 = -4.6\times10^{-7}\frac{m^2}{s^3}$$

$$a_0(a=1) \approx 200H_0 u_0 \approx 1.2\times10^{-10}\, m/s^2$$

Energy cascade

Potential connection with dark energy??

$$\rho_{vac} = \frac{\Lambda c^2}{8\pi G} = \frac{3\pi}{2G}\left(\frac{(3\pi)^2\varepsilon_u}{u_0}\right)^2 = \frac{3\pi}{2}\frac{a_0^2 H_0}{GH} \propto \frac{a_0^2}{H}$$

$$a_0(z=0) \approx c\frac{(\Lambda/3)^{1/2}}{2\pi}$$

- Ideal gas pressure P ∝ temperature T ∝ velocity fluctuation
- DE density ∝ a_0^2 ∝ acceleration fluctuation (implies an entropic origin?)

The deep MOND behavior

- Fluctuation of acceleration introduces a scale of acceleration a_0;
- Deep MOND for particles with acceleration $a_p \ll a_0$.
- Consider a one-dimensional dark matter flow with a velocity scale v_0 and acceleration scale a_0

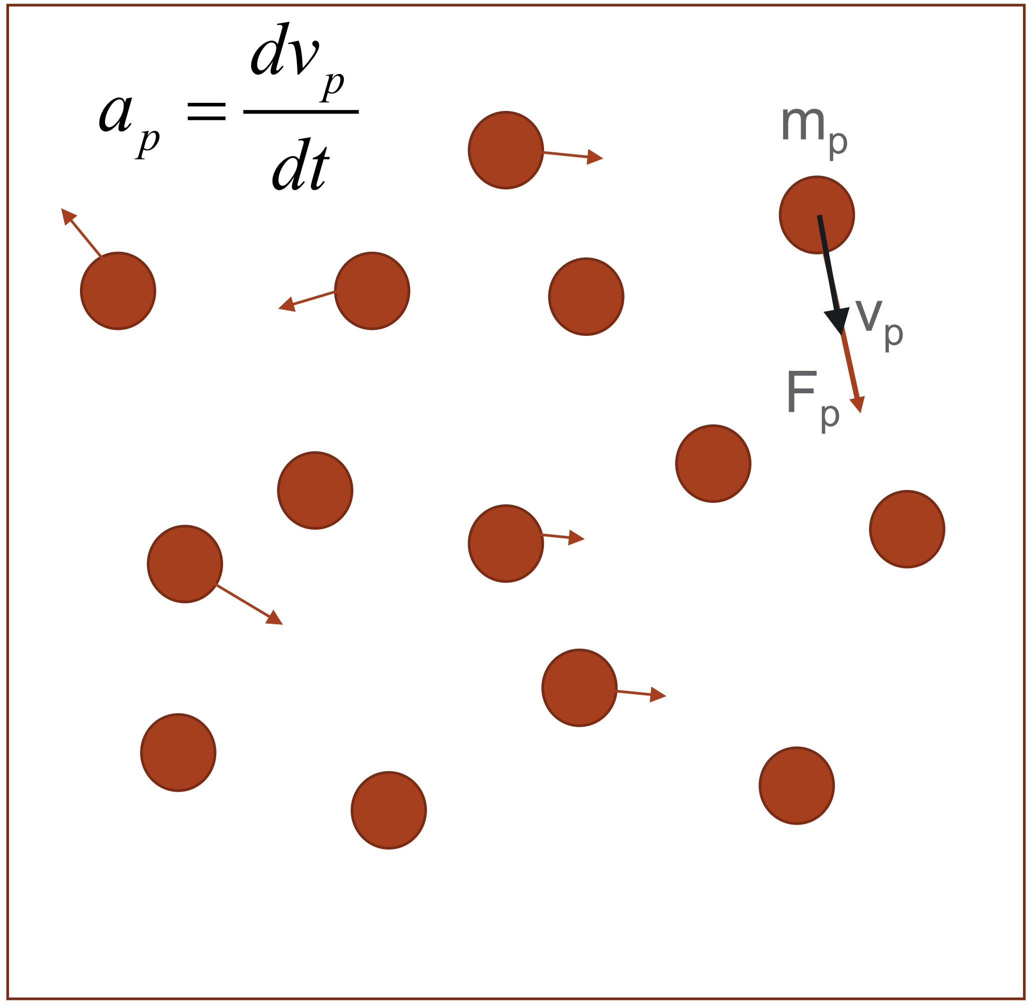
$$\frac{1}{2}\frac{dv_p^2}{dt} = v_p \frac{dv_p}{dt} = a_p v_p = a_0 v_0 = -\varepsilon_u$$

Constant rate of Energy cascade

$$\varepsilon_K(v) = v_0 v_p$$

Maximum entropy distribution: particle kinetic energy is proportional to velocity

$$F_p v_p = m_p \frac{d\varepsilon_K}{dt} \Rightarrow F_p = m_p \frac{v_0}{v_p} a_p = m_p \frac{a_p^2}{a_0} \propto a_p^2$$



Baryonic mass subject to external force F_p is suspended in and in equilibrium with dark matter flow

Summary and keywords

Modified Newtonian Dynamics	Constant rate of energy cascade	Maximum entropy distribution
Critical MOND acceleration	Mass/energy cascade	Deep MOND

- Direct energy cascade from large to small scales in hydrodynamic turbulence
- Inverse energy cascade in dark matter flow from small to large mass scales with a constant rate of cascade
- Long-range interaction of dark matter flow leads to a fluctuation in acceleration with a typical scale a_0
- The acceleration fluctuation in N-body simulation exactly matches the value of critical MOND acceleration
- The acceleration fluctuation in dark matter flow as the origin of MOND acceleration that can be related to the constant rate of energy flux.
- Suggest dark energy density might be also related to the acceleration fluctuation.
- Both Newtonian dynamics and "deep-MOND" behavior can be recovered based on the maximum entropy distribution and energy cascade in dark matter flow.

The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow

Xu Z., 2022, arXiv:2203.06899v1 [astro-ph.GA]
https://doi.org/10.48550/arXiv.2203.06899

Introduction

- The existence of dark matter (DM) is supported by numerous astronomical observations:
  - Flat rotation curves of spiral galaxies
  - Motion of galaxies in galaxy clusters
  - Gravitational lensing, Bullet clusters, CMB ……
- Though the nature of dark matter is still unclear, dark matter is believed to be cold (non-relativistic), collisionless, dissipationless, non-baryonic, barely interacting with baryons except through gravity, and sufficiently smooth with a fluid-like behavior.
- Total galaxy baryonic mass = stellar mass + cold gas.
- Stellar-to-halo mass relation (SHMR)
  - halo abundance matching approach
- Baryonic-to-halo mass relation (BHMR)

- Baryonic Tully and Fisher relation (BTFR):

$$v_f^4 = G m_b a_0$$

observed baryonic mass

- Halo mass m_h can be related to the halo virial radius r_h through constant density ratio Δ_c

$$m_h = \frac{4}{3}\pi\left(r_h\right)^3 \Delta_c \bar{\rho}_0\left(a\right)$$

- The BHMR (m_b and m_h) can be obtained only if the relation between v_f and r_h is known.
- The BHMR from the mass and energy cascade of dark matter flow?
- What is the average mass fraction of baryons in all halos?
- What is the fraction of total baryons residing in all galaxies?

Energy cascade in hydrodynamic turbulence

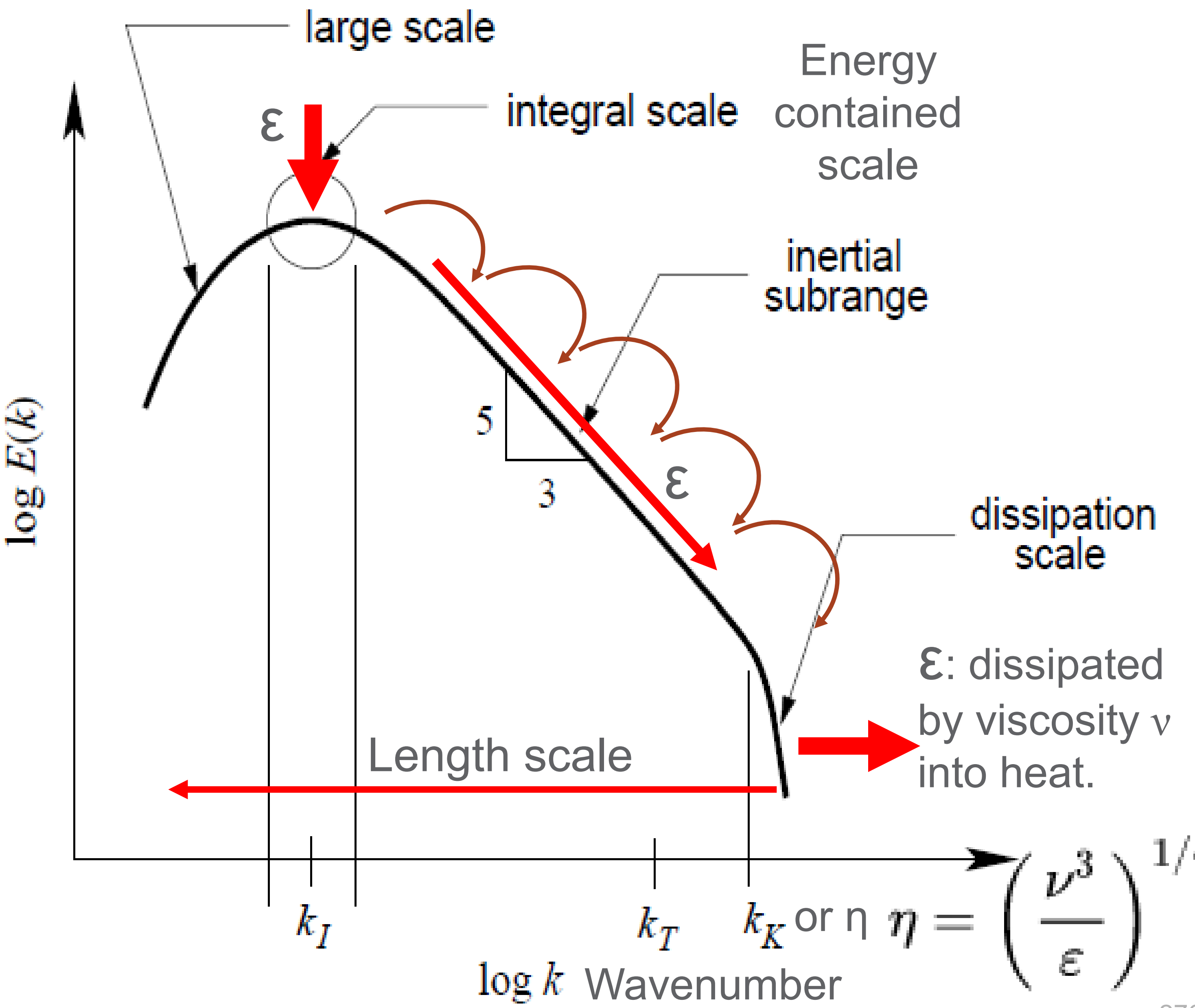
- There exist an inertial range with a scale-independent rate of energy cascade (ε does not depend on eddy size *l*) for eddy size η< l <*L*. η is a dissipative scale determined by viscosity ν and ε.

- In this range, inertial force is dominant over viscous force. For eddies with a characteristic velocity u and size l , the lifetime (turnaround time) of eddy is l/u. The rate ε can be computed as the kinetic energy passed per eddy lifetime.

$$\varepsilon \approx \frac{u^2}{(l/u)} \approx \frac{u^2}{l} u \quad \Rightarrow \quad u^3 \propto l$$

turnaround time acceleration

Big whirls have little whirls, That feed on their velocity;
And little whirls have lesser whirls, And so on to viscosity.



$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

Mass/Energy cascade in dark matter flow (SG-CFD)

- Collisionless nature and long-range interaction.
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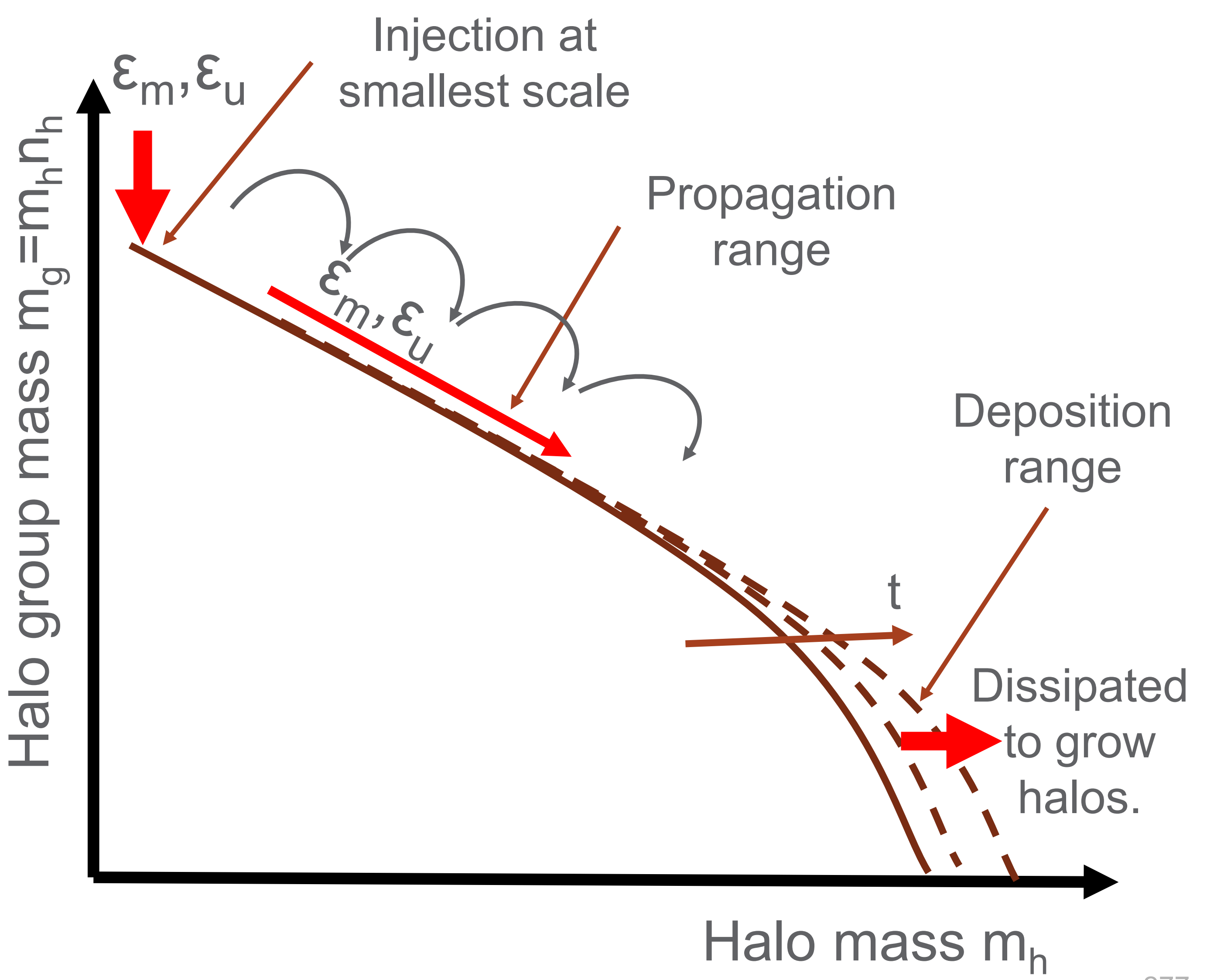

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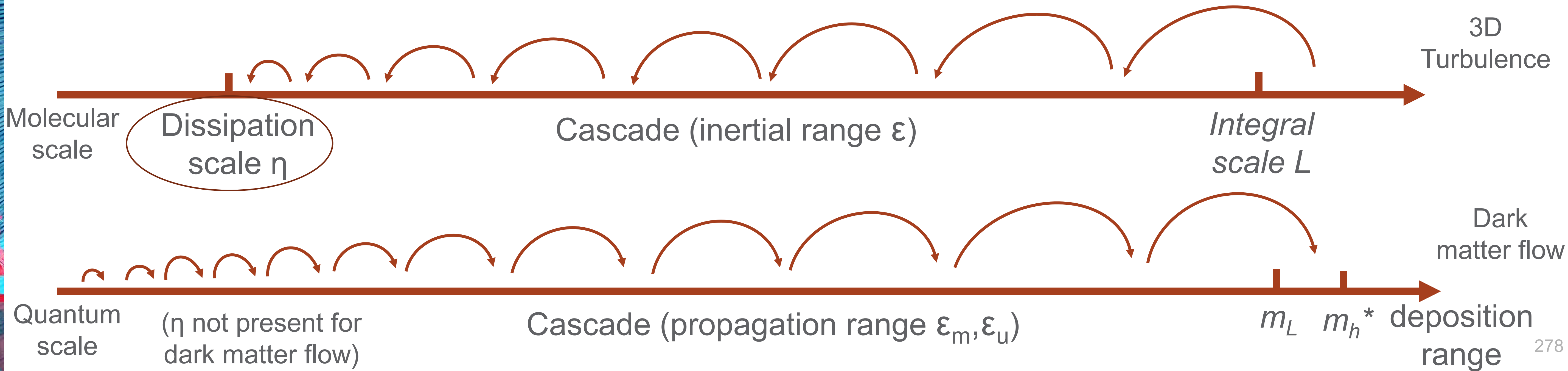

Constant (time and scale independent) rate of energy cascade

Power-law time evolution for energy in terms of rate of energy cascade ε_u:

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Power-law for Peculiar kinetic energy

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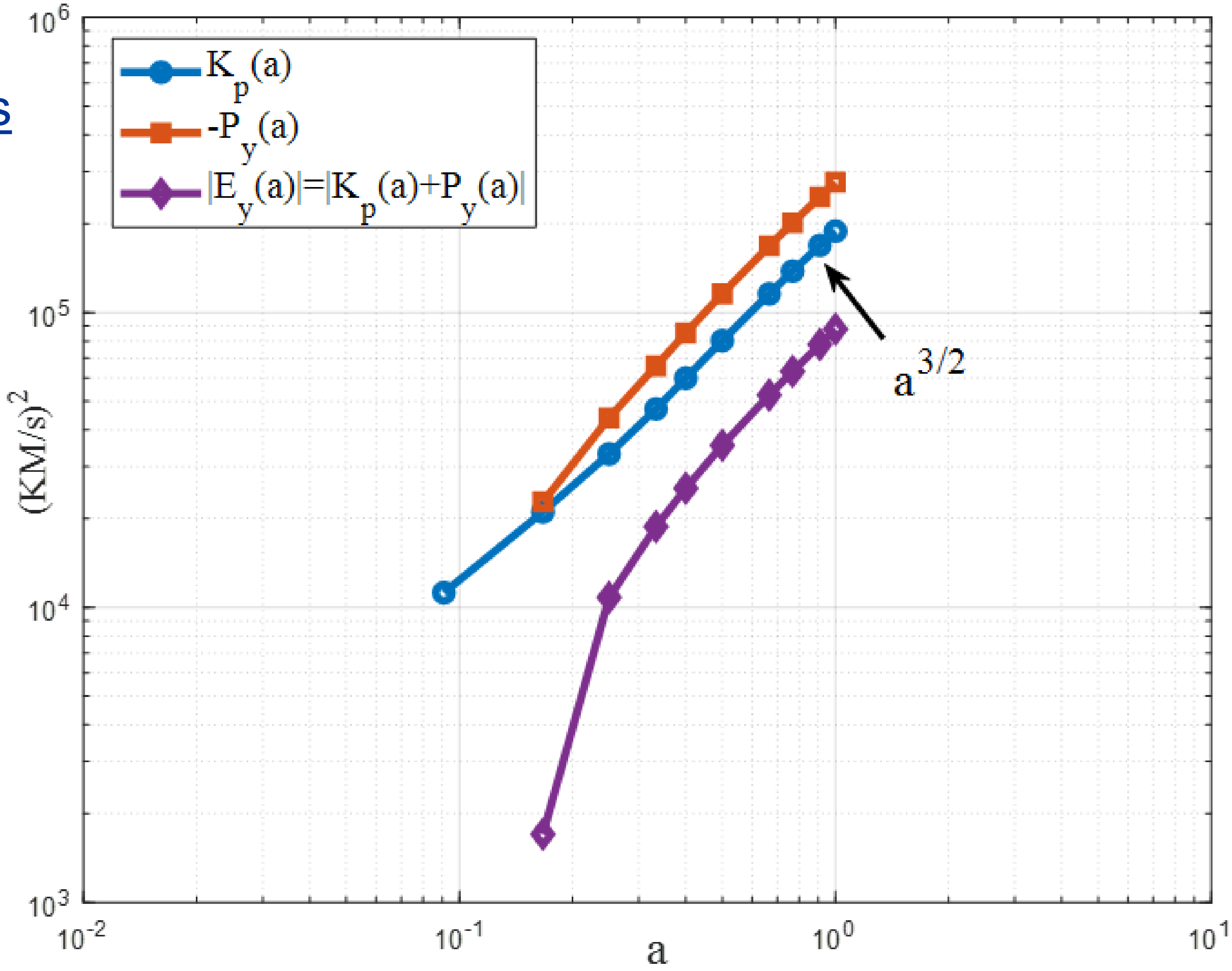
Power-law for potential energy

$$\varepsilon_u = -\frac{K_p}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} \approx -4.6\times 10^{-7}\,\frac{m^2}{s^3}$$

Also see detail analysis for inverse kinetic energy cascade.



The time variation of specific kinetic and potential energies from *N*-body simulation.

Dimensional analysis for critical mass scales

The smallest mass scale (dark matter particle mass)

At the smallest scale, three fundamental constants:

Gravitational constant	$G = 6.67\times10^{-11}\, m^3/\left(kg\cdot s^2\right)$
Rate of energy cascade	$\varepsilon_u = -4.6\times10^{-7}\, m^2/s^3$
Planck constant	$\hbar = 1.05\times10^{-34}\, kg\cdot m^2/s$

Simple dimensional analysis predicts:

Mass scale:
$$m_X \propto \left(-\varepsilon_u\hbar^5/G^4\right)^{\frac{1}{9}} \approx 8.7\times10^{-16}\, kg = 0.5GeV$$

Length scale: $l_X \propto \left(-G\hbar/\varepsilon_u\right)^{\frac{1}{3}}$

Time scale: $t_X \propto \left(G^2\hbar^2/\varepsilon_u^5\right)^{\frac{1}{9}}$

The largest mass scale (critical halo mass)

Three fundamental constants:

Gravitational constant	$G = 6.67\times10^{-11}\, m^3/\left(kg\cdot s^2\right)$
Rate of energy cascade	$\varepsilon_u = -4.6\times10^{-7}\, m^2/s^3$
Velocity dispersion or Hubble constant H	$u_0 \equiv u\left(a=1\right) = 354.61\, km/s$

Simple dimensional analysis predicts:

Mass scale: $m_L \propto -u_0^5/\left(G\varepsilon_u\right) \approx 9.14\times10^{13} M_\odot$

Length scale: $l_L \propto -u_0^3/\varepsilon_u \approx 3.14 Mpc$

Time scale: $t_L \propto u_0^2/\varepsilon_u \approx 8.7\times10^9\, yr$

The baryonic-to-halo mass ratio from energy cascade

Baryonic Tully-Fisher relation (BTFR):

$$v_f^4 = G m_b a_0$$

Halo mass and halo size relation:

$$m_h = \frac{4}{3}\pi r_h^3 \Delta_c \bar{\rho}_0 a^{-3}$$

Baryonic Tully-Fisher relation (BTFR):

$$v_f^4 = G m_b a_0$$

Halo mass and halo size relation:

$$m_h = \frac{4}{3}\pi r_h^3 \Delta_c \bar{\rho}_0 a^{-3}$$

Rate of energy cascade

$$\varepsilon_u = -\beta_f \frac{u^2}{r_h/v_f} a^q$$

Small halos <m_L: Baryonic mass in equilibrium with DM, i.e. same kinetic energy u^2

$$v_{cir} = \frac{4}{9}\sqrt{\frac{\Delta_c}{2}}\beta_f v_f a^q \propto (m_h)^{1/3} a^{-1/2}$$

$$r_h = \frac{4}{9}\beta_f v_f H^{-1} a^q \propto (m_h)^{1/3} a^1$$

$$v_f = \frac{9}{4\beta_f}\left(\frac{2}{\Delta_c}\right)^{\frac{1}{3}} (G m_h H)^{1/3} a^{-q} \propto (m_h)^{1/3} a^0$$

$$\varepsilon_u = -\alpha_f \frac{v_f^2}{r_h/v_f} a^p$$

Turnaround time

Large halos >m_L: Baryonic mass and DM are two miscible phases sharing same rate of cascade.

$$v_{cir} = \frac{4}{9}\sqrt{\frac{\Delta_c}{2}}\alpha_f \frac{v_f^3}{u^2} a^p \propto (m_h)^{1/3} a^{-1/2}$$

$$r_h = \frac{4}{9}\alpha_f \frac{v_f^3}{H u^2} a^p \propto (m_h)^{1/3} a^1$$

$$v_f = \left(\frac{3}{2\sqrt{\alpha_f}}\right)^{\frac{2}{3}} \left(\frac{2}{\Delta_c}\right)^{\frac{1}{9}} (G m_h H)^{1/9} u^{2/3} a^{-p/3} \propto (m_h)^{1/9} a^{(1-p)/3}$$

Critical scales and Baryonic-Halo-Mass Ratio

Critical rotation speed: $v_{fc} = ua^{(q-p)/2}\sqrt{\beta_f/\alpha_f}$

Critical circular speed: $v_{cc} = \frac{4}{9}\sqrt{\frac{\Delta_c}{2}}\sqrt{\frac{\beta_f^3}{\alpha_f}}ua^{(3q-p)/2}$

Critical halo size: $r_{hc} = \frac{4}{9}a^{(3q-p)/2}uH^{-1}\beta_f\sqrt{\beta_f/\alpha_f}$

Critical halo mass: $m_{hc} = \frac{16}{81}\left(\frac{\beta_f^3}{\alpha_f}\right)^{3/2}\left(\frac{\Delta_c}{2}\right)\left(\frac{u^5}{G\varepsilon_u}\right)a^{\frac{3}{2}(3q-p)}$

Critical baryonic mass: $m_{bc} = \frac{2}{\Delta_c}\left(\frac{\beta_f}{\alpha_f}\right)^2\left(\frac{u^5}{G\varepsilon_u}\right)a^{2(q-p)}$

Mass scale m_L

The baryonic mass in small halos:

$$m_b = \left(M_{c1}\right)^{-1/3}\left(m_h\right)^{4/3} \qquad M_{c1}(a) = \left(\frac{2}{3}\right)^{16}\left(\beta_f a^q\right)^{12}\left(\frac{\Delta_c}{2}\right)^7\left(\frac{u^5}{G\varepsilon_u}\right)$$

The baryonic mass in large halos:

$$m_b = \left(M_{c2}\right)^{5/9}\left(m_h\right)^{4/9} \qquad M_{c2}(a) = \left(\frac{2}{3}\right)^{-\frac{16}{5}}\left(\alpha_f a^p\right)^{-\frac{12}{5}}\left(\frac{2}{\Delta_c}\right)^{\frac{13}{5}}\left(\frac{u^5}{G\varepsilon_u}\right)$$

The baryonic-halo-mass ratio in critical halos:

$$A(z) \equiv \frac{m_{bc}}{m_{hc}} = \left(\frac{M_{c2}}{M_{c1}}\right)^{5/24} = \frac{81\left(2/\Delta_c\right)^2}{16\left(\alpha_f\right)^{1/2}\left(\beta_f\right)^{5/2}}a^{-(5q+p)/2}$$

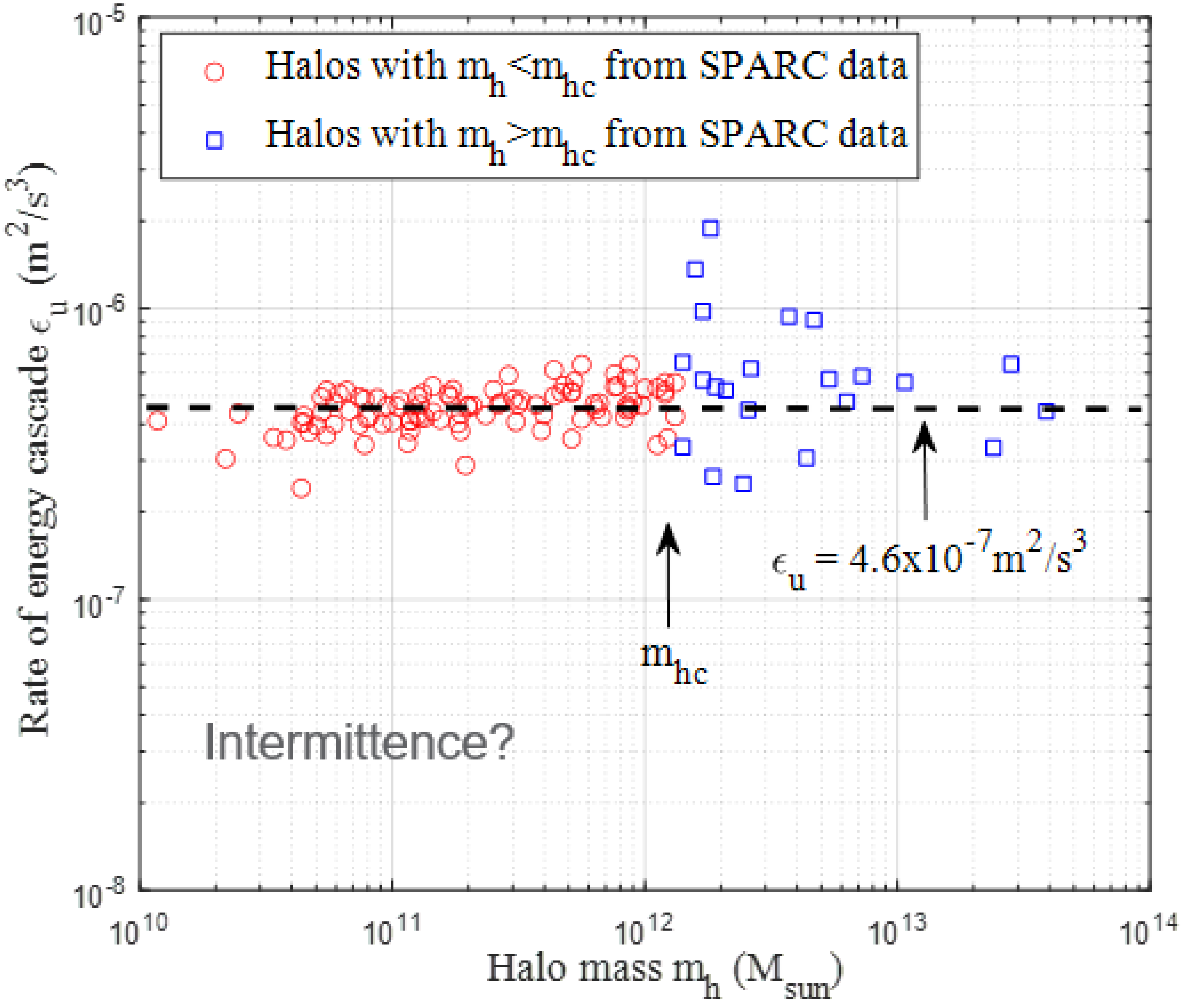
$$A(z=0) \approx 0.076$$

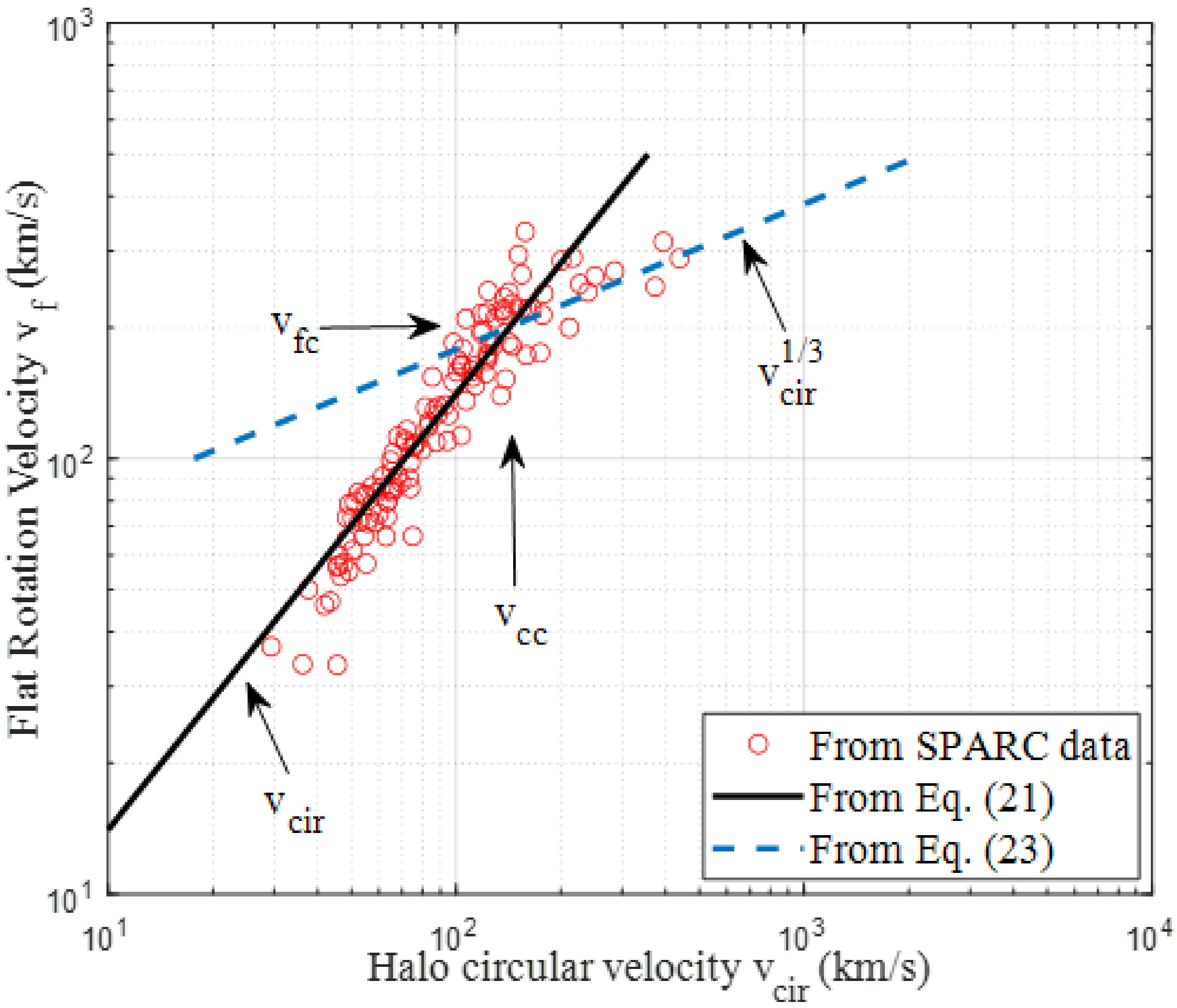
Relevant parameters for baryonic-to-halo mass ratio

Table 2. Parameters for deriving baryonic-to-halo mass ratio

Δ_c	200	p	$7/4$	M_{c1}	$3.01\times10^{15}a^{-9/4}M_{sun}$
ε_u	$4.6\times10^{-7}\,m^2/s^3$	q	$-1/2$	M_{c2}	$1.29\times10^{10}a^{-9/20}M_{sun}$
H_0	$1.62\times10^{-18}\,1/s$	α_f	0.5	m_{hc}	$1.33\times10^{12}a^{-9/8}M_{sun}$
u_0	$354.61\,km/s$	β_f	0.16	m_{bc}	$1.01\times10^{11}a^{-3/4}M_{sun}$
$a_0(z=0)$	$1.2\times10^{-10}\,m/s^2$	m	4	$A(z)$	$0.0761a^{3/8}$
η_0	0.76	q_0	0.556	m_h^*	$4\times10^{13}a^{3/2}M_{sun}$ [27]

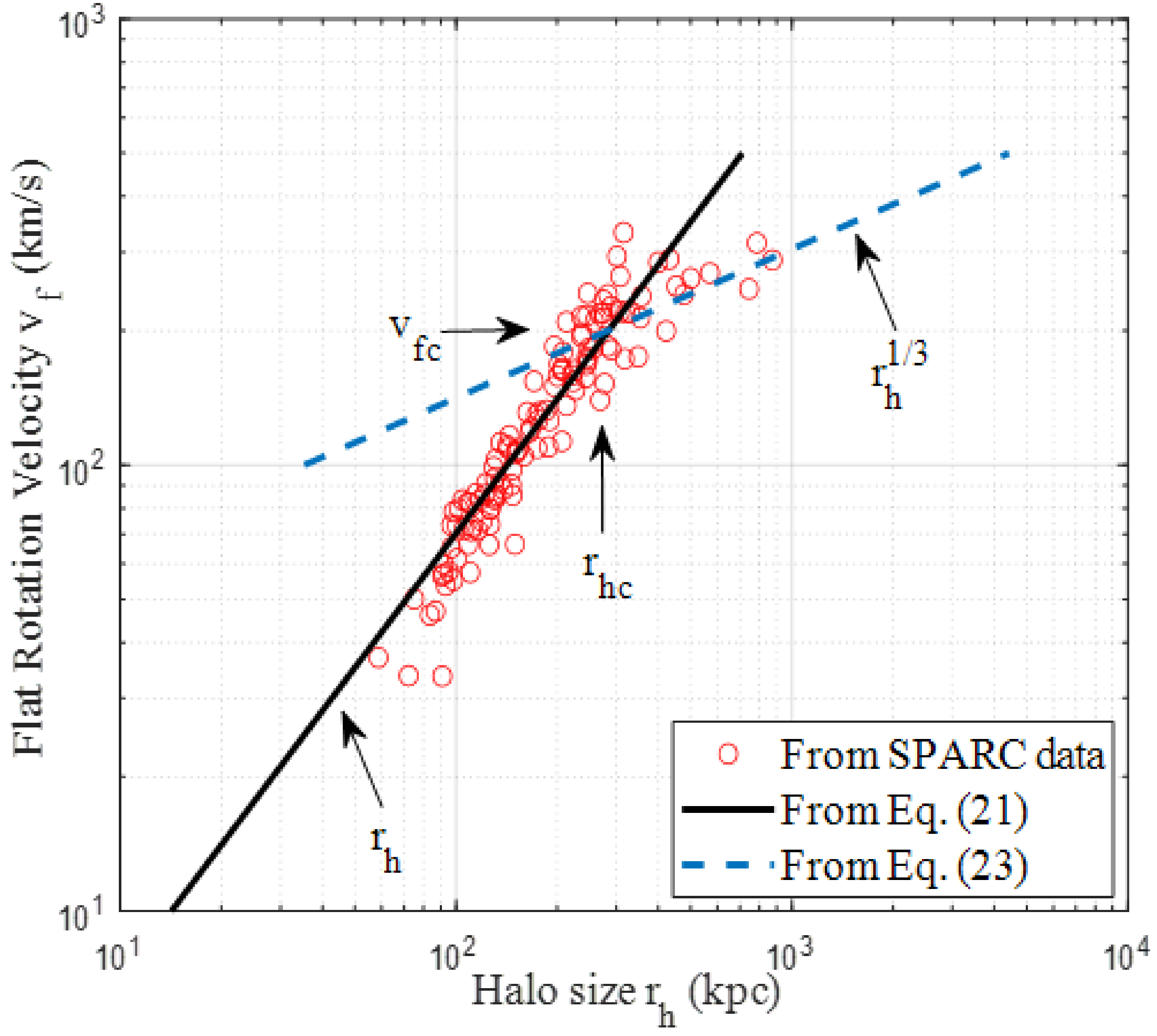
SPARC (Spitzer Photometry & Accurate Rotation Curves) data and model

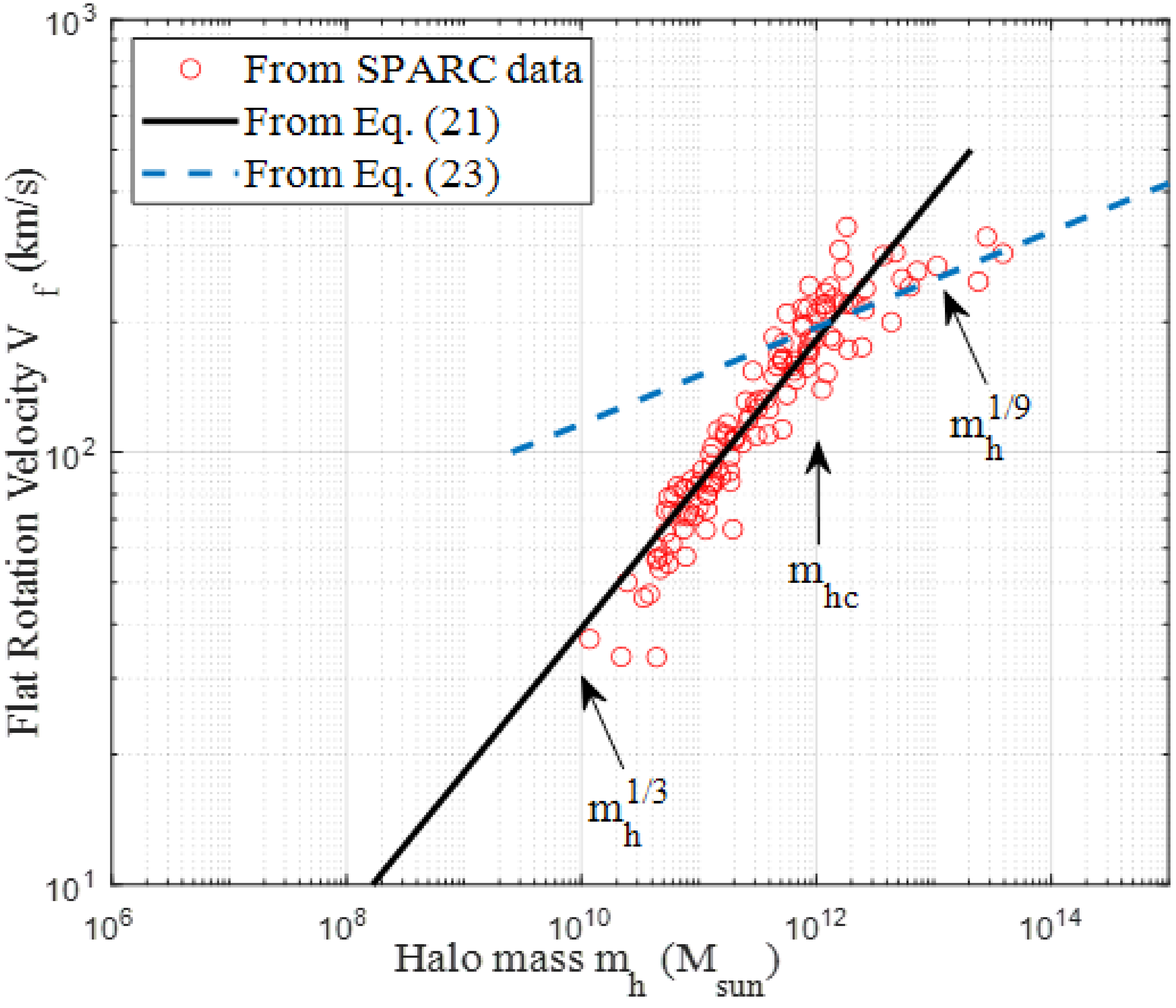




Halos have different rate of energy cascade with an average around ε_u (spatial intermittence in dark matter flow?)

SPARC data and model





SPARC data and model

Baryonic mass in small halos: $m_b = (M_{c1})^{-1/3} (m_h)^{4/3}$

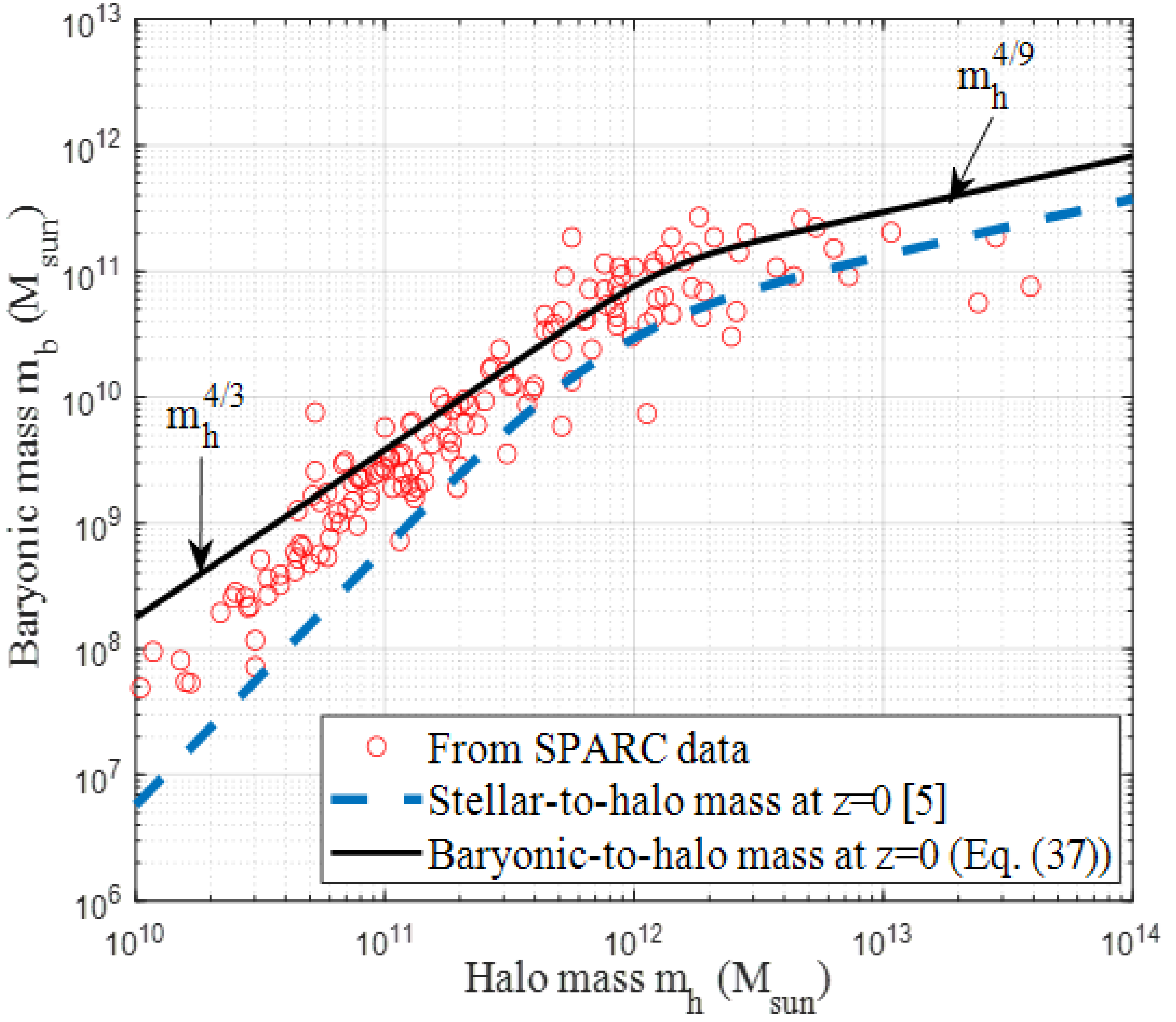
Baryonic mass in large halos: $m_b = (M_{c2})^{5/9} (m_h)^{4/9}$

Model incorporate two limits:

$$\frac{m_b}{m_h} = 2^{\frac{1}{m}} A(z) \left[\left(\frac{m_h}{m_{hc}(z)} \right)^{-\frac{m}{3}} + \left(\frac{m_h}{m_{hc}(z)} \right)^{\frac{5m}{9}} \right]^{-\frac{1}{m}}$$

- Dash line: the stellar-to-halo mass ratio obtained from halo abundance matching approach (require to match the stellar mass function)

- The scaling 4/9 law for both SHMR and BHMR



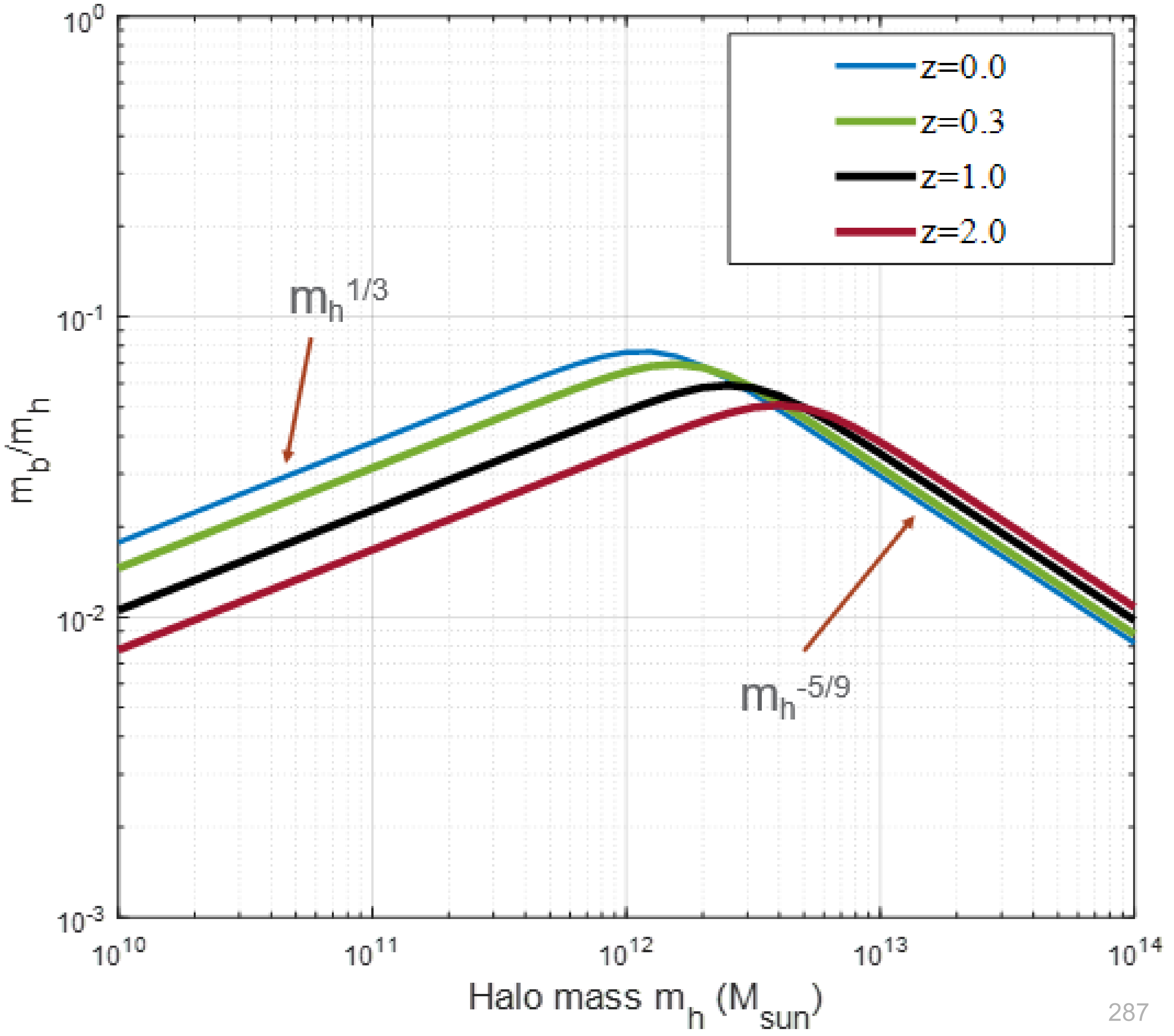
Redshift variation of baryonic-to-halo mass ratio

Models for baryonic-to-halo mass ratio:

$$\frac{m_b}{m_h} = 2^{\frac{1}{m}} A(z) \left[\left(\frac{m_h}{m_{hc}(z)} \right)^{-\frac{m}{3}} + \left(\frac{m_h}{m_{hc}(z)} \right)^{\frac{5m}{9}} \right]^{-\frac{1}{m}}$$

m is a parameter to adjust the transition;

- There exist a maximum BHMR ~0.076 at critical halo mass $m_{hc}=1.33\times10^{12}$ M_{sun}
- The critical halo mass decreases with time
- The maximum BHMR increases with time



Redshift evolution of baryonic-halo-mass relation

Overall cosmic baryonic-to-DM mass ratio (including both halos and out-of-halo) is ~18.8% in ΛCDM model:

Baryonic-to-DM mass ratio in out-of-halos

Baryonic-to-halo mass ratio in all halos

$$A_{boh}(z) = \frac{0.188 - A_{dh}(z)A_{bh}(z)}{1 - A_{dh}(z)}$$

Fraction of DM mass in halos

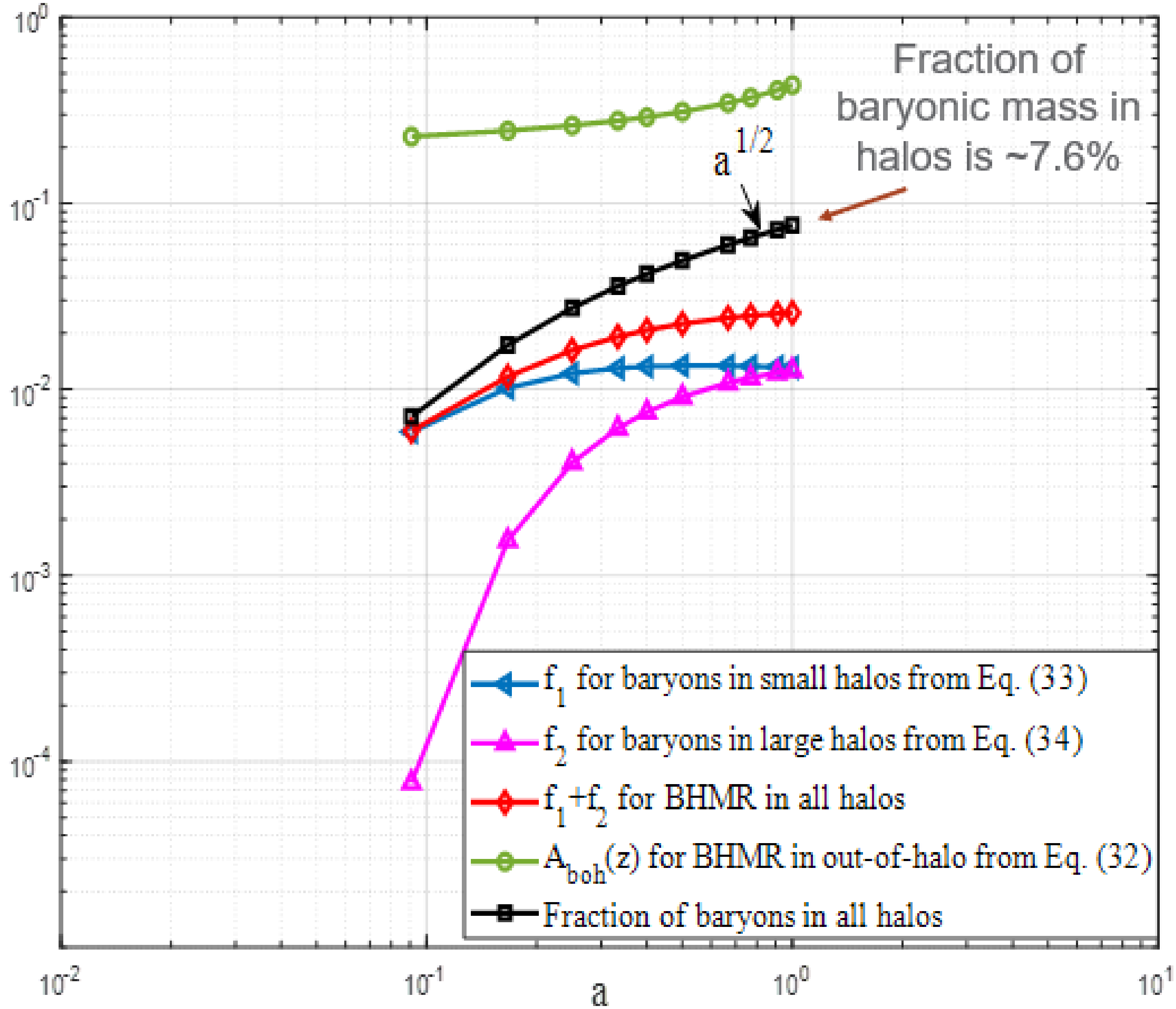
Use double-λ mass function to compute:

$$f_1 = \int_0^{\nu_c} f_{D\lambda}(\nu)(M_{c1})^{-1/3}(\nu^{3/2}m_h^*)^{1/3}\,d\nu$$

The baryonic-to-halo mass ratio in small halos

$$f_2 = \int_{\nu_c}^{\infty} f_{D\lambda}(\nu)(M_{c2})^{5/9}(\nu^{3/2}m_h^*)^{-5/9}\,d\nu$$

The baryonic-to-halo mass ratio in large halos



Redshift evolution of BHMR

Summary and keywords

Halo mass function	Mass/energy cascade	Tully-Fisher relation
Modified Newtonian Dynamics	Stellar-to-halo mass relation	Baryonic-to-halo mass relation

- Review direct energy cascade from large to small scales in hydrodynamic turbulence
- Reveal inverse mass and energy cascade that is unique for dark matter flow
- Present a fundamental theory for baryonic-to-halo mass ratio based on the mass/energy cascade in dark matter flow (agrees well with SPARC data)
- Predict a maximum baryonic-to-halo mass ratio ~0.076 for halos with a critical mass (agrees with SPARC data) and an average ratio ~0.024 for all halos
- Predict two distinct regimes for small and large halos, respectively, with critical halo mass and size explicitly derived (agrees with observations of stellar-to-halo mass ratio).
- Predict the fraction of total baryons in all galaxies is ~7.6% and that fraction increases with time (agrees very well with astronomical surveys including optical Sloan Digital Sky Survey and HIPASS). Most baryons (~92.4%) are not in galaxies.

Backup Slides

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