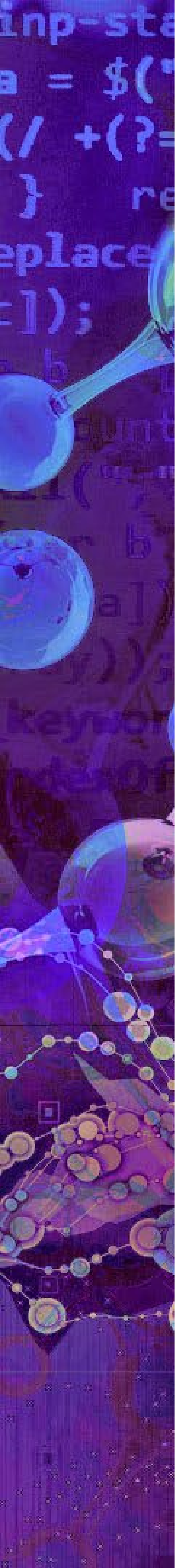

A comparative study of dark matter flow & hydrodynamic turbulence and its applications

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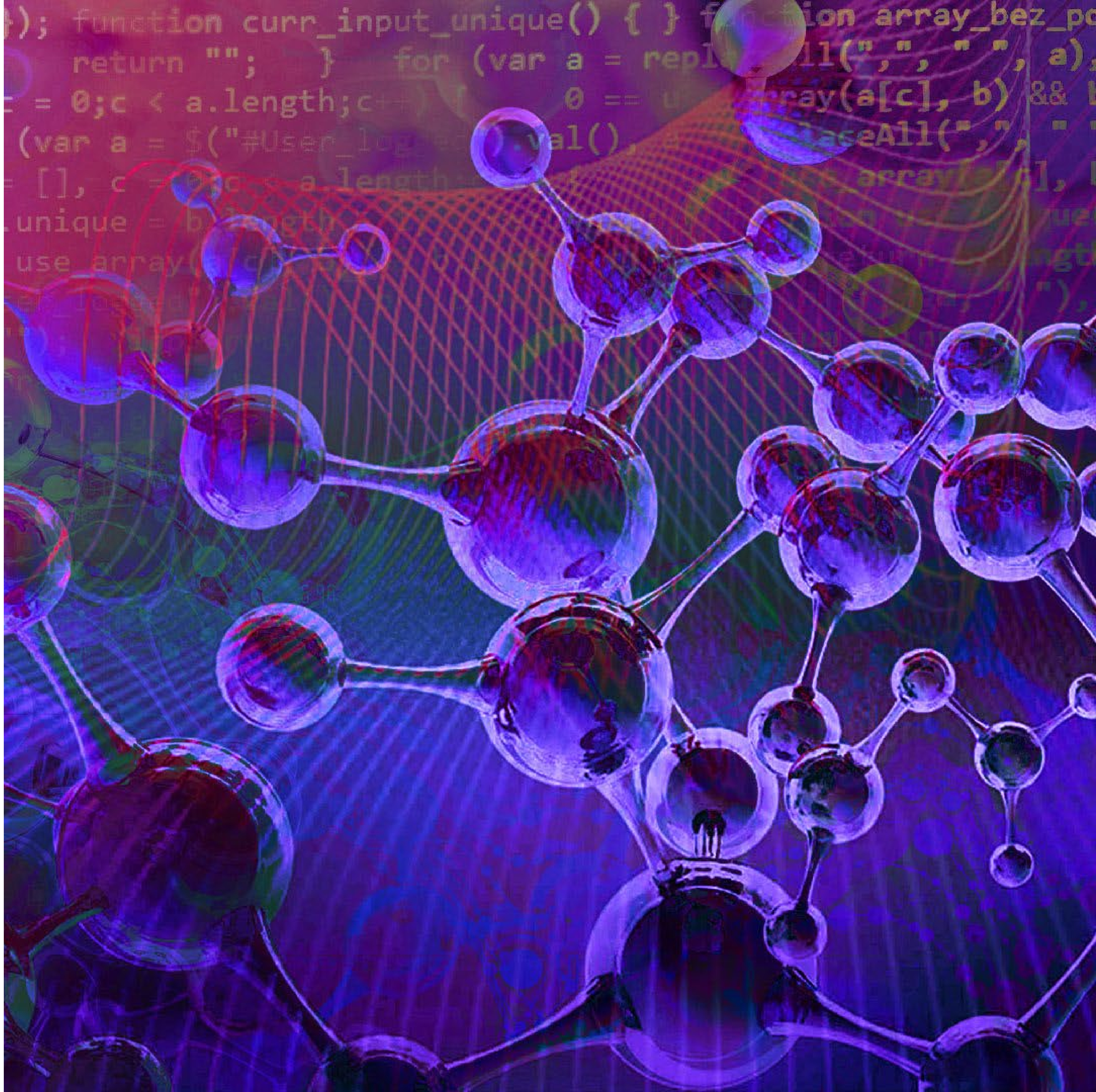

Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!

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Data repository and relevant publications

Structural (halo-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6541230
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6569898
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

Statistical (correlation-based) approach for dark matter flow

The statistical theory of dark matter flow (second order)

Xu Z., 2022, arXiv:2202.00910 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2202.00910

Introduction

Review:

Statistical theory in hydrodynamic turbulence

- Kinematic relations between statistical measures
  - Correlation functions
  - Structure functions
  - Power spectrum functions
- Incompressible on all scales
  - Divergence-free
  - Constant density

- N-body simulations are invaluable to understand dark matter flow (DMF).
- Fundamental problems when projecting N-body velocity field onto structured grids:
  - Velocity field is only sampled by N-body simulations at discrete locations of particles.
  - The sampling has a poor quality at locations with low particle density
  - Velocity field can be multi-valued and discontinuous due to the collisionless nature.

Goal 1: what are the kinematic relations in dark matter flow?

Goal 2: what is the nature of dark matter flow on different scales?

Approach:

- Use pairwise average for real-space two-point statistics to avoid projecting
- Take advantage of symmetry implied by the assumptions of homogeneity and isotropy.
- Develop kinematic relations between different statistical measures
- Identify the nature of DM flow, i.e. incompressible, constant divergence, or irrotational flow.

Two-point first order velocity correlation tensor

General correlation tensor between velocity field and a scalar field p(x):

$$Q_i(\mathbf{x},\mathbf{r}) = \left\langle u_i(\mathbf{x})\, p(\mathbf{x}')\right\rangle \qquad \mathbf{x}' = \mathbf{x}+\mathbf{r}$$

Pairwise average: Averaging over all particle pairs with the same separation r.

Reduced to function of r due to homogeneity and isotropy:

$$Q_i(\mathbf{x},\mathbf{r}) \equiv Q_i(\mathbf{r}) \equiv Q_i(r) = A_1(r)\, r_i$$

Divergence of first order tensor:

$$\frac{\partial Q_i(r)}{\partial r_i} = -\left\langle \left(\nabla\cdot\mathbf{u}(\mathbf{x})\right) p(\mathbf{x}')\right\rangle = 3A_1 + \frac{\partial A_1}{\partial r} r$$

Curl of first order tensor (always zero):

$$\nabla\times\mathbf{Q}(\mathbf{x},\mathbf{r}) = \left\langle \left(\nabla\times\mathbf{u}(\mathbf{x})\right) p(\mathbf{x}')\right\rangle = -\varepsilon_{ijk}\left(A_1\delta_{ik} + \frac{r_i r_k}{r}\frac{\partial A_1}{\partial r}\right) = 0$$

the Levi-Civita symbol satisfies the identity $\varepsilon_{ijk}\delta_{jk} = 0 \quad \varepsilon_{ijk} r_j r_k = \mathbf{r}\times\mathbf{r} = 0$

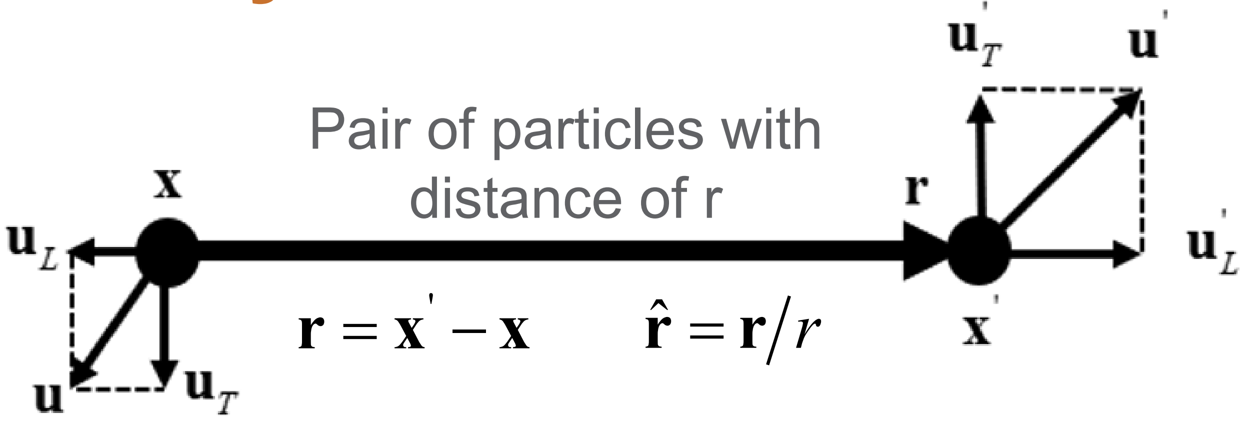
Incompressible flow	Constant divergence
$A_1(r) = 0$	$A_1(r) = -\theta\left\langle p(\mathbf{x})\right\rangle/3$
$Q_i(r) = 0$	$Q_i(r) = -\left(\frac{\theta}{3}\left\langle p(\mathbf{x})\right\rangle\right) r_i$

- The first order correlation tensor must vanish for incompressible flow
- The curl of the first order correlation tensor is always zero for any flow

Two-point second order velocity correlation tensors

Second order velocity correlation tensor:

$$Q_{ij}(\mathbf{r}) = Q_{ij}(r) = \left\langle u_i(\mathbf{x}) u_j(\mathbf{x}') \right\rangle$$



General form of isotropic second order tensor:

$$Q_{ij}(\mathbf{r}) = Q_{ij}(r) = A_2(r) r_i r_j + B_2(r)\delta_{ij}$$

Divergence of second order tensor:

$$Q_{ij,i} = \left(4A_2 + \frac{\partial A_2}{\partial r} r + \frac{1}{r}\frac{\partial B_2}{\partial r}\right) r_j$$

Used to derive
Kinematic relations

$$Q_{ij,i} = -\left\langle \left(\nabla \cdot u_i(\mathbf{x})\right) u_j(\mathbf{x}') \right\rangle = 0$$ Incompressible flow

$$Q_{ij,i} = -\left\langle \left(\nabla \cdot u_i(\mathbf{x})\right) u_j(\mathbf{x}') \right\rangle = -\theta \left\langle u_j(\mathbf{x}') \right\rangle = 0$$ Constant divergence flow

Curl of second order tensor:

$$\nabla \times Q_{ij}(r) = \varepsilon_{imj} r_m \left(A_2 - \frac{1}{r}\frac{\partial B_2}{\partial r}\right) = 0$$ Irrotational flow

Longitudinal velocity:	Transverse velocity:
$u_L = \mathbf{u}\cdot\hat{\mathbf{r}} = u_i \hat{r}_i$	$\mathbf{u}_T = -(\mathbf{u}\times\hat{\mathbf{r}}\times\hat{\mathbf{r}})$
$u'_L = \mathbf{u}'\cdot\hat{\mathbf{r}} = u'_i \hat{r}_i$	$\mathbf{u}'_T = -(\mathbf{u}'\times\hat{\mathbf{r}}\times\hat{\mathbf{r}})$

Velocity difference or Pairwise velocity: $\Delta u_L = u'_L - u_L$

Velocity sum: $\sum u_L = u'_L + u_L$

Same even order kinematic relations for incompressible flow and constant divergence flow

Two-point second order velocity correlation functions

Using index contraction of second order tensor to define three scalar correlation functions

Total correlation function:

$$R_2(r) = Q_{ij}\delta_{ij} = \langle \mathbf{u}\cdot\mathbf{u}' \rangle = \langle u_i u_i' \rangle = A_2 r^2 + 3B_2$$

Longitudinal correlation function

$$L_2(r) = Q_{ij}\, r_i r_j / r^2 = \langle u_L u_L' \rangle = A_2 r^2 + B_2$$

Transverse correlation function

$$T_2(r) = Q_{ij} n_i n_j = \langle \mathbf{u}_T \cdot \mathbf{u}_T' \rangle / 2 = B_2(r)$$

$$R_2(r) = 2R(r) = L_2(r) + 2T_2(r)$$

Two correlation coefficients can be defined for longitudinal and transverse velocity:

$$\rho_L(r) = \frac{\langle u_L u_L' \rangle}{\langle u_L^2 \rangle} \quad \text{and} \quad \rho_T(r) = \frac{\langle \mathbf{u}_T \cdot \mathbf{u}_T' \rangle}{\langle |\mathbf{u}_T|^2 \rangle}$$

The velocity power spectrum and correlation function form Fourier transform pair

$$R(r) = \int_0^\infty E_u(k)\frac{\sin(kr)}{kr}dk$$

$$E_u(k) = \frac{2}{\pi}\int_0^\infty R(r)\, kr \sin(kr)\, dr$$

Integral scale: the length scale within which velocities are appreciably correlated

$$l_{u0} = \frac{1}{u^2}\int_0^\infty R(r)\,dr = \frac{\pi}{2u^2}\int_0^\infty E_u(k)k^{-1}dk$$

One-dimensional RMS (root-mean-square) velocity:

$$u(a) = \left(\frac{1}{3}\langle \mathbf{u}(\mathbf{x})\cdot\mathbf{u}(\mathbf{x})\rangle\right)^{1/2}$$

Kinematic relations for correlation functions

For incompressible flow or constant divergence flow:

$$T_2 = \frac{1}{2r}\left(r^2 L_2\right)_{,r} \qquad R_2 = \frac{1}{r^2}\left(r^3 L_2\right)_{,r}$$

$$Q_{ij}(r) = -\frac{1}{2r}\left[\left(L_2\right)_{,r} r_i r_j - \left(r^2 L_2\right)_{,r}\delta_{ij}\right]$$

$$L_2(r) = \int_0^\infty E_u(k)\frac{2 j_1(kr)}{kr}dk$$

$$T_2(r) = \int_0^\infty E_u(k)\left(j_0(kr) - \frac{j_1(kr)}{kr}\right)dk$$

$$l_{u0} = \frac{1}{u^2}\int_0^\infty R(r)\,dr = \frac{1}{u^2}\int_0^\infty L_2(r)\,dr$$

Relations between correlation functions

Correlation tensor in terms of correlations

Relations to power spectrum function

Integral length scale

For irrotational flow:

$$R_2 = \frac{1}{r^2}\left(r^3 T_2\right)_{,r} \qquad L_2 = \left(r T_2\right)_{,r}$$

$$Q_{ij}(r) = \left(T_2\right)_{,r}\frac{r_i r_j}{r} + T_2\delta_{ij}$$

$$L_2(r) = 2\int_0^\infty E_u(k)\left(j_0(kr) - 2\frac{j_1(kr)}{kr}\right)dk$$

$$T_2(r) = \int_0^\infty E_u(k)\frac{2 j_1(kr)}{kr}dk$$

$$l_{u0} = \frac{1}{u^2}\int_0^\infty R(r)\,dr = \frac{1}{u^2}\int_0^\infty T_2(r)\,dr$$

nth order spherical Bessel function of the first kind: $j_n(kr)$

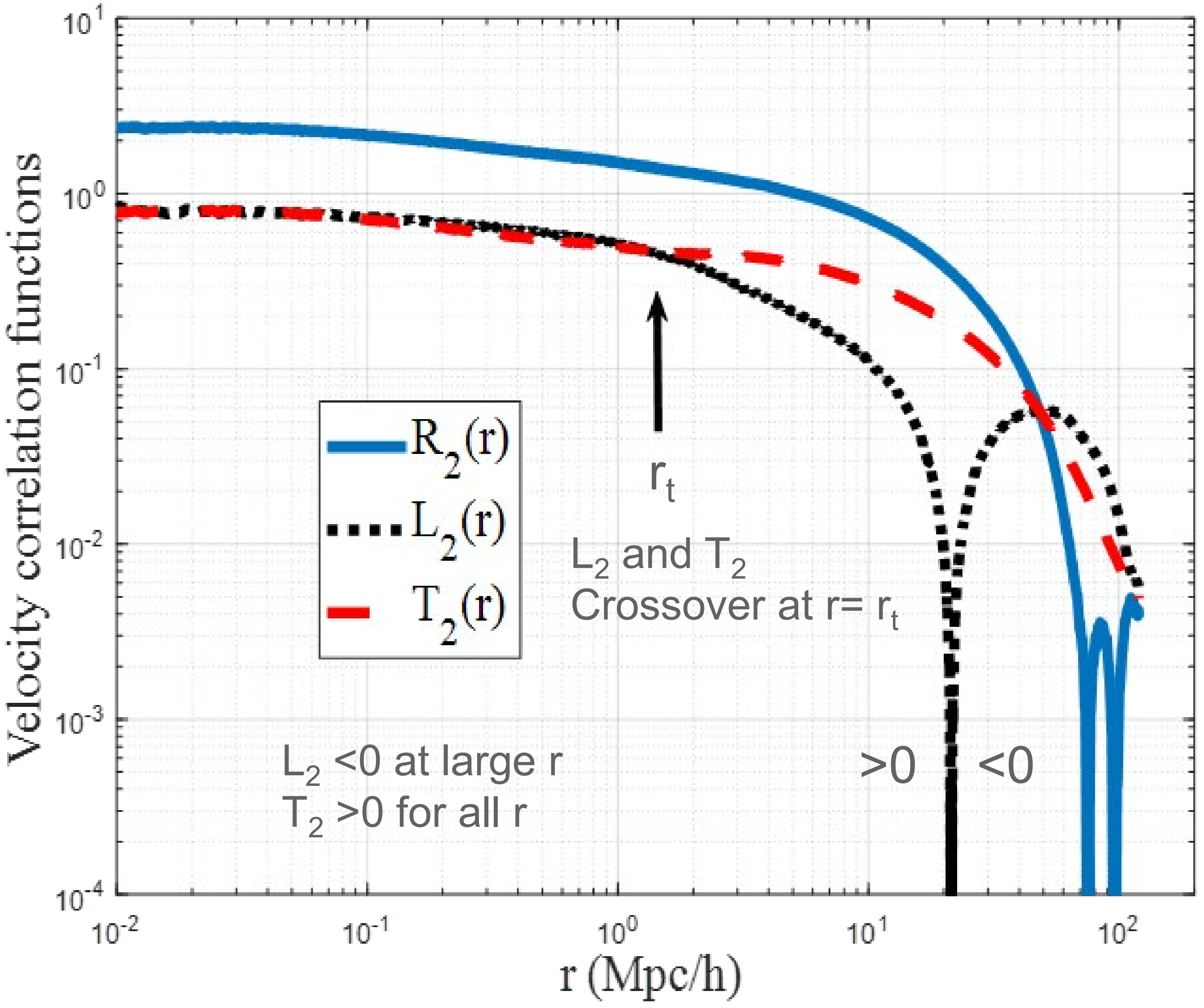
Characterizing the type of flow

$$R_{2i} = \frac{1}{r^3}\int_0^r R_2(y)\,y^2 dy$$

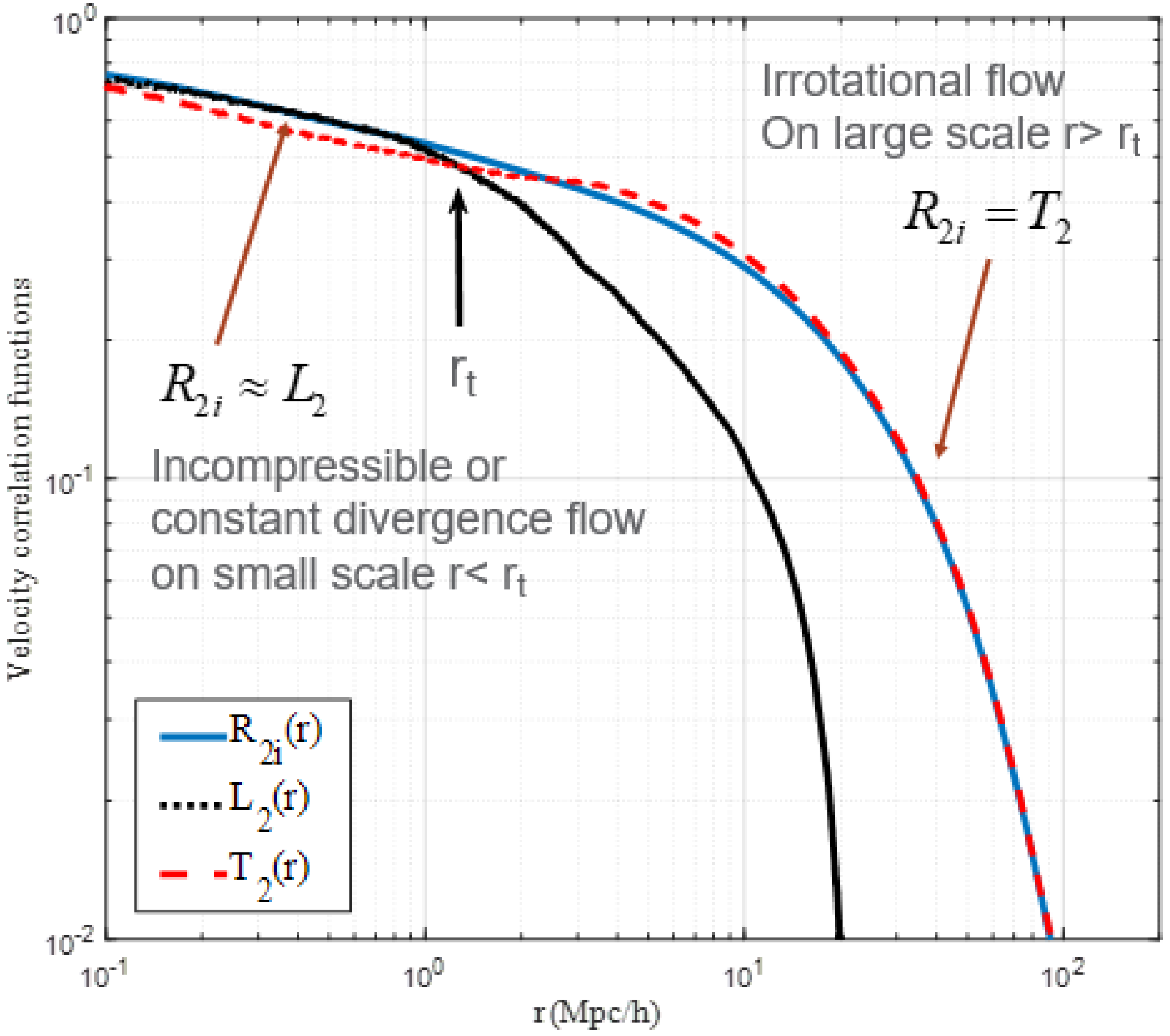
For incompressible or constant divergence flow: $R_{2i} = L_2$

For irrotational flow: $R_{2i} = T_2$

Correlation functions from N-body simulation and nature of dark matter flow

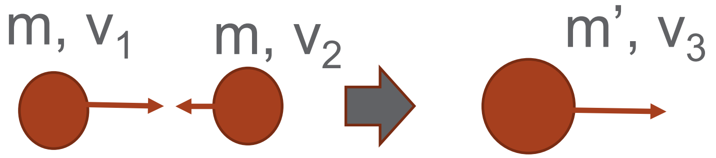


The variation of two-point second order velocity correlation functions (normalized by u2) with scale r at z=0



Using correlation functions to characterize different types of flow.

Velocity correlation and collisionless particle "annihilation"



annihilation

Momentum conservation:

$$mv_1 + mv_2 = m^{'}v_3 \Rightarrow v_3 = \frac{m}{m^{'}}\left(v_1 + v_2\right)$$

$$\left\langle v_3^2 \right\rangle = \left(\frac{m}{m^{'}}\right)^2 \left\langle \left(v_1 + v_2\right)^2 \right\rangle = 2\left(\frac{m}{m^{'}}\right)^2 \left\langle u_L^2 \right\rangle \left(1 + \boxed{\rho_{L0}}\right)$$

Mass-energy conservation:

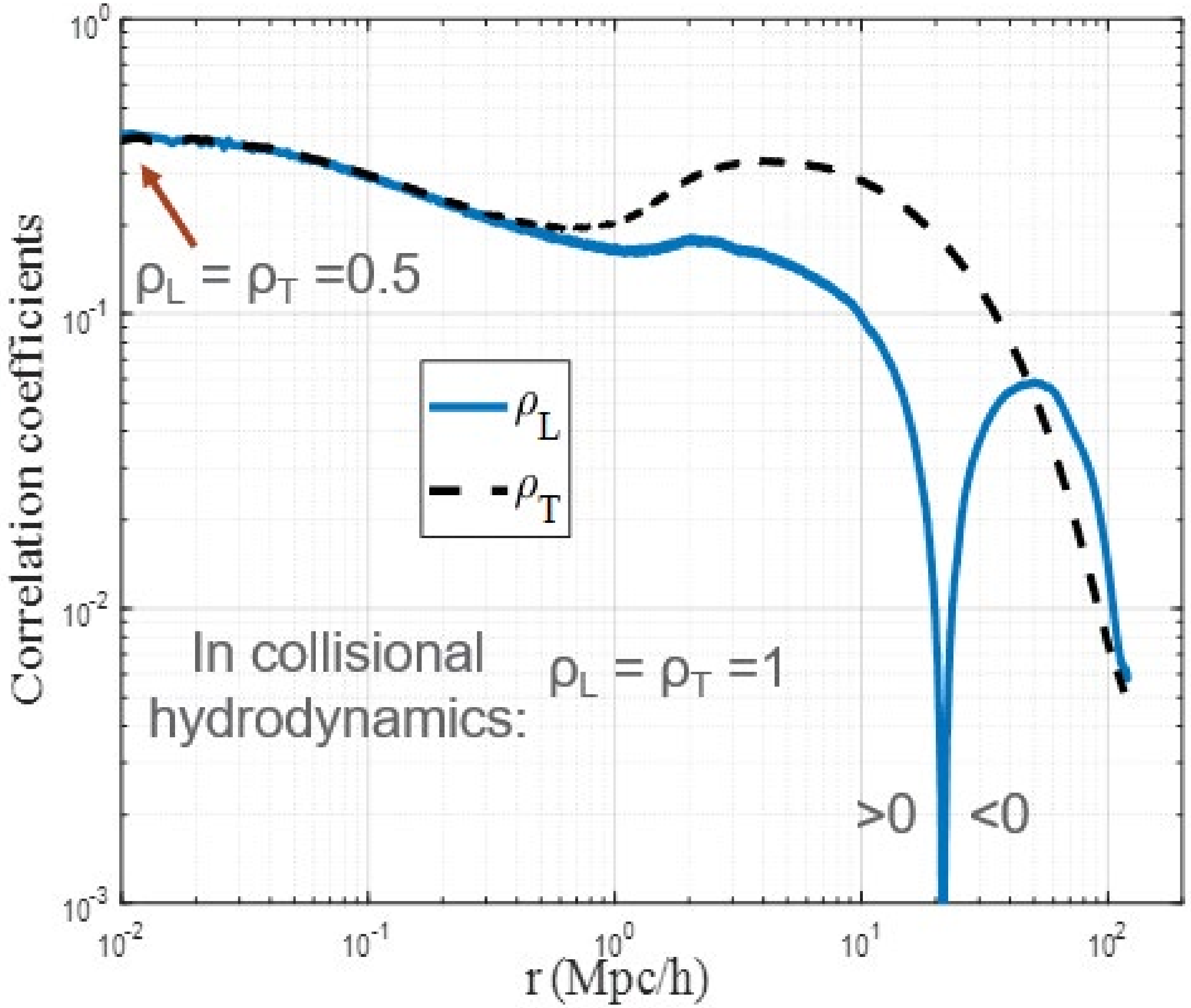
$$mc^2 + \frac{1}{2}mv_1^2 + mc^2 + \frac{1}{2}mv_2^2 = m^{'}c^2 + \frac{1}{2}m^{'}v_3^2$$

$$m^{'} = m\left[1 + \frac{\left\langle u_L^2 \right\rangle}{2c^2} + \sqrt{1 - \rho_{L0}\frac{\left\langle u_L^2 \right\rangle}{c^2} + \frac{\left\langle u_L^2 \right\rangle^2}{4c^4}}\right] \approx m\left[2 + \left(1 - \rho_{L0}\right)\frac{\left\langle u_L^2 \right\rangle}{2c^2}\right]$$

Particle "annihilation" (r=0) leads to extra mass converted from kinetic energy if gravity is the only interaction and no radiation is produced from that "annihilation".

Equipartition: halo T and halo group T

$$\sigma^2\left(m_h\right) = \sigma_v^2\left(m_h\right) + \sigma_h^2\left(m_h\right) \quad \rho_{cor} = \left\langle \sigma_h^2 \right\rangle / \left\langle \sigma^2 \right\rangle \approx 1/2$$



The correlation coefficients for longitudinal velocity and for transverse velocity

Modeling velocity correlation functions on large scale

On large scale, transverse velocity correlation can be well modelled by exponential function:

$$T_2(r,a) = a_0 u^2 \exp(-r/r_2) \propto a \qquad a_0 (u/u_0)^2 = 0.45a$$

$$r_2 \approx 21.4\, Mpc/h$$

Redshift-independent length scale, might be related to the size of sound horizon

Using kinematic relations for irrotational flow on large scale

$$L_2(r,a) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(1-\frac{r}{r_2}\right)$$

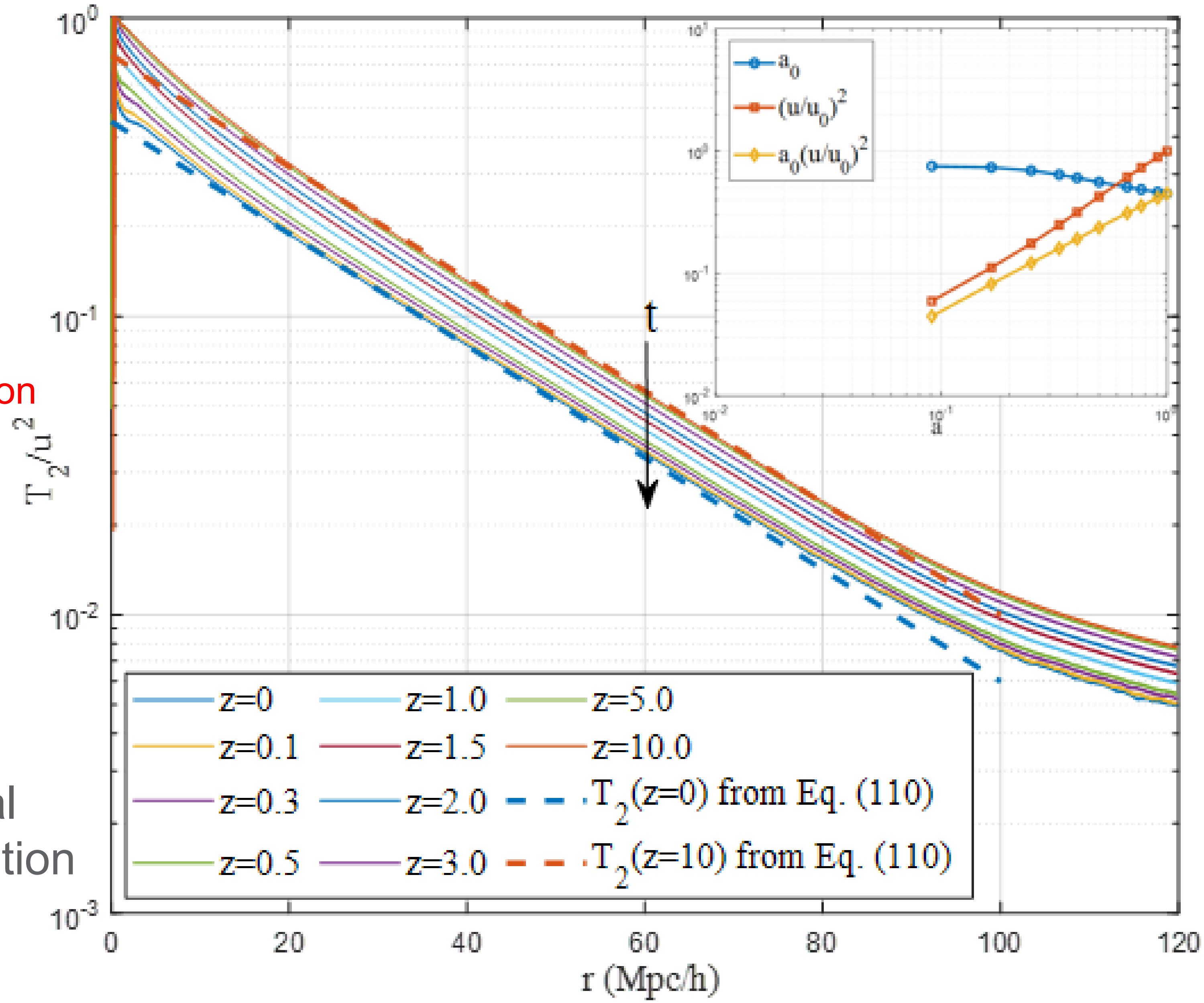
Longitudinal correlation

$$R_2(r,a) = \langle \mathbf{u}\cdot\mathbf{u}' \rangle = 2R(r) = a_0 u^2 \exp\left(-\frac{r}{r_2}\right)\left(3-\frac{r}{r_2}\right)$$

Total correlation

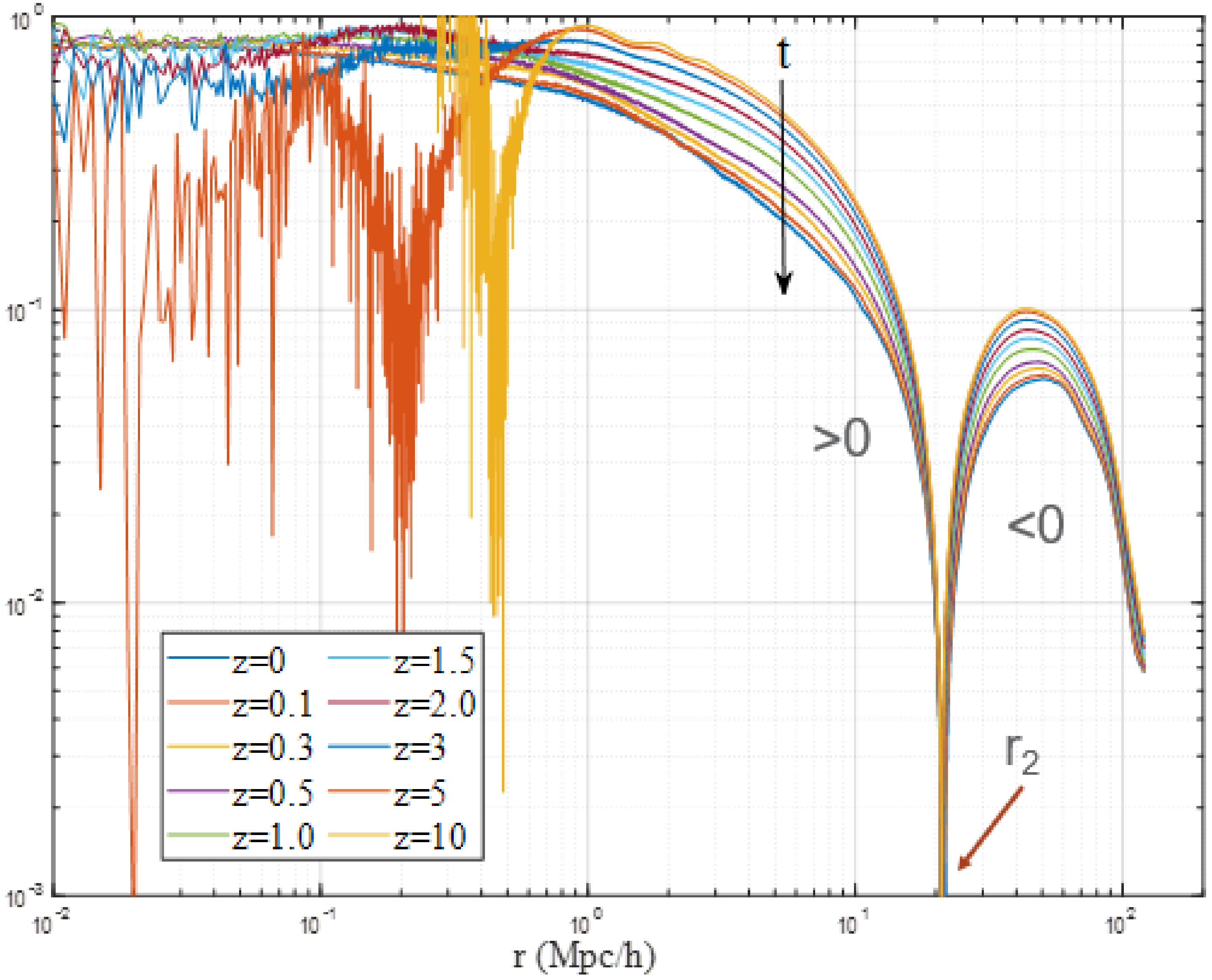
$$l_{u0} = \frac{1}{u^2}\int_0^\infty R(r)\,dr = \frac{1}{2u^2}\int_0^\infty R_2(r)\,dr = 2a_0 r_2$$

Correlation length

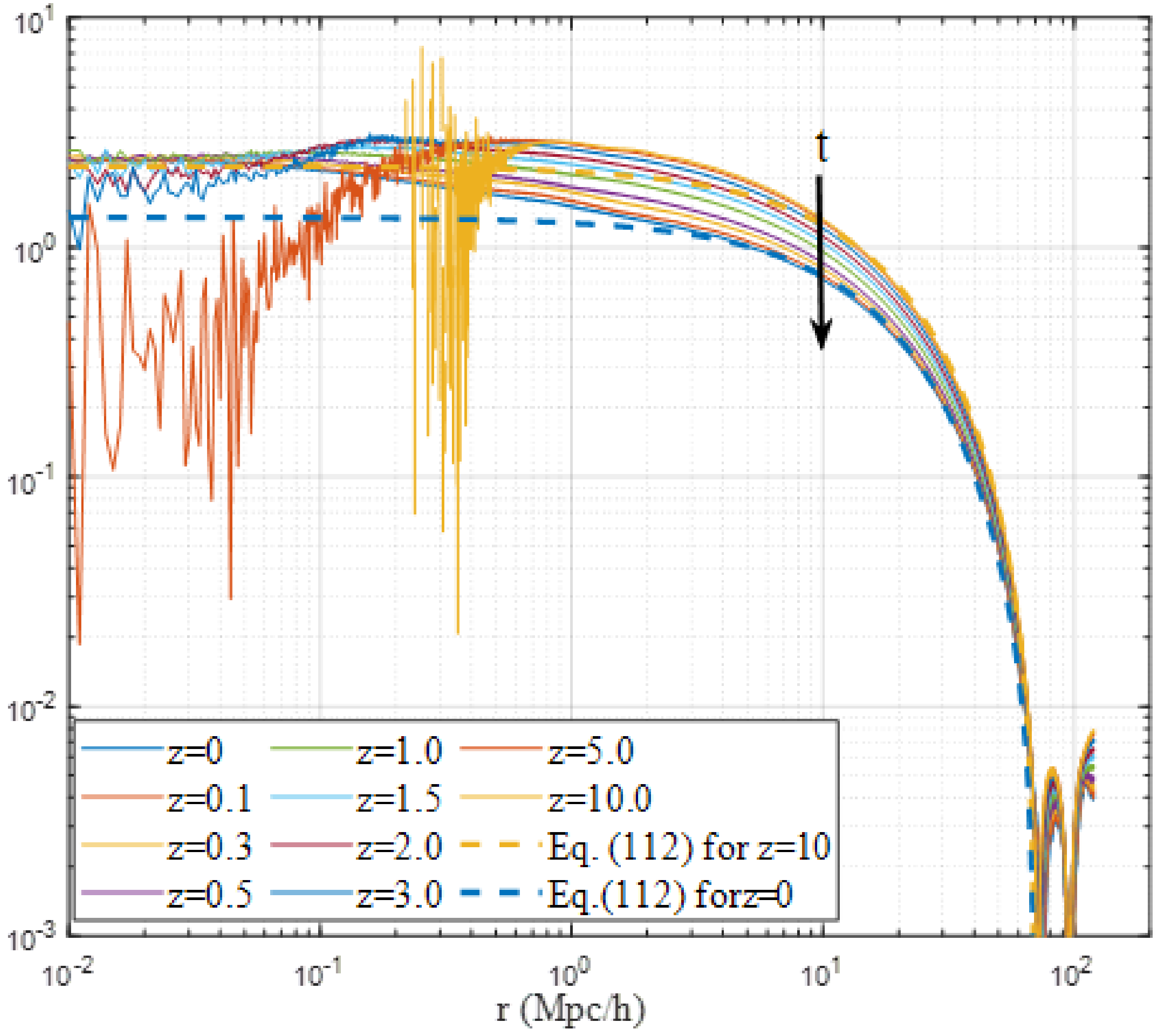


Transverse velocity correlation function T_2 varying with r at different redshifts z

Longitudinal and total velocity correlation



The variation of longitudinal velocity correlation function L_2 with scale r and redshift z



The variation of total velocity correlation function R_2 with scale r and redshift z

Density and potential correlations on large scale

Using kinematic relations and exponential transverse velocity correlation, we can analytically derive all correlations for velocity, density and potential on large scale.

Linear perturbation theory and Zeldovich approximation on large scale:

$$\delta \approx \eta = -\frac{\nabla \cdot \mathbf{u}}{aHf(\Omega_m)} \qquad \mathbf{u} = -\frac{Hf(\Omega_m)\nabla\phi}{4\pi G\rho a} \qquad \delta \approx \eta = \frac{\nabla^2\phi}{4\pi G\rho a^2}$$

Log-density field: $\eta(\mathbf{x}) = \log(1+\delta) \approx \delta$

Density correlation:

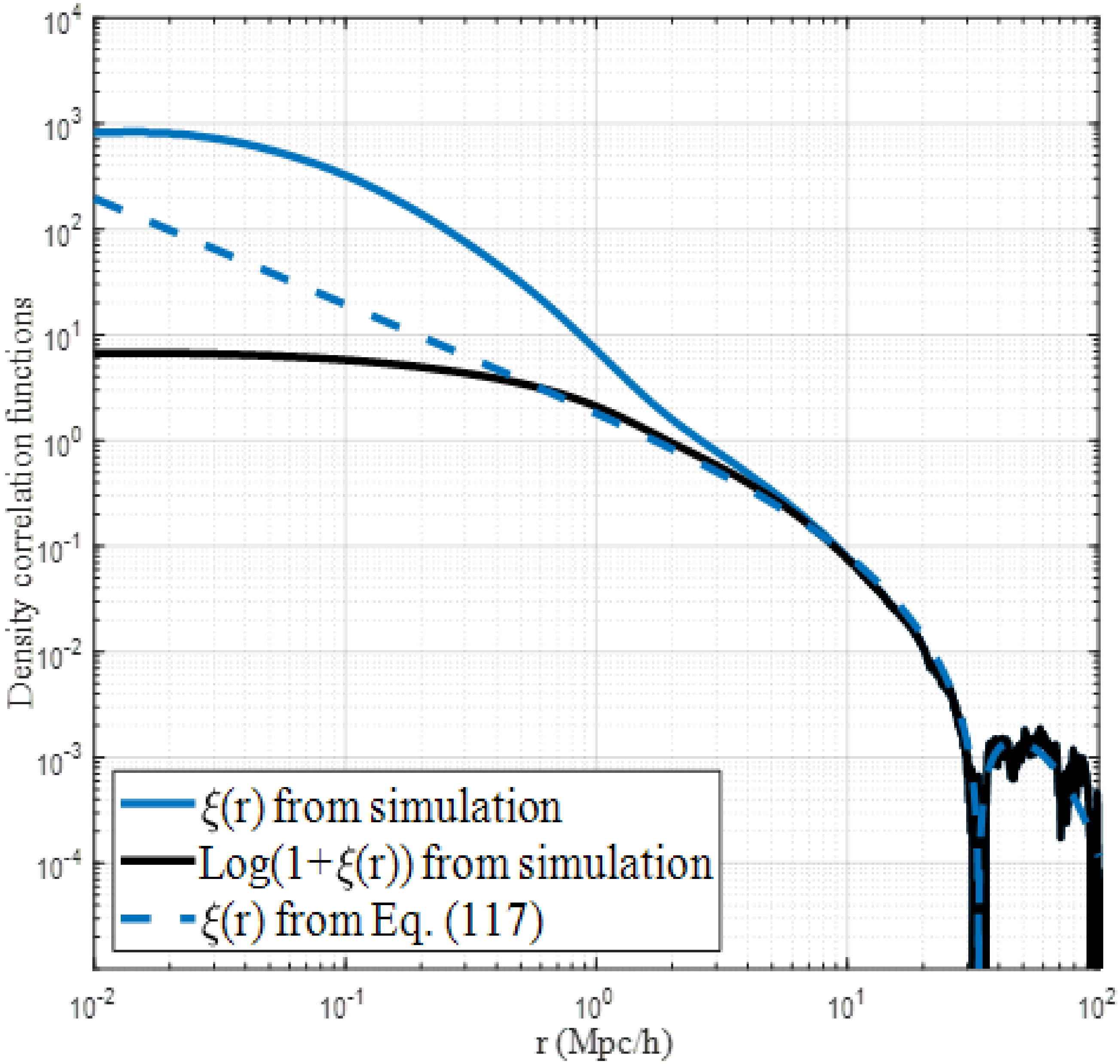
$$\xi(r,a) = \frac{1}{(aHf(\Omega_m))^2} \cdot \frac{a_0 u^2}{r r_2} \exp\left(-\frac{r}{r_2}\right)\left[\left(\frac{r}{r_2}\right)^2 - 7\left(\frac{r}{r_2}\right) + 8\right]$$

Averaged density correlation:

$$\bar{\xi}(r,a) = \frac{3}{r^3}\int_0^r \xi(y,a) y^2 dy == \frac{a_0 u^2}{(aHf(\Omega_m))^2} \frac{3}{r r_2} \exp\left(-\frac{r}{r_2}\right)\left(4 - \frac{r}{r_2}\right)$$

Potential correlation:

$$R_\phi = \frac{1}{2}\langle \phi(\mathbf{x}) \cdot \phi(\mathbf{x}') \rangle = \frac{9}{8}\left(\frac{aH}{f(\Omega_m)}\right)^2 a_0 u^2 r_2^2 \exp\left(-\frac{r}{r_2}\right)\left[\left(\frac{r}{r_2}\right) + 1\right] \propto a^0$$



Density correlation at z=0 and comparison with model

Velocity/density/potential spectrum functions on large scale

Velocity spectrum function:

$$E_u(k) = a_0 u^2 \frac{8}{\pi r_2} \frac{k^{-2}}{\left(1+1/(kr_2)^2\right)^3}$$

$$k_{\max} r_2 = \sqrt{2}$$

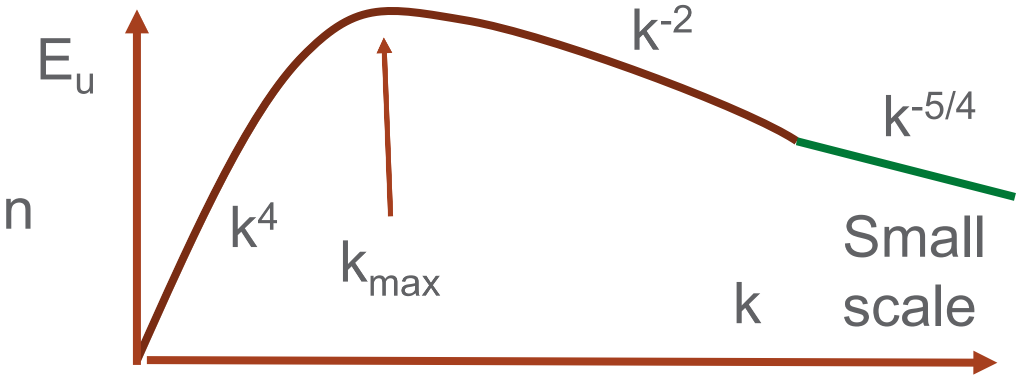
$$E_u(k_{\max}) = \frac{256}{125\pi} r_2 a_0 u^2$$

$E_u(k) \propto k^4$ for $kr_2 \ll 1$

$E_u(k) \propto k^{-2}$ for $kr_2 \gg 1$

[k^4 spectrum due to vanishing linear momentum]

Signature of Burger's equation in weakly nonlinear regime



Density spectrum function:

$$E_\delta(k) = \frac{16 a_0 u^2}{\left(aHf(\Omega_m)\right)^2 \pi r_2} \frac{1}{\left(1+1/(kr_2)^2\right)^3}$$

Potential spectrum function:

$$E_\phi(k) = \frac{18}{\pi r_2}\left(\frac{aH}{f(\Omega_m)}\right)^2 \frac{a_0 u^2 k^{-4}}{\left(1+1/(kr_2)^2\right)^3}$$

Matter power spectrum:

$$P_\delta(k,a) = 2\pi^2 E_\delta(k,a)/k^2 = \frac{32\pi a_0 u^2 r_2}{\left(aHf(\Omega_m)\right)^2} \frac{1}{(kr_2)^2\left(1+1/(kr_2)^2\right)^3}$$

$$P_\delta(k_{\max},a) = \frac{128\pi a_0 u^2 r_2}{27\left(aHf(\Omega_m)\right)^2}$$

Second order velocity dispersion functions and energy distribution in real space

Dispersion function for smoothed velocity (energy contained in scales above r):

$$\sigma_u^2(r) = \frac{1}{3}\int_{-\infty}^{\infty} E_u(k) W(kr)^2\, dk = \int_r^{\infty} E_{ur}(r')\, dr'$$

Window function for tophat spherical filter:

$$W(x) = \frac{3}{x^3}\left[\sin(x) - x\cos(x)\right] = 3\frac{j_1(x)}{x}$$

$$E_{ur}(r) = -\frac{\partial \sigma_u^2(r)}{\partial r}$$

Energy contained in scales between [r, r+dr]

Energy contained in scales below r:

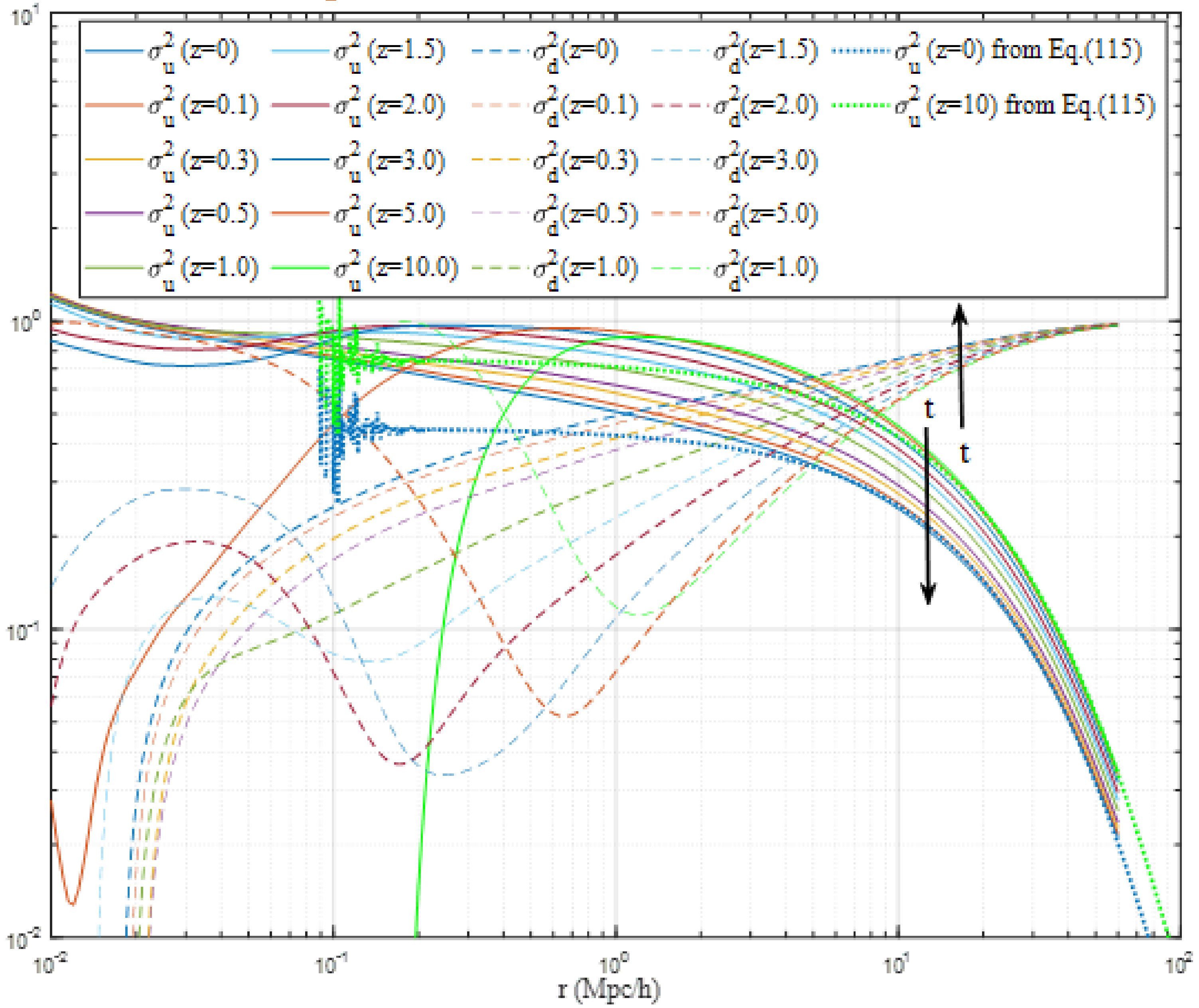
$$\sigma_d^2(r) = \frac{1}{3}\int_{-\infty}^{\infty} E_u(k)\left[1 - W(kr)^2\right] dk$$

$$\sigma_u^2(r) + \sigma_d^2(r) = u^2$$

Energy decomposed into scales below and above r:

Relations to velocity correlation function:

$$R_2(2r) = \frac{1}{24r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\sigma_u^2(r) r^4\right)\right)\right)$$



Variation of two dispersion functions with scale r (simulation). Fraction of energy contained in large scale decreases with time.

Second order velocity structure functions

Longitudinal Structure functions are moments of pairwise velocity:

$$S_m^{lp}(r) = \left\langle \left(\Delta u_L\right)^2 \right\rangle = \left\langle \left(u_L^{'} - u_L\right)^m \right\rangle$$

$$S_1^{lp}(r) = \left\langle \Delta u_L \right\rangle = \left\langle u_L^{'} - u_L \right\rangle$$

Second order longitudinal structure function (pairwise velocity dispersion):

$$S_2^{lp}(r) = \left\langle \left(\Delta u_L\right)^2 \right\rangle = \left\langle \left(u_L^{'} - u_L\right)^2 \right\rangle = 2\left(\left\langle u_L^2 \right\rangle - L_2(r)\right)$$

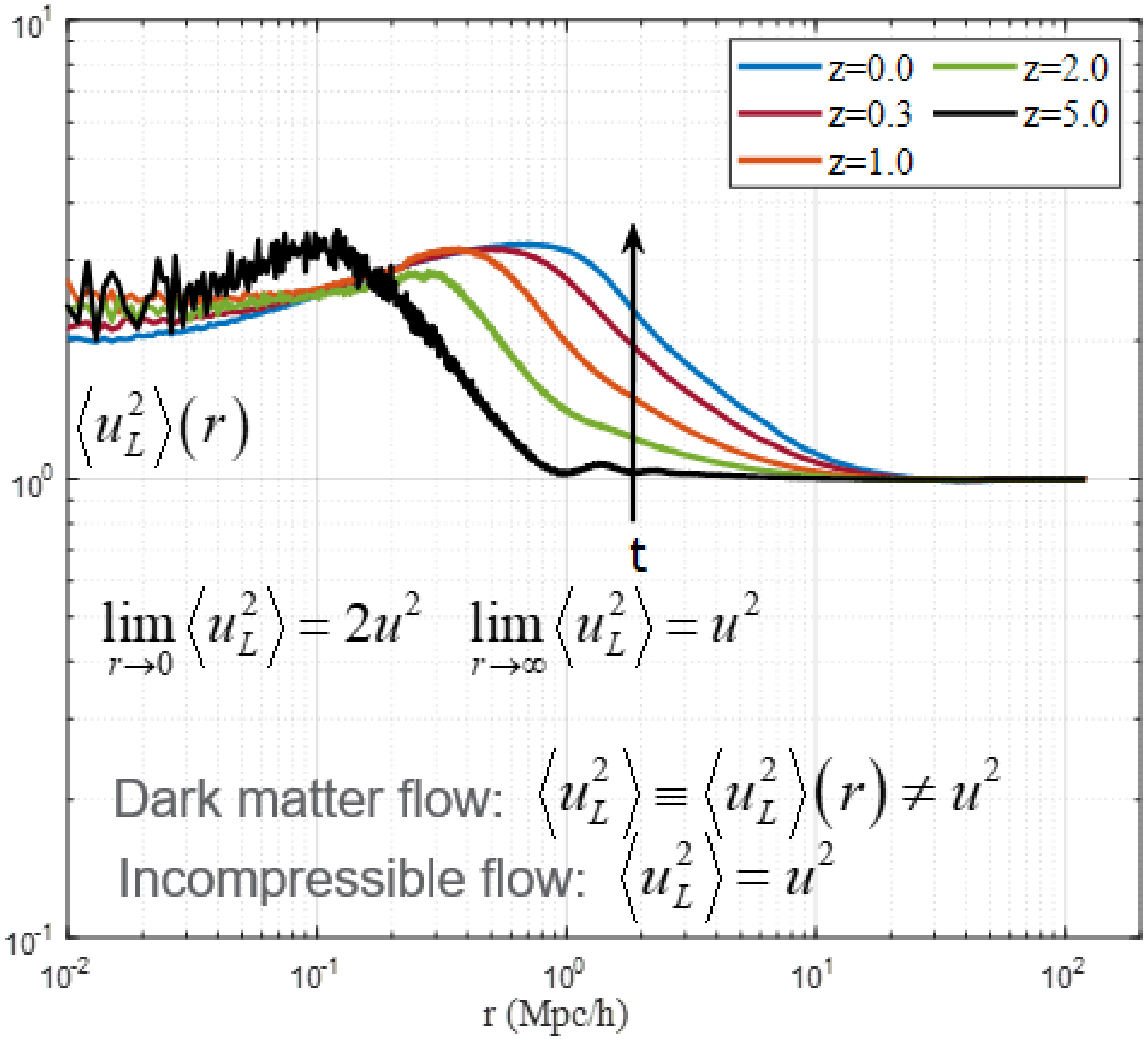
Second order longitudinal structure function (modified):

$$S_2^{l}(r) = 2\left(\lim_{r\to 0}\left\langle u_L u_L^{'} \right\rangle - L_2(r)\right) = 2\left(u^2 - L_2(r)\right)$$

$S_2^{lp}(r) \neq S_2^{l}(r)$ because of $\left\langle u_L^2 \right\rangle \neq u^2$

$$\lim_{r\to 0}\left\langle u_L^2 \right\rangle = 2u^2 \quad \lim_{r\to \infty}\left\langle u_L^2 \right\rangle = u^2 \qquad \lim_{r\to 0} S_2^{lp} = \lim_{r\to \infty} S_2^{lp} = 2u^2$$

$$\lim_{r\to 0} L_2(r) = \lim_{r\to 0} T_2(r) = u^2 \qquad \lim_{r\to \infty} L_2(r) = \lim_{r\to \infty} T_2(r) = 0$$



The variation of longitudinal velocity dispersion $\langle \Delta u_L^2 \rangle$ with scale r at different redshifts z

Second order velocity structure functions

Total velocity structure function:

$$S_2^{ip}(r) = \langle \Delta \mathbf{u}^2 \rangle = \left\langle \left(\mathbf{u}' - \mathbf{u}\right)^2 \right\rangle = 6\langle u_L^2 \rangle - 2R_2(r)$$

Total velocity structure function (modified):

$$S_2^i(r) = 6u^2 - 2R_2(r)$$

$$S_2^{ip}(r) \neq S_2^i(r) \quad \text{because of} \quad \langle u_L^2 \rangle \neq u^2$$

Relation to velocity spectrum function:

$$S_2^i(r) = 4\int_0^\infty E_u(k)\left(1 - j_0(kr)\right)dk$$

Relation to velocity dispersion function:

$$S_2^i(2r) = \frac{1}{12r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\sigma_d^2(r)r^4\right)\right)\right)$$

Structure function for enstrophy and real space enstrophy distribution:

Enstrophy: $$E_n = \int_0^\infty E_u(k)k^2dk$$

Enstrophy of smoothed velocity by a filter of size r:

$$\frac{S_2^x(r)}{2r^2} = \frac{1}{3}\int_0^\infty E_u(k)k^2W^2(kr)dk = \int_r^\infty E_{nr}(r')dr'$$

Real space distribution of enstrophy between [r r+dr]:

$$E_{nr}(r) = -\frac{\partial}{\partial r}\left[S_2^x(r)/\left(2r^2\right)\right]$$

Relation to total structure function:

$$\frac{1}{3r^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(S_2^x(r)r^4\right)\right) = \frac{\partial S_2^i(2r)}{\partial r}$$

Kinematic relations for structure functions

For incompressible flow or constant divergence flow:

$$S_2^l(r) = \frac{4}{3}\int_0^\infty E_u(k)\left(1 - 3\frac{j_1(kr)}{kr}\right)dk$$

Relation between different structure functions:

$$S_2^i(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 S_2^l(r)\right]$$

Relation to velocity dispersion functions:

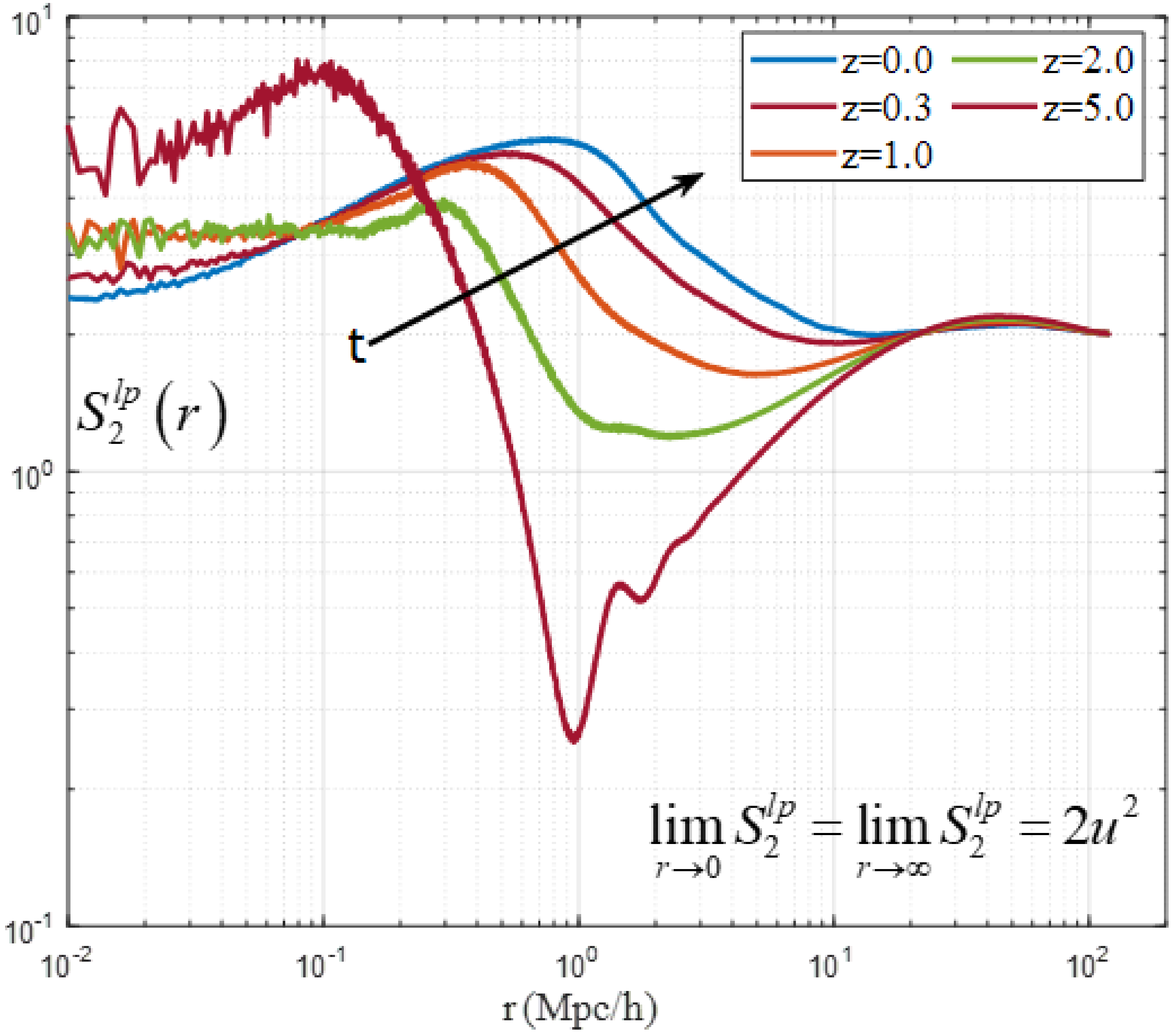
$$S_2^l(2r) = \frac{1}{12r^5}\frac{\partial}{\partial r}\left(r^3\frac{\partial}{\partial r}\left(\sigma_d^2(r)r^4\right)\right)$$

For irrotational flow:

$$S_2^l(r) = \frac{4}{3}\int_0^\infty E_u(k)\left(1 - 3j_0(kr) + 6\frac{j_1(kr)}{kr}\right)dk$$

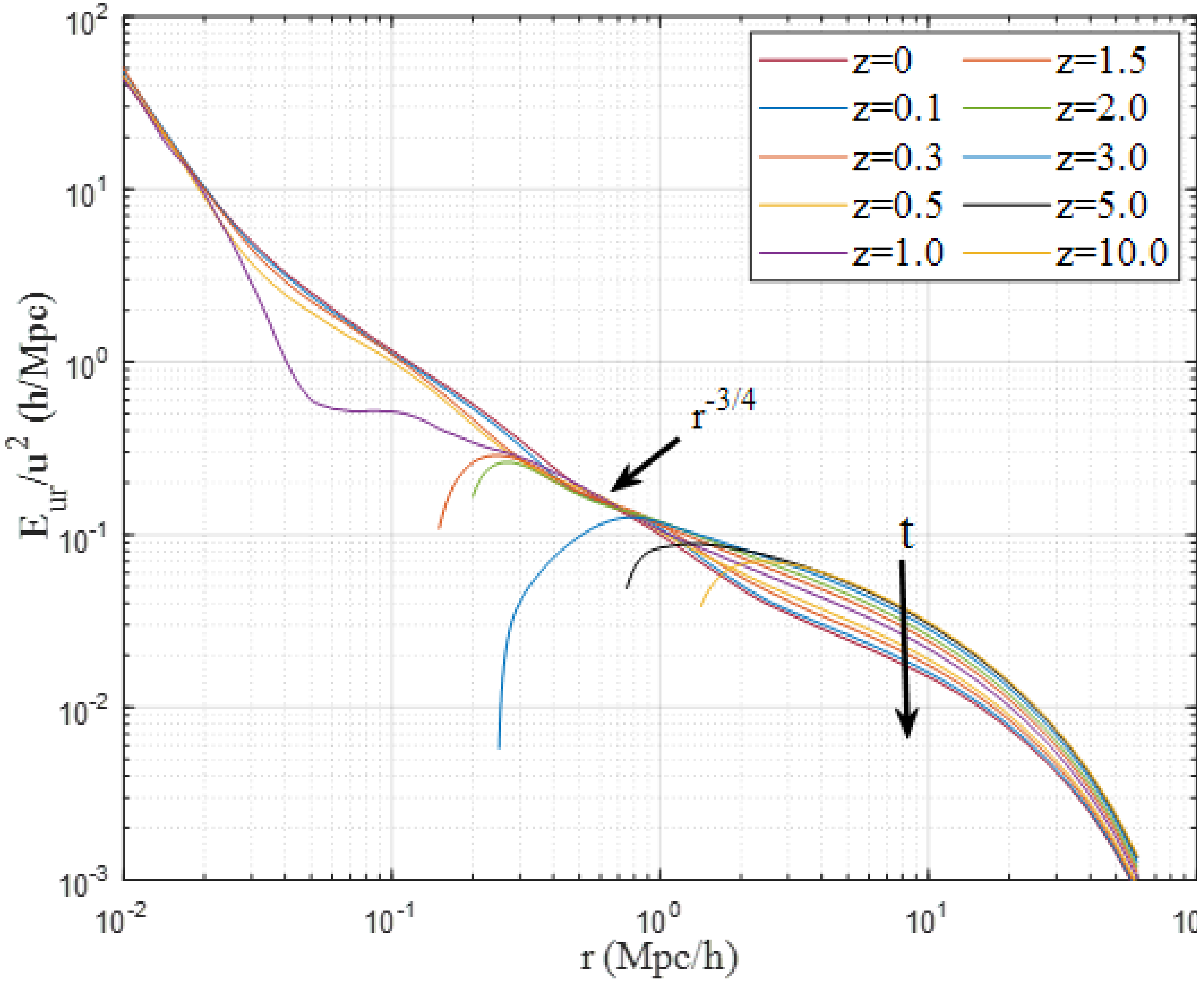
Relation between different structure functions:

$$\frac{\partial\left[rS_2^i(r)\right]}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 S_2^l(r)\right]$$

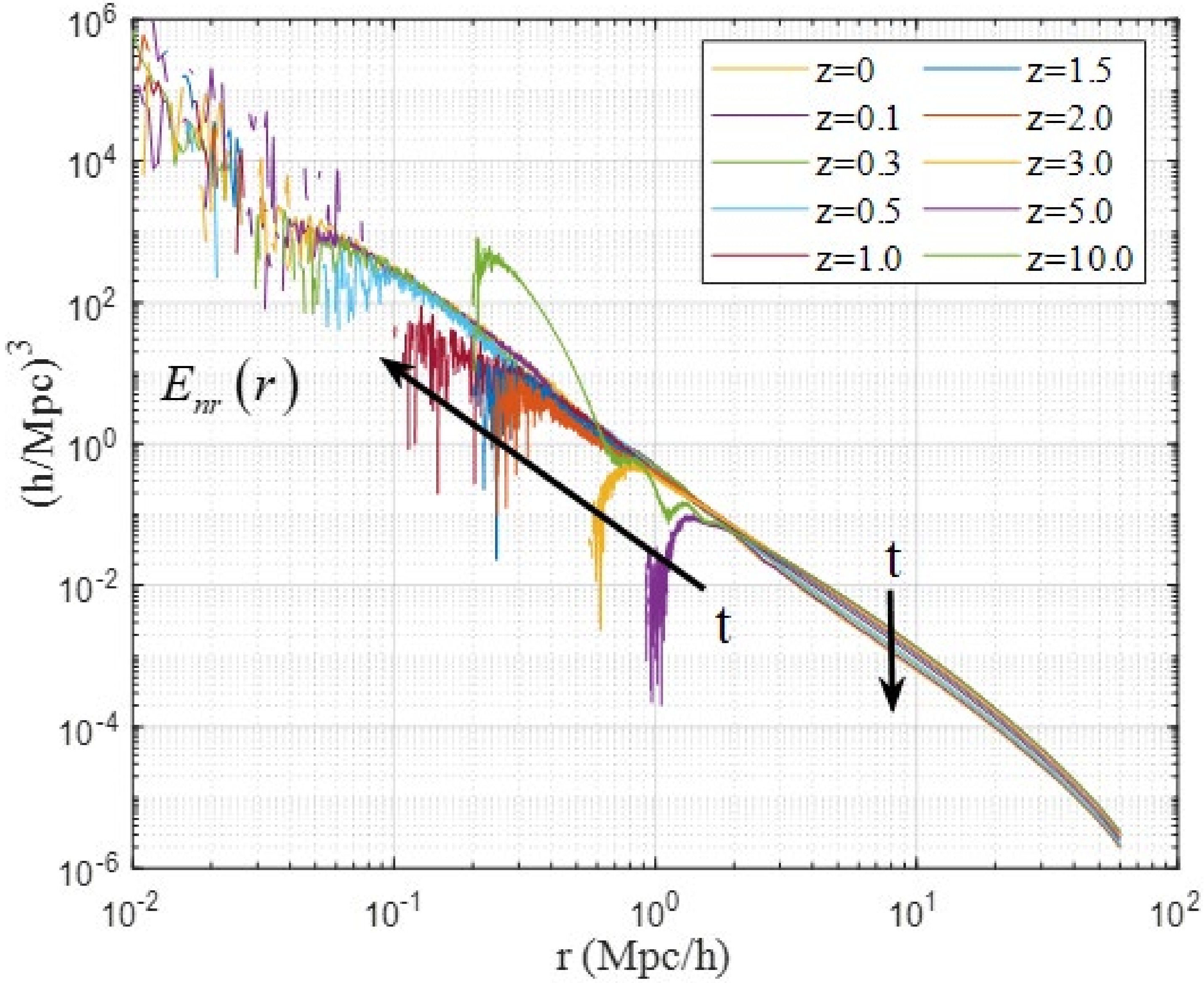


The variation of longitudinal velocity structure function S^{lp}_2 with scale r at different redshifts z

Energy and enstrophy distribution in real space



The real space distribution of energy on scale r at different redshifts



The real space distribution of enstrophy on scale r at different redshifts

Correlation functions of velocity gradients and Kinematic relations

Divergence of velocity:

$$\theta(\mathbf{x}) = \nabla \cdot \mathbf{u}(\mathbf{x})$$

Vorticity (curl):

$$\boldsymbol{\omega}(\mathbf{x}) = \nabla \times \mathbf{u}(\mathbf{x})$$

$$-\nabla^2 \left\langle \mathbf{u} \cdot \mathbf{u}' \right\rangle = \left\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \right\rangle + \left\langle \theta \cdot \theta' \right\rangle = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial R_2}{\partial r} \right]$$

$$-\nabla^2 \left\langle \mathbf{u} \cdot \mathbf{u}' \right\rangle = 2\int_0^\infty E_u(k) k^2 \frac{\sin(kr)}{kr} dk$$

Divergence and vorticity correlations:

$$R_\theta + R_{\boldsymbol{\omega}} = \frac{1}{4r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} S_2^i(r) \right)$$

$$R_\theta + R_{\boldsymbol{\omega}} = \frac{1}{96r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 \frac{\partial}{\partial r}\left(S_2^x(r) r^2 \right)\right)\right)$$

$$R_{\boldsymbol{\omega}} = \frac{\left\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \right\rangle}{2} = \frac{1}{r^2}\left[r^2 \left(A_2 r - \frac{\partial B_2}{\partial r} \right) \right]_{,r}$$

$$R_\theta = \frac{\left\langle \theta \cdot \theta' \right\rangle}{2} = -\frac{1}{2r^2}\left[r^2 \left(4A_2 r + \frac{\partial A_2}{\partial r} r^2 + \frac{\partial B_2}{\partial r} \right) \right]_{,r}$$

For incompressible flow or constant divergence flow:

Vorticity correlation (divergence is zero):

$$\left\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \right\rangle = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial R_2}{\partial r} \right] = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \left(r^3 L_2 \right)}{\partial r} \right) \right]$$

$$R_{\boldsymbol{\omega}}(r) = \frac{1}{2}\left\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \right\rangle = \int_0^\infty E_u(k) k^2 \frac{\sin(kr)}{kr} dk$$

For irrotational flow:

Divergence correlation (vorticity is zero):

$$\left\langle \theta \cdot \theta' \right\rangle = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial R_2}{\partial r} \right] = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial \left(r^3 T_2 \right)}{\partial r} \right) \right]$$

$$R_\theta(r) = \frac{1}{2}\left\langle \theta \cdot \theta' \right\rangle = \int_0^\infty E_u(k) k^2 \frac{\sin(kr)}{kr} dk$$

Modeling the longitudinal structure function on large scale

Structure function (pairwise velocity dispersion):

$$S_2^{lp}(r)=\left\langle\left(\Delta u_L\right)^2\right\rangle=2\left(\left\langle u_L^2\right\rangle-L_2(r)\right)$$

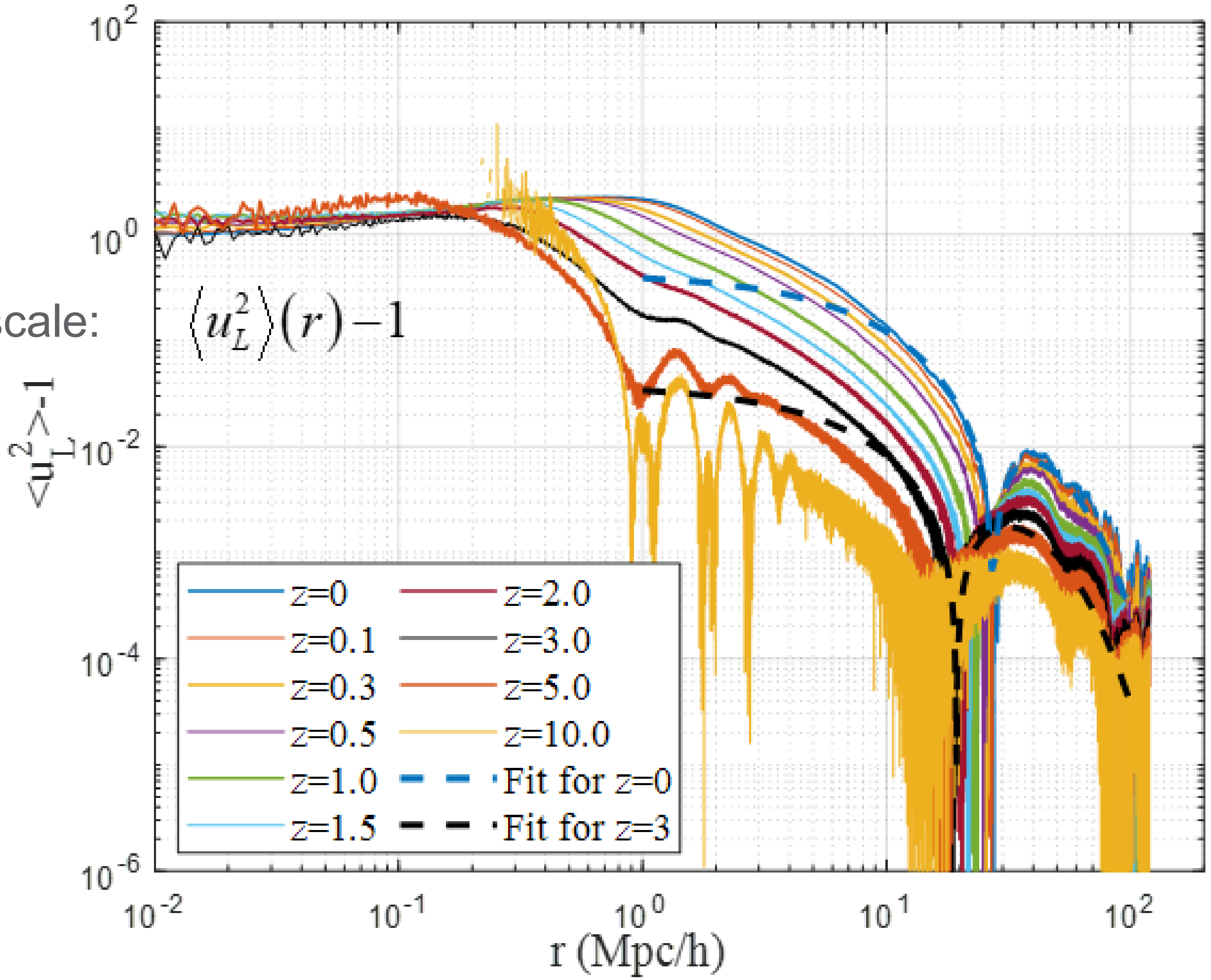
Modeling longitudinal velocity dispersion on large scale:

$$\left\langle u_L^2\right\rangle=u^2\left[1+a_d\exp\left(-\frac{r}{r_{d1}}\right)\left(1-\frac{r}{r_{d2}}\right)\right]$$

$$a_d=0.44a^{7/4}$$

$$r_{d1}=11.953\,Mpc/h$$

$$r_{d2}=27.4a^{1/4}\,Mpc/h$$



The variation of normalized longitudinal velocity dispersion

Modeling the longitudinal structure function on small scale (two-thirds 2/3 law)

Second order structure function (pairwise velocity dispersion):

$$S_2^{lp}(r)=\left\langle(\Delta u_L)^2\right\rangle=2\left(\left\langle u_L^2\right\rangle-L_2(r)\right) \quad \text{with} \quad \lim_{r\to 0} S_2^{lp}=2u^2$$

For hydrodynamic turbulence: $\lim_{r\to 0} S_2^{lp}=0$

Construct reduced structure function that is purely determined by the rate of energy cascade ε_u:

$$S_{2r}^{lp}=S_2^{lp}(r)-2u^2 \quad (m^2/s^2) \quad \text{and} \quad \varepsilon_u : (m^3/s^2)$$

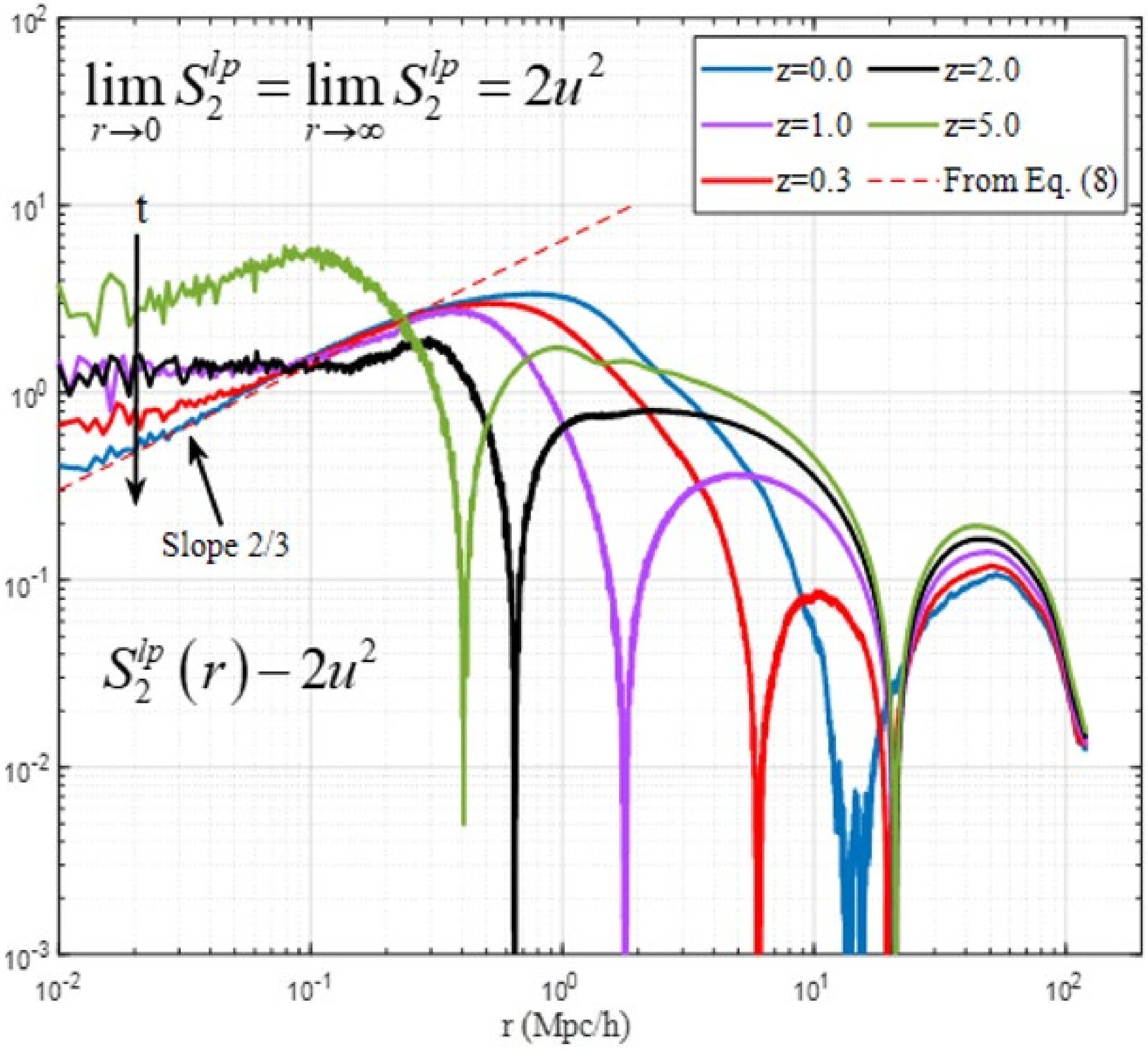
Dimensional analysis leads to two-thirds law for S_{2r}^{lp}

$$S_{2r}^{lp}\propto(-\varepsilon_u)^{2/3} r^{2/3} \quad \text{or} \quad S_{2r}^{lp}=a^{3/2}\beta_2^*(-\varepsilon_u)^{2/3} r^{2/3}$$

By introducing a length scale r_s: upper limit for two-thirds law

$$S_2^{lp}(r)=S_{2r}^{lp}+2u^2=u^2\left[2+\beta_2^*(r/r_s)^{2/3}\right]$$

$$r_s=-\frac{u_0^3}{\varepsilon_u}=\frac{4}{9}\frac{u_0}{H_0}=\frac{2}{3}u_0 t_0\approx 1.58\,Mpc/h \quad \text{and} \quad \beta_2^*\approx 9.5$$



Variation of normalized reduced longitudinal structure function and two-thirds law

Two-thirds law might be used to predict dark matter particle properties

Modeling the longitudinal structure function on small scale (one-fourth ¼ law)

1/4 law for (modified) structure function on small scale:

Also see slides for additional information.

Potential energy for a sphere of radius r:

$$U(r) = -\frac{\int_0^r \frac{G}{y} M(y)\rho(y)4\pi y^2 dy}{\int_0^r \rho(x)4\pi x^2 dx}$$

Virial theorem

$$2T(r)+\gamma U(r)=0$$

$$\xi(r)=a^m\left(r/r_\xi\right)^{-n} \Rightarrow \rho(y)=\rho_0\left(1+\xi(y)\right)\approx \rho_0 a^m\left(y/r_\xi\right)^{-n}$$

$$U(r)=-\frac{3-n}{5-2n}\frac{GM(r)}{r}=-\frac{3a^m H_0^2 r^2}{2(5-2n)}\left(\frac{r}{r_\xi}\right)^{-n}$$

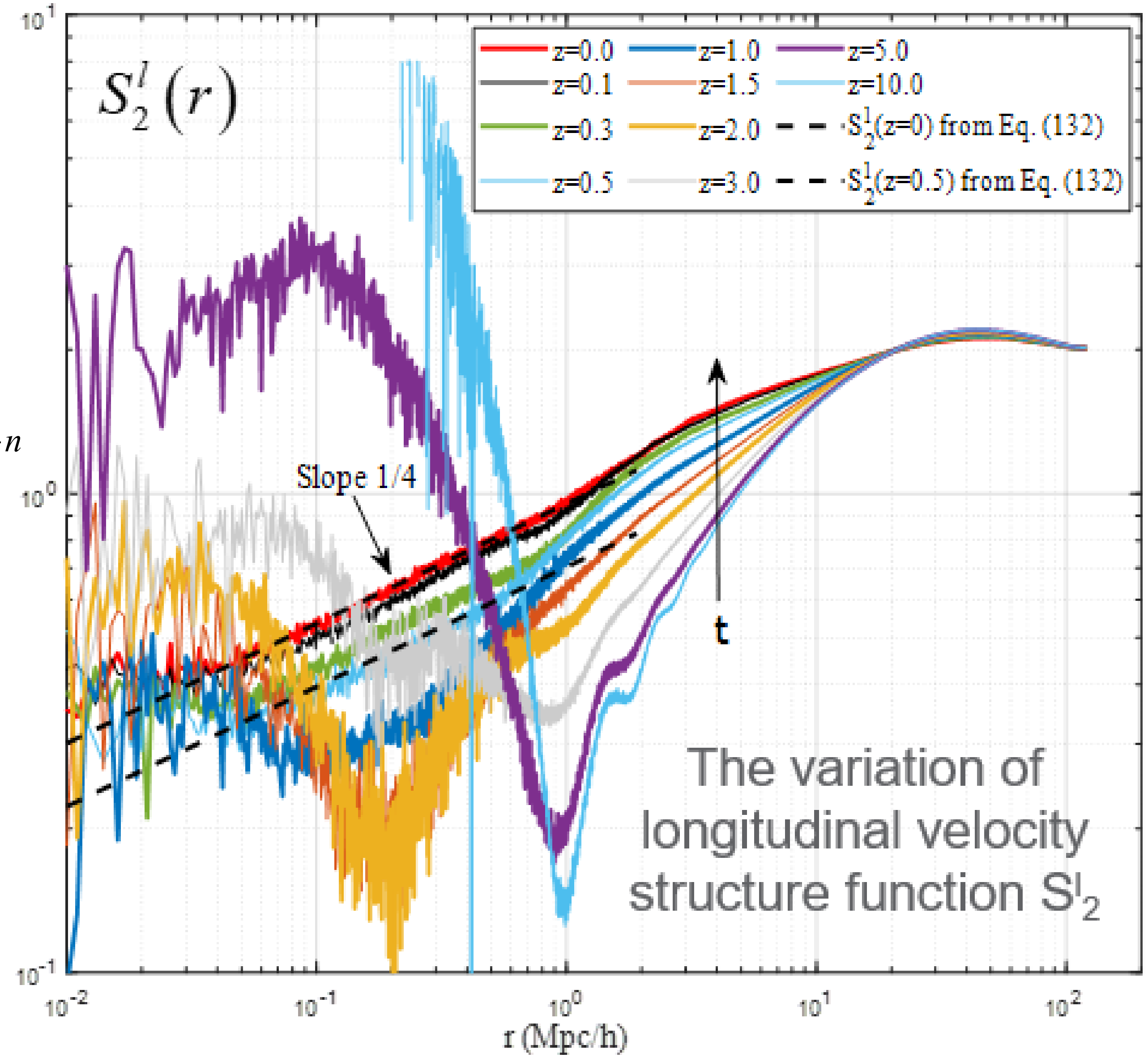
Use virial theorem:

$$\sigma_d^2(r)=\frac{\gamma\left(H_0 r\right)^2}{2r^5}\frac{\int_0^r\left(1+\xi(y)\right)\left(1+\bar{\xi}(y)\right)y^4 dy}{\left(1+\bar{\xi}(r)\right)}$$

$$\sigma_d^2(r)=\frac{\gamma a^m\left(H_0 r\right)^2}{2(5-2n)}\left(\frac{r}{r_\xi}\right)^{-n}$$

Use kinematic relation

$$S_2^l(r)=\frac{\gamma(6-n)(8-n)}{24(5-2n)2^{2-n}}a^m\left(H_0 r\right)^2\left(\frac{r}{r_\xi}\right)^{-n} \Rightarrow S_2^l(r)\propto r^{1/4}$$



Modeling velocity correlation functions on small scale

1/4 law for (modified) longitudinal structure function can be used to derive all other velocity correlations on small scale:

$$S_2^l = 2u^2\left(r/r_1\right)^n \quad \text{with} \quad n \approx 1/4$$

$$r_1(a) \approx r_1^* a^{-3} \quad \text{and} \quad r_1^* \approx 19.4\,Mpc/h$$

Using kinematic relations on small scale:

Longitudinal correlation
$$L_2(r) = u^2\left[1-\left(\frac{r}{r_1}\right)^n\right]$$

Transverse correlation
$$T_2 = u^2\left[1-\frac{2+n}{2}\left(\frac{r}{r_1}\right)^n\right]$$

Total correlation
$$R_2 = u^2\left[3-(3+n)\left(\frac{r}{r_1}\right)^n\right]$$

Velocity dispersion function for energy contained below scale r:

$$\sigma_d^2(r) = \frac{24\cdot 2^n}{(4+n)(6+n)}u^2\left(\frac{r}{r_1}\right)^n \approx 1.0745u^2\left(\frac{r}{r_1}\right)^n$$

Total structure function
$$S_2^i(r) = 2(3+n)u^2\left(\frac{r}{r_1}\right)^n$$

Structure function for enstrophy
$$S_2^x(r) = \frac{6n(3+n)\cdot 2^n}{(4+n)(2+n)}u^2\left(\frac{r}{r_1}\right)^n = 0.6063u^2\left(\frac{r}{r_1}\right)^n$$

Vorticity correlation
$$R_{\boldsymbol{\omega}} = \frac{1}{2}\left\langle \boldsymbol{\omega}(\mathbf{x})\cdot\boldsymbol{\omega}(\mathbf{x}')\right\rangle = \frac{n(1+n)(3+n)}{2r^2}u^2\left(\frac{r}{r_1}\right)^n$$

Velocity & vorticity spectrum
$$E_u(k) = Cu^2 r_1^{-n} k^{-(1+n)} \qquad E_{\boldsymbol{\omega}}(k) = Cu^2 r_1^{-n} k^{(1-n)}$$

Proportional constant
$$C = -\frac{2(3+n)\Gamma\left((n+3)/2\right)}{2^{1-n}\Gamma(3/2)\Gamma(-n/2)} = 0.4485$$

Modeling the velocity correlations on entire range

- Correlation functions are modelled on both large and small scales
- Need smooth and differentiable velocity correlations for the entire range of scales
- Correlations of vorticity and divergence can be obtained as derivatives of velocity correlations

$$f_1(r) = R_{2s}(r) = 3 - (3+n)\left(\frac{r}{r_1}\right)^n$$

Correlation function on small scale

$$f_2(r) = R_{2l}(r) = a_0 \exp\left(-\frac{r}{r_2}\right)\left(3 - \frac{r}{r_2}\right)$$

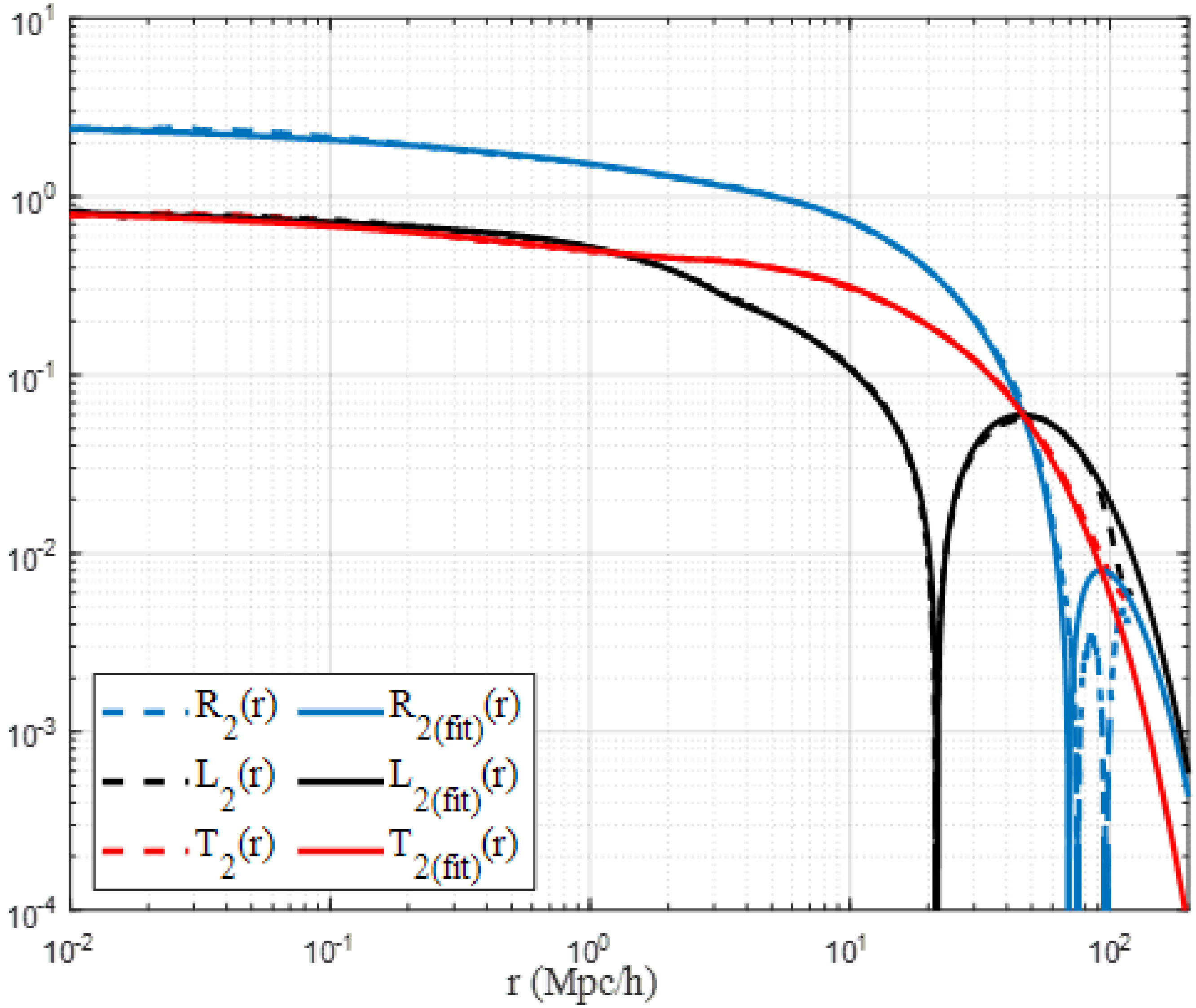
Correlation function on large scale

$$s(r) = \frac{1}{1 + x_b e^{-(r-x_c)/x_a}}$$

Interpolation function for smooth connection

$$R_{2(fit)}(r) = R_{2s}\left(1 - s(r)\right)^{n1} + R_{2l}\left(s(r)\right)^{n2}$$

Final fitted correlation function is obtained by parameter optimization using correlations from N-body simulation



The fitted velocity correlation functions compared to original correlations from N-body simulation

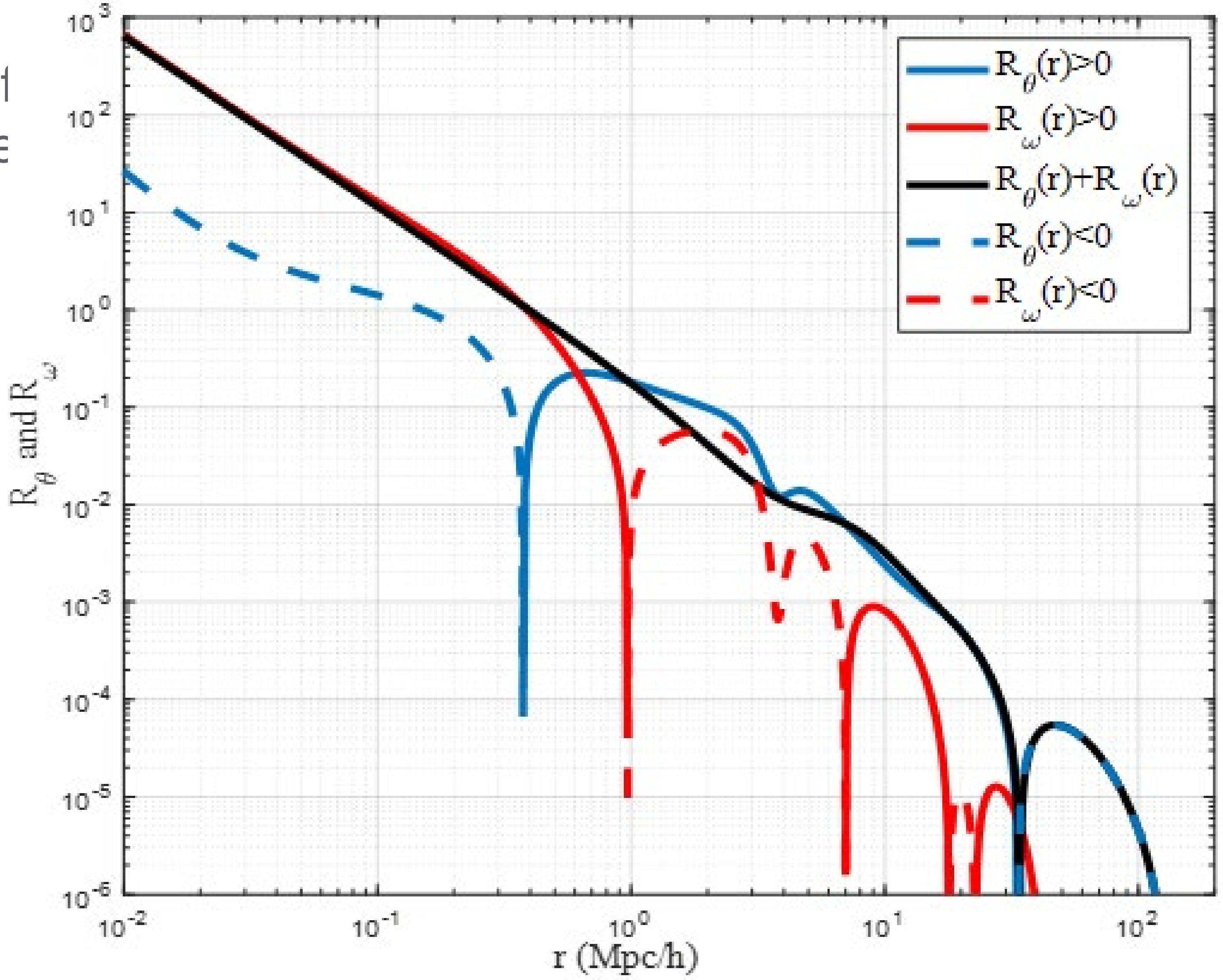
Modeling divergence and vorticity correlations on entire range of scales

- With correlation functions modelled on entire range of scales, correlations of divergence and vorticity can be obtained using kinematic relations.

- Divergence is negatively correlated on scale $r > 30\,Mpc/h$

- Vorticity is negatively correlated for scale r between 1Mpc/h and 7Mpc/h (pair of particles mostly from different halos) and positively correlated on small scale (pair of particles from the same halo).

- Vorticity is dominant on small scale while divergence dominant on large scale.



Variation of correlation functions of divergence and vorticity with scale *r* at z=0

Summary and keywords

Velocity correlation tensor	Longitudinal velocity	Two-thirds law / one-fourth law
Kinematic relations	Transverse velocity	Spectrum functions
Correlation functions	Structure functions	Dispersion functions

- Identify connections with homogeneous isotropic turbulence for the development of the statistical theory in terms of correlation, structure, dispersion, and spectrum functions
- Identify the nature of peculiar velocity in dark matter flow: constant divergence flow on small scale and irrotational flow on large scale.
- Develop kinematic relations between different statistical measures
- The limiting correlation coefficient of velocity $\rho=1/2$ on the smallest scale ($r=0$) is a unique feature of dark matter flow ($\rho=1$ for incompressible flow) along with the implications for particle annihilation
- On large scale, the transverse velocity correlation has an exponential form with a comoving length scale $r_2=21.3$Mpc/h. All correlation/structure/dispersion/spectrum functions for velocity, density, and potential can be derived analytically using kinematic relations for irrotational flow.
- On small scale, the longitudinal structure function follows a one-fourth law $S_2^l \sim r^{1/4}$, along with other correlation/structure/dispersion/spectrum functions obtained from kinematic relations for constant divergence flow.