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A comparative study of dark matter flow & hydrodynamic turbulence and its applications

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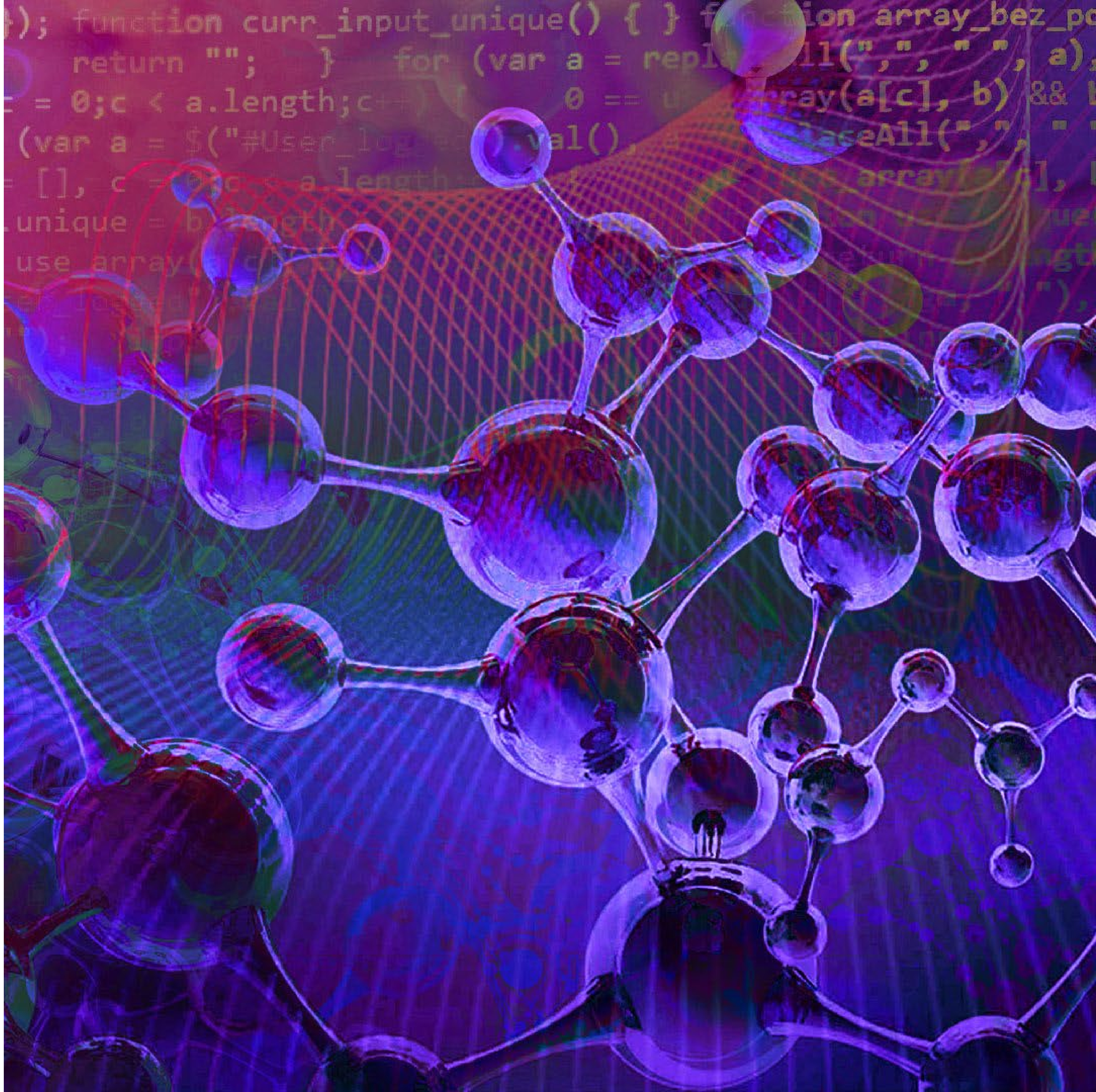


Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!

Data repository and relevant publications

Structural (halo-based) approach:

0.	**Data https://dx.doi.org/10.5281/zenodo.6541230**
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	**Data https://dx.doi.org/10.5281/zenodo.6569898**
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

Structural (halo-based) approach for dark matter flow

Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion

arXiv:2202.04054 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2202.04054

Introduction

Review: In freely decaying turbulence, there is no energy injection on large scale and total energy is continuously decaying with time.

- Both integral scale l (energy-contained scale) and energy dissipation rate ε vary with time.
- What is the large-scale dynamics of freely decaying turbulence? How does energy evolve with time?

$$\varepsilon \equiv A\frac{u^2}{(l/u)} = A\frac{u^3}{l}$$

Loitsyansky integral invariant (integral of velocity correlation):

$$\int \langle \mathbf{u}\cdot\mathbf{u}' \rangle \mathbf{r}^2 d\mathbf{r} \approx u^2 l^5 = const$$

$$u^2 \sim t^{-10/7}$$

$$l \sim t^{2/7}$$

$$\varepsilon \sim t^{-17/7}$$

Due to the formation and virilization of halos, the kinetic energy in dark matter flow continuously increases with time. In this regard, dark matter flow is a freely growing turbulence.

What is the large-scale dynamics of dark matter flow? How do energy/momentum evolve with time?

- Goal 1: Formulate large scale dynamics in dark matter flow (how energy and momentum evolves?)
- Goal 2: Energy, momentum and spin parameter in halos
- Goal 3: Formulate integral "constants" on large and halo scales (are they still constants?)

Equations of motion in comoving and transformed systems

Equation of motion in a comoving system with expanding background → Equation of motion in a transformed system with fixed damping in static background

$$\frac{d^2\mathbf{x}_i}{dt^2}+2H\frac{d\mathbf{x}_i}{dt}=-\frac{Gm_p}{a^3}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^3}$$

Potential with an arbitrary exponent of ***n*** for particle-particle interacting

$$V_p(r)=-G_n m_p^2/r^{-n}$$

$$\frac{d^2\mathbf{x}_i}{dt^2}+2H\frac{d\mathbf{x}_i}{dt}=\frac{nG_n m_p}{a^3}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}$$

Introduce a new transformed time scale **s**

$$ds/dt=a^p$$

- If p=-2, *s* is the time variable for integration in N-body simulation.
- Transformed system: fixed damping and no scale factor *a*;

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{d\mathbf{x}_i}{ds}(p+2)a^{-p}H=\frac{nG_n m_p}{a^{3+2p}}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}$$

$$\dot{H}=-3H^2/2$$

$$H^2=8\pi G\bar{\rho}_y(a)/3$$

$$H_0^2=H^2a^3$$

$p=-3/2$ → Matter dominant

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{H_0}{2}\frac{d\mathbf{x}_i}{ds}=nG_n m_p\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}=\frac{\mathbf{F}_i}{m_p}$$

Peculiar velocity in comoving:

$$\mathbf{u}_i=a\frac{d\mathbf{x}_i}{dt}=\frac{d\mathbf{r}_i}{dt}-H\mathbf{r}_i=a^{-1/2}\mathbf{v}_i$$

Velocity in time scale **s**:

$$\mathbf{v}_i=\frac{d\mathbf{x}_i}{ds}=a^{3/2}\frac{d\mathbf{x}_i}{dt}=a^{1/2}\mathbf{u}_i$$

Energy evolution in transformed system

Starting from equation of motion in transformed system:

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{H_0}{2}\frac{d\mathbf{x}_i}{ds}=nG_nm_p\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}=\frac{\mathbf{F}_i}{m_p}$$

Express force as potential gradient:

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{1}{2}H_0\frac{d\mathbf{x}_i}{ds}+N\frac{\partial P_s}{\partial \mathbf{x}_i}=0$$

$\cdot\left(\frac{d\mathbf{x}_i}{ds}\right)$ Dot product on both sides:

Time evolution of energy in ***s***:

$$\frac{\partial\left(P_s+K_s\right)}{\partial s}+H_0K_s=0$$

Exactly same as damped oscillator. Need additional relation to close.

Time evolution of energy in ***t***:

$$\frac{\partial}{\partial t}\left(K_p+a^{-n-1}P_y\right)+H\left(2K_p+a^{-n-1}P_y\right)=0$$

With n=-1

$$\frac{\partial}{\partial t}\left(K_p+P_y\right)+H\left(2K_p+P_y\right)=0$$

Standard cosmic energy equation

Specific potential energy (radial moment):

$$P_s=\frac{1}{N}\sum_i^N\phi\left(\mathbf{x}_i\right)=a^{-n}P_y$$

Potential energy in physical coordinate

Specific kinetic energy of entire system:

$$K_s=\frac{1}{2N}\sum_{i=1}^N\mathbf{v}_i^2=\frac{a}{2N}\sum_{i=1}^N\mathbf{u}_i^2=aK_p$$

Peculiar kinetic energy

\+ Recall solution from Two-body collapse model (TBCM): exponential evolution of energy

$$K_s=\alpha\exp\left(-\frac{H_0s}{1+\beta/\alpha}\right)\quad P_s=\beta\exp\left(-\frac{H_0s}{1+\beta/\alpha}\right)$$

Energy evolution in comoving system and ε_u

Transformation back to comoving system:

$$ds/dt = a^{-3/2} \Rightarrow s = t_0 \ln(t/t_i)$$

Exponential in ***s*** corresponds to power-law in ***t***:

$$\exp(\tau H_0 s) \rightarrow (a/a_i)^{\tau}$$

Power-law time evolution for energy in terms of rate of energy cascade ε_u:

$$K_p = -\varepsilon_{\mathbf{u}} t^{-\frac{2}{3}\frac{(2+\beta/\alpha)}{(1+\beta/\alpha)}} = -\varepsilon_{\mathbf{u}} a^{-\frac{(2+\beta/\alpha)}{(1+\beta/\alpha)}}$$

Power-law for Peculiar kinetic energy

$$P_y = a^n P_s = \frac{\beta}{\alpha}\varepsilon_{\mathbf{u}} a^{n-\frac{1}{(1+\beta/\alpha)}}$$

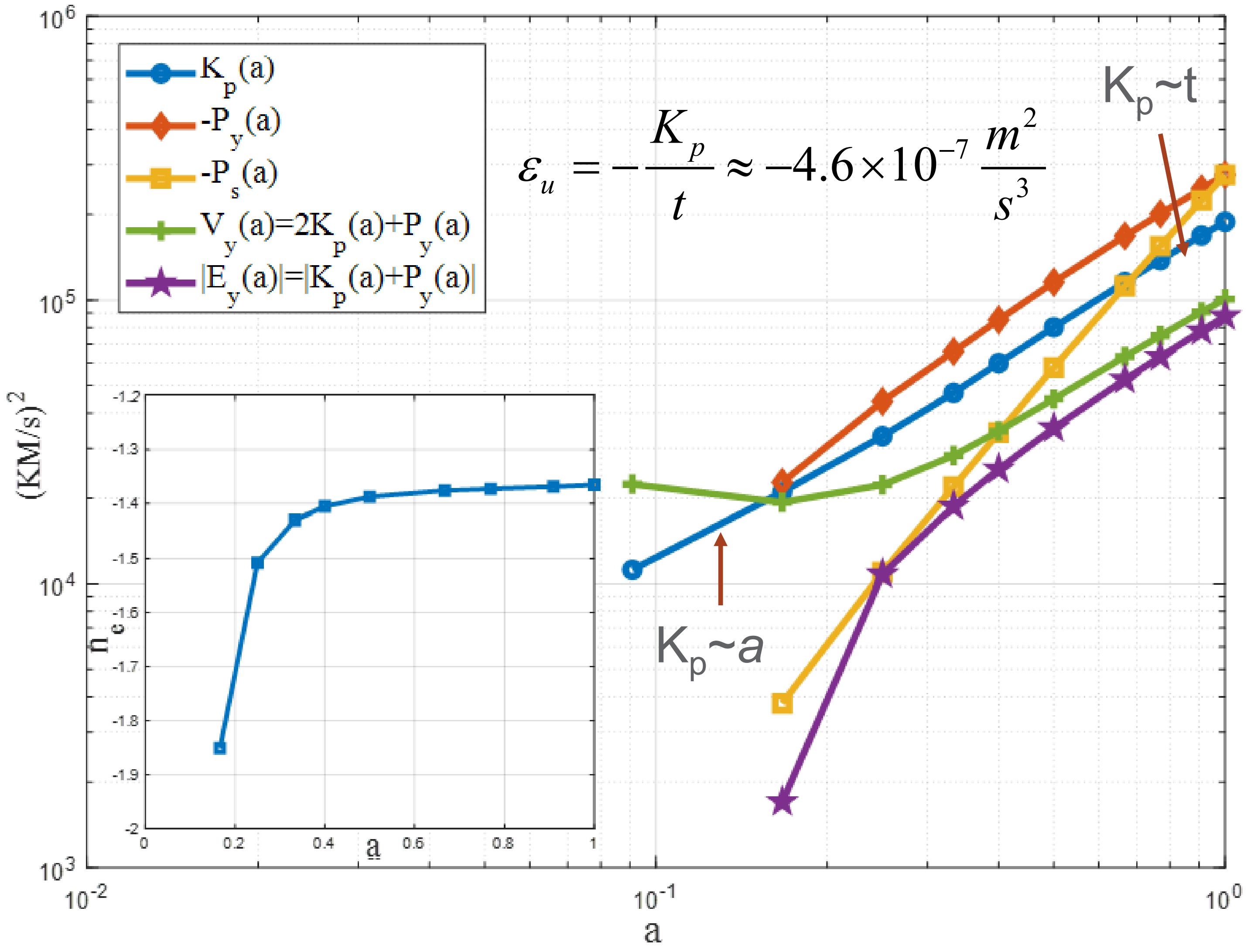
Power-law for potential energy

N-body simulation Early time: $K_p \propto a$; Early time: $K_p \propto t$;

Effective exponent for virial theorem:

$$n_e = \frac{2K_p}{P_y} = -\frac{10}{7} \neq -1$$

Mostly from Halo surface energy



The variation of kinetic and potential energies with scale factor *a* from a N-body simulation. Both energies exhibit a power-law scaling.

Momentum evolution in transformed system

Starting from equation of motion in transformed system:

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{H_0}{2}\frac{d\mathbf{x}_i}{ds}=nG_n m_p\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}=\frac{\mathbf{F}_i}{m_p}$$

Express force as potential (P_s) gradient:

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{1}{2}H_0\frac{d\mathbf{x}_i}{ds}+N\frac{\partial P_s}{\partial \mathbf{x}_i}=0$$

Dot product on both sides $\bullet\mathbf{x}_i$

Time evolution of virial quantity in ***s***:

$$\frac{dG_s}{ds}+\frac{1}{2}H_0 G_s=2K_s-nP_s$$

Time evolution of virial quantity in ***t***:

$$\frac{dG_p}{dt}+HG_p=\frac{2aK_p-na^{-n}P_y}{a^2}$$

Specific virial quantity (radial moment):

$$G_s=\frac{1}{N}\sum_i^N\mathbf{v}_i\cdot\mathbf{x}_i=a^{1/2}\frac{1}{N}\sum_i^N\mathbf{u}_i\cdot\mathbf{x}_i=a^{1/2}G_p$$

Peculiar virial quantity

Specific angular momentum:

$$\mathbf{H}_s=\frac{1}{N}\sum_i^N\mathbf{x}_i\times\mathbf{v}_i=\frac{1}{N}\sum_i^N\mathbf{x}_i\times\mathbf{u}_i a^{1/2}=a^{1/2}\mathbf{H}_p$$

Peculiar angular momentum

Taking cross product on both sides

$$\frac{d^2\mathbf{x}_i}{ds^2}+\frac{1}{2}H_0\frac{d\mathbf{x}_i}{ds}+N\frac{\partial P_s}{\partial \mathbf{x}_i}=0 \qquad \times\mathbf{x}_i$$

Time evolution of angular momentum in ***s***:

$$\frac{d\mathbf{H}_s}{ds}+\frac{H_0}{2}\mathbf{H}_s=-\sum_i^N\mathbf{x}_i\times\frac{\partial P_s}{\partial \mathbf{x}_i}=\frac{1}{m_p}\sum_i^N\mathbf{x}_i\times\mathbf{F}_i=0$$

- This is for open system without boundary.
- Extra care is needed for N-body systems with periodic boundaries

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The evolution of momentum on halo and large scale

Virial quantity in entire N-body system:

$$G_p = \frac{1}{N}\sum_{i}^{N}\mathbf{u}_i \cdot \mathbf{x}_i = \frac{1}{a}G_{py}$$

Subscript
- "p" for Comoving
- "py" for physical coordinate

Angular momentum in entire N-body system:

$$\mathbf{H}_p = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i \times \mathbf{u}_i = \frac{1}{a}\mathbf{H}_{py}$$

Decompose both position x and velocity u:

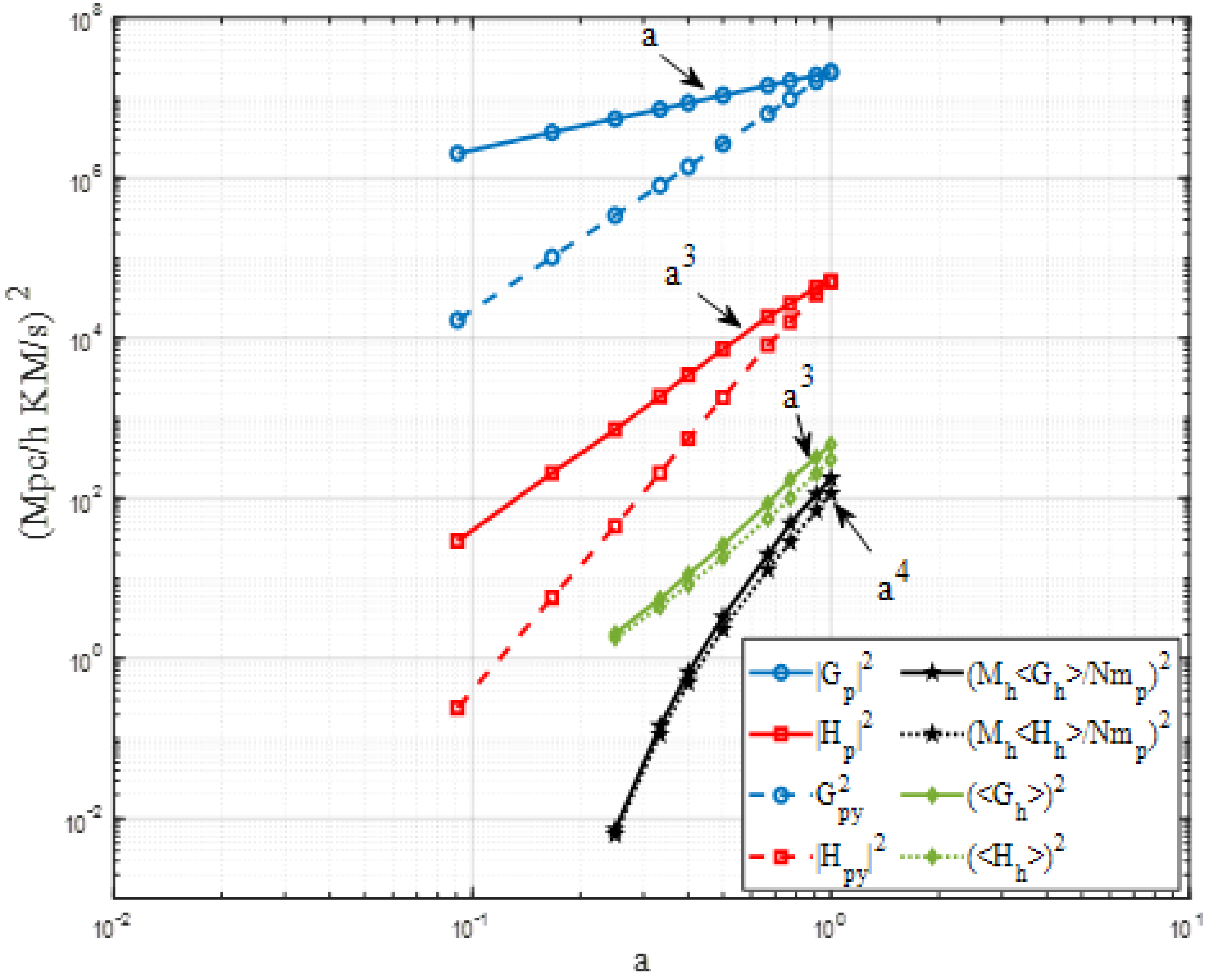
$$\mathbf{x}_i = \mathbf{x}_h + \mathbf{x}_i^{'} \qquad \mathbf{u}_i = \mathbf{u}_h + \mathbf{u}_i^{'}$$

Halo virial quantity (radial momentum):

$$G_{hc} = \frac{1}{n_p}\sum_{i=1}^{n_p}\left(\mathbf{x}_p^{'} \cdot \mathbf{u}_p^{'}\right) = \frac{1}{a}G_h$$

Halo angular momentum:

$$\mathbf{H}_{hc} = \frac{1}{n_p}\sum_{i=1}^{n_p}\left(\mathbf{x}_p^{'} \times \mathbf{u}_p^{'}\right) = \frac{1}{a}\mathbf{H}_h$$



On large scale (see here for proof)

$$G_{py} \propto a^{3/2}$$
$$\left|\mathbf{H}_{py}\right| \propto a^{5/2}$$

On halo scale (Consistent with previous results)

$$\langle G_h \rangle \propto a^{3/2} \propto t$$
$$\langle |\mathbf{H}_h| \rangle \propto a^{3/2} \propto t$$

The variation of energy in halos

- Identify all halos of different sizes
- Group halos according to halo size n_p or m_h
- Compute mean square radius r_g for each halo
- Compute halo virial kinetic energy σ_v^2 for each halo
- Compute intra-halo potential energy Φ_h
- Compute the group average and std.

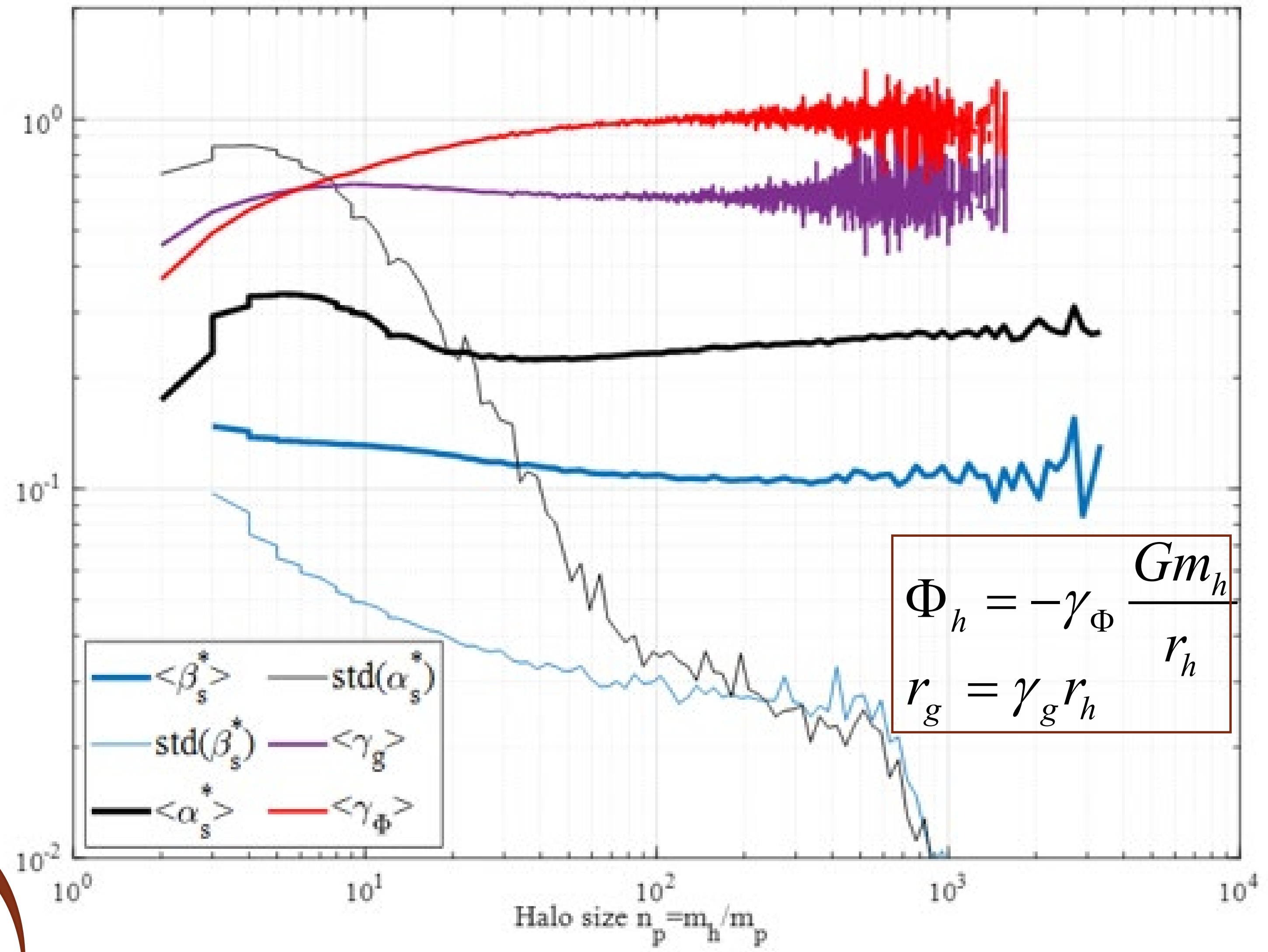
Halo virial kinetic energy: $$\sigma_v^2 = \frac{1}{3n_p}\sum_{k=1}^{n_p}\left|\mathbf{u}_k - \mathbf{u}_h\right|^2$$

Halo velocity: $$\mathbf{u}_h = \frac{1}{n_p}\sum_{k=1}^{n_p}\mathbf{u}_k$$

$$\beta_s^* = \frac{Hr_g}{\sigma_v} \qquad \alpha_s^* = \frac{\sigma_v^2 r_g}{Gm_h} \qquad \Delta_c = 18\pi^2 \qquad \gamma_v = -\frac{3\sigma_v^2}{\Phi_h}$$

Angle of incidence | Ratio of kinetic to potential energy | Critical density ratio | Halo virial ratio

$$\gamma_g = \left(\frac{1}{2}\alpha_s^*\beta_s^{*2}\Delta_c\right)^{1/3} \qquad \gamma_\Phi = \frac{6\left(\alpha_s^*\beta_s^{*2}\Delta_c/2\right)^{2/3}}{\gamma_v\beta_s^{*2}\Delta_c}$$



$$\Phi_h = -\gamma_\Phi\frac{Gm_h}{r_h}$$
$$r_g = \gamma_g r_h$$

Small halos of same size are generated at different time (large std). Large halos are synchronized and generated at the same time (small std).

The variation of momentum in halos

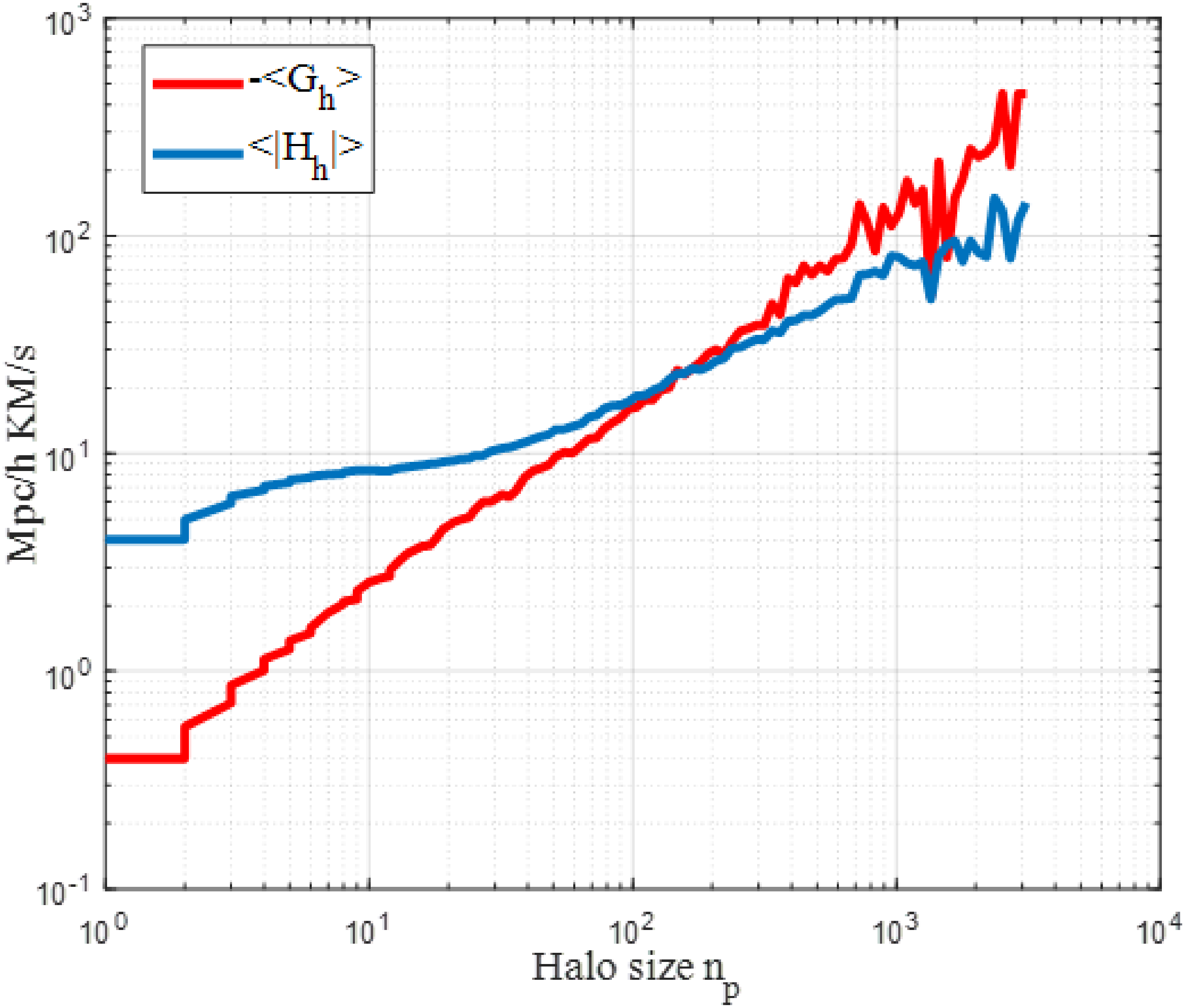
- Compute mean square radius r_g for each halo
- Compute halo virial kinetic energy σ_v^2 for each halo
- Compute radial momentum G_h for each halo
- Compute angular momentum H_h for each halo
- Compute the group average and std.

$$G_h = -\tau_s^* \sigma_v r_g a^{-1} \qquad \left|\mathbf{H}_h\right| = \eta_s^* \sigma_v r_g a^{1/2}$$

$$n_s^* = \frac{2K_h}{\Phi_h} = \frac{3\sigma_v^2}{\Phi_h} = -\gamma_v \qquad z_s^* = \frac{E_h}{\sigma_v^2} = \frac{K_h + \Phi_h}{\sigma_v^2}$$

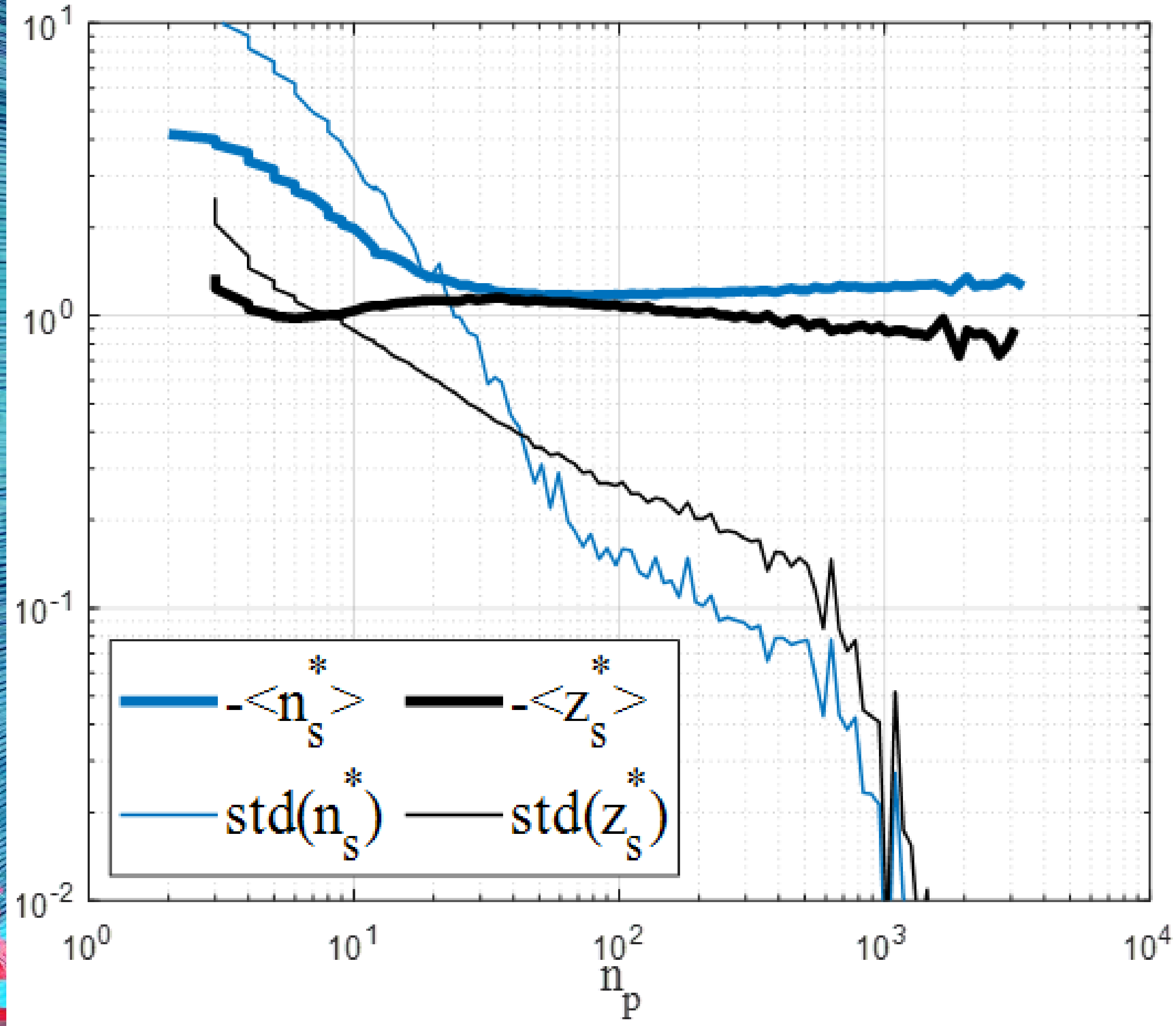
$$\lambda_p = a^{1/2} \alpha_s^* \eta_s^* \sqrt{\left|z_s^*\right|}$$

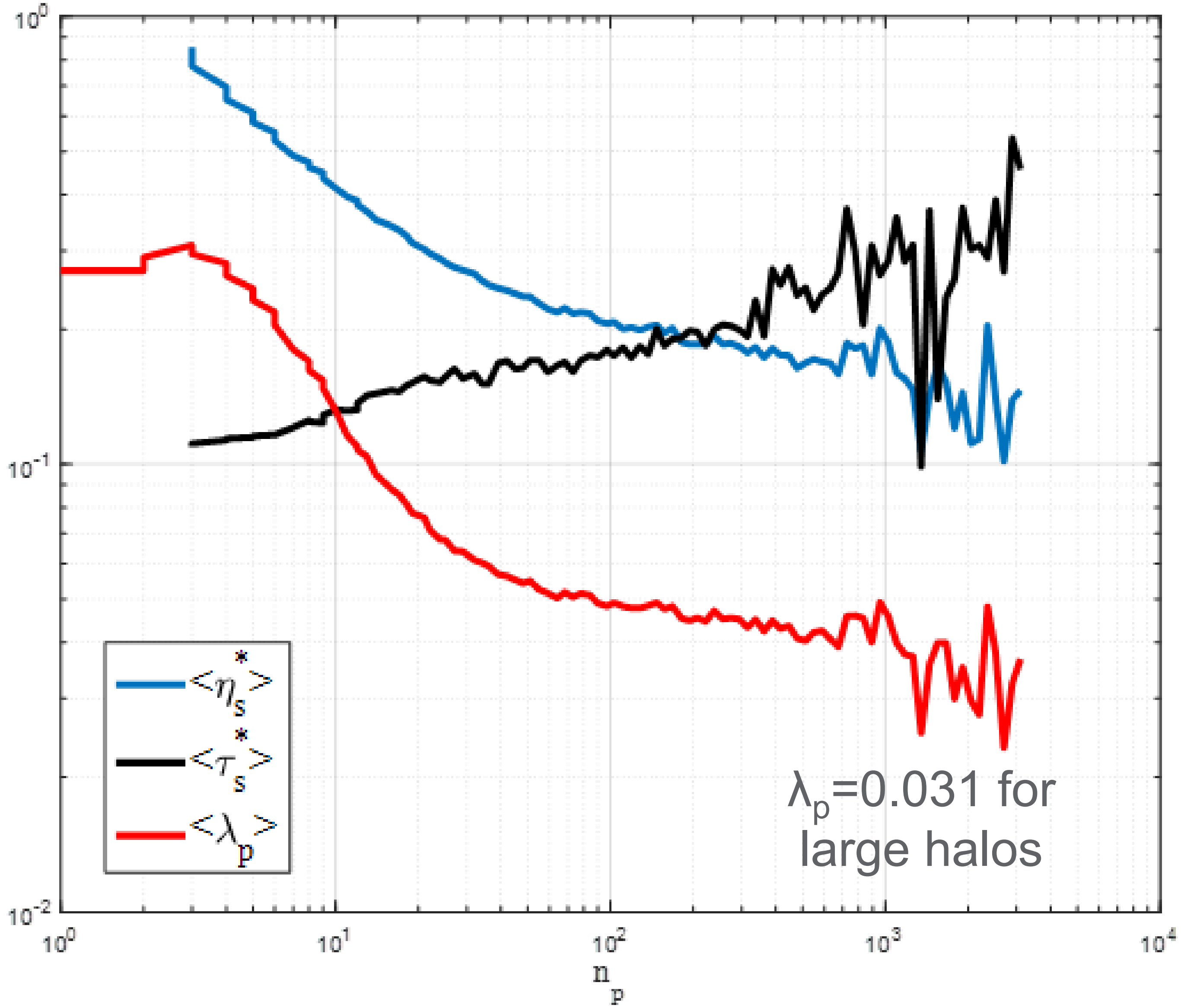
$$\lambda_p = \frac{\sqrt{2}}{2} \frac{\left(m_1 m_2\right)^{3/2}}{\left(m_1 + m_2\right)^3} = \frac{\sqrt{2}}{16} \approx 0.0884$$



The variation of momentum with halo size from a N-body simulation.

The variation of momentum in halos





λ_p=0.031 for large halos

Relevant parameters for halo energy and momentum

Table 2. Relevant parameters for halo energy, momentum and spin from theory and simulations

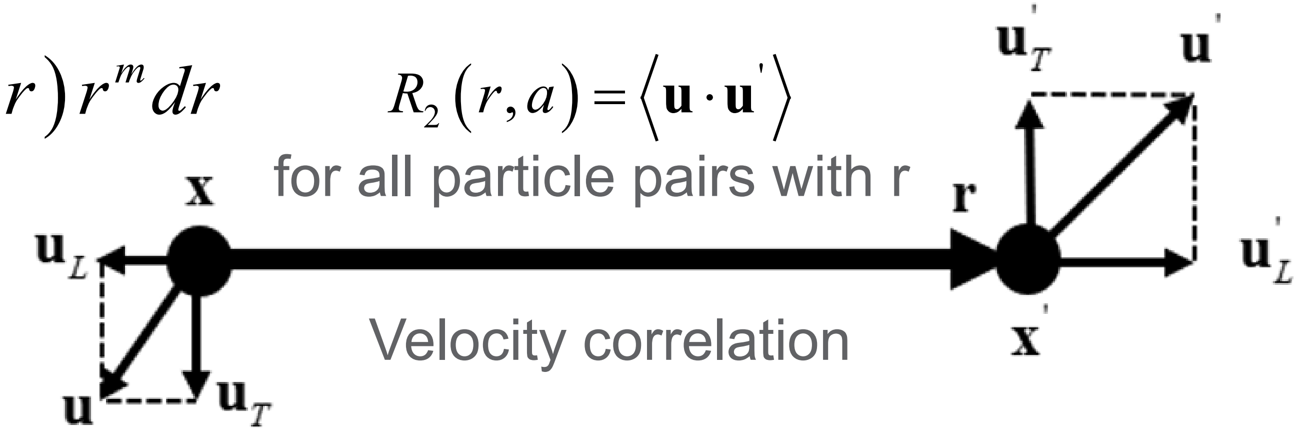
Type of halos	γ_Φ	γ_g	γ_v	Δ_c	α_s^* Eq.(54)	β_s^* Eq.(54)
Two-body halos (theory)	1/4	1/2	1.0	$18\pi^2$	1/24	$\sqrt{3}/(3\pi)$
Large halos (NFW)	0.936 Eq.(49)	0.567 Eq.(48)	1.3	$18\pi^2$	0.230	0.095
Large halos (isothermal)	1 Eq.(45)	$\sqrt{3}/3$ Eq.(43)	1.5	$18\pi^2$	$\sqrt{3}/6$ Eq.(54)	$\sqrt{2/3}/(3\pi)$ Eq.(54)

	n_s^* Eq.(68)	z_s^* Eq.(68)	η_s^* Eq.(63)	τ_s^* Eq.(63)	$f_H(m_h)$ Eq.(63)	$f_G(m_h)$ Eq.(63)	λ_p Eq.(70)
Two-body halos (theory)	-1.0	-1.5	$\sqrt{3}$	$\sqrt{3}/(3\pi)$	3π	1	$\sqrt{2}/16$
Large halos (NFW)	-1.3	-0.81	0.151	0.103	1.59	1.08	0.031
Large halos (isothermal)	-1.5	-0.5	$\sqrt{2/3}/(3\pi)$	$\sqrt{2/3}/(3\pi)$	1	1	$1/(18\pi)$

Integral constants I_m and physical meaning of I_2

The virial quantity (radial momentum) and angular momentum are intimately related to integral constants for dynamics of dark matter flow. Starting from the velocity correlation function R_2, defining

$$I_m = \int \left\langle \mathbf{u}\cdot\mathbf{u}' \right\rangle r^{m-2} d\mathbf{r}^3 = \int R_2(r)\, r^{m-2} d\mathbf{r}^3 = \int_0^\infty 4\pi R_2(r)\, r^m dr$$



Energy spectrum is Fourier transform of R_2:

$$E_u(k) = \frac{1}{\pi}\int_0^\infty R_2(r)\, kr \sin(kr)\, dr$$

Integral constant I_m is the derivative of spectrum at long wave-length limit (large scale):

$$I_m = 4\pi^2 \frac{(-1)^{1+m/2}}{m} \left.\frac{\partial^m E_u}{\partial k^m}\right|_{k=0} \propto a \quad \text{with} \quad E_u(k\to 0) \propto a$$

I_2 is related to the linear momentum. This leads to a k^4 velocity spectrum on large scale.

Assume linear momentum vanishes: $\int_V \mathbf{u}\, d\mathbf{x}^3 = 0$

$$I_2 = \int \left\langle \mathbf{u}\cdot\mathbf{u}' \right\rangle d\mathbf{r}^3 = \lim_{V\to\infty} \frac{1}{V}\int_V \int_V \left\langle \mathbf{u}\cdot\mathbf{u}' \right\rangle d\mathbf{x}^3 d\mathbf{x}'^3 = \lim_{V\to\infty} V \left\langle \left(\frac{1}{V}\int_V \mathbf{u}\, d\mathbf{x}^3\right)^2 \right\rangle = 0 \Rightarrow \left.\frac{\partial^2 E_u}{\partial k^2}\right|_{k=0} = 0$$

Physical meaning of integral constants I_4

Defined in comoving coordinates:

$$G = \frac{1}{V}\int_V \mathbf{x}\cdot\mathbf{u}dx^3$$ Virial quantity

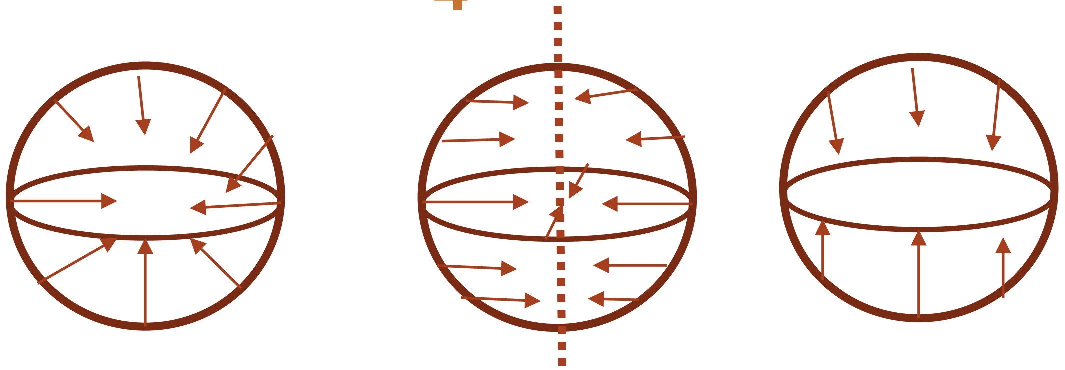
$$\mathbf{H} = \frac{1}{V}\int_V \mathbf{x}\times\mathbf{u}dx^3$$ Angular momentum

$$\mathbf{M} = \frac{1}{V}\int_V \mathbf{x}\otimes\mathbf{u}dx^3$$ Momentum tensor

$$\mathbf{I} = \frac{1}{V}\int_V \mathbf{x}\otimes\mathbf{x}dx^3$$ Inertial tensor

$$T = \mathbf{M}:\mathbf{M}$$ Contraction of momentum tensor

$\bar{u}_{k,k}$ is mean divergence



$$\mathbf{M} = \begin{bmatrix} G/3 & 0 & 0 \\ 0 & G/3 & 0 \\ 0 & 0 & G/3 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} G/2 & 0 & 0 \\ 0 & G/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} G & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\alpha_T=3$ $\qquad$ $\alpha_T=2$ $\qquad$ $\alpha_T=1$

$$I_4 = -2\lim_{V\to\infty}\left(\langle T\rangle V\right) = \lim_{V\to\infty}\left[-\left|\mathbf{H}\right|^2 + \left(\mathbf{M}:\mathbf{I}\right)\bar{u}_{k,k}\right]V$$

$\alpha_T=3$ if structure collapsing into a point,
$\alpha_T=2$ if collapsing into a filament (N-body)
$\alpha_T=1$ if collapsing into a plane.

For incompressible flow with vanishing divergence ($u_{k,k}=0$), I_4 is related to the angular momentum of entire system

$$I_4 = -\lim_{V\to\infty}\left(\left\langle\left|\mathbf{H}\right|^2\right\rangle V\right)$$

For dark matter flow with vanishing **H** on large scale, Both **M** and **I** are diagonal. I_4 is related to the virial quantity (radial momentum) of entire system.

$$\alpha_T T = \alpha_T\left(\mathbf{M}:\mathbf{M}\right) \approx G^2 \gg \left|\mathbf{H}\right|^2$$

$$I_4 = -\frac{2}{\alpha_T}\lim_{V\to\infty}\left(\left\langle G^2\right\rangle V\right)$$

Evolution of momentum on large scale

Integral constant on large scale:

$$I_4 = -\frac{2}{\alpha_T}\lim_{V\to\infty}\left(\left\langle G^2\right\rangle V\right) = \lim_{V\to\infty}\left(\left\langle (\mathbf{M}:\mathbf{I})\bar{u}_{k,k}\right\rangle V\right)$$

$$\left\langle G^2\right\rangle = -\frac{\alpha_T}{2}\left\langle (\mathbf{M}:\mathbf{I})\bar{u}_{k,k}\right\rangle$$

$$\mathbf{I} = \frac{1}{V}\int_V x_i x_j d\mathbf{x}^3 = \frac{1}{3}\left(\frac{L}{2}\right)^2 \delta_{ij}$$

Inertial tensor on large scale

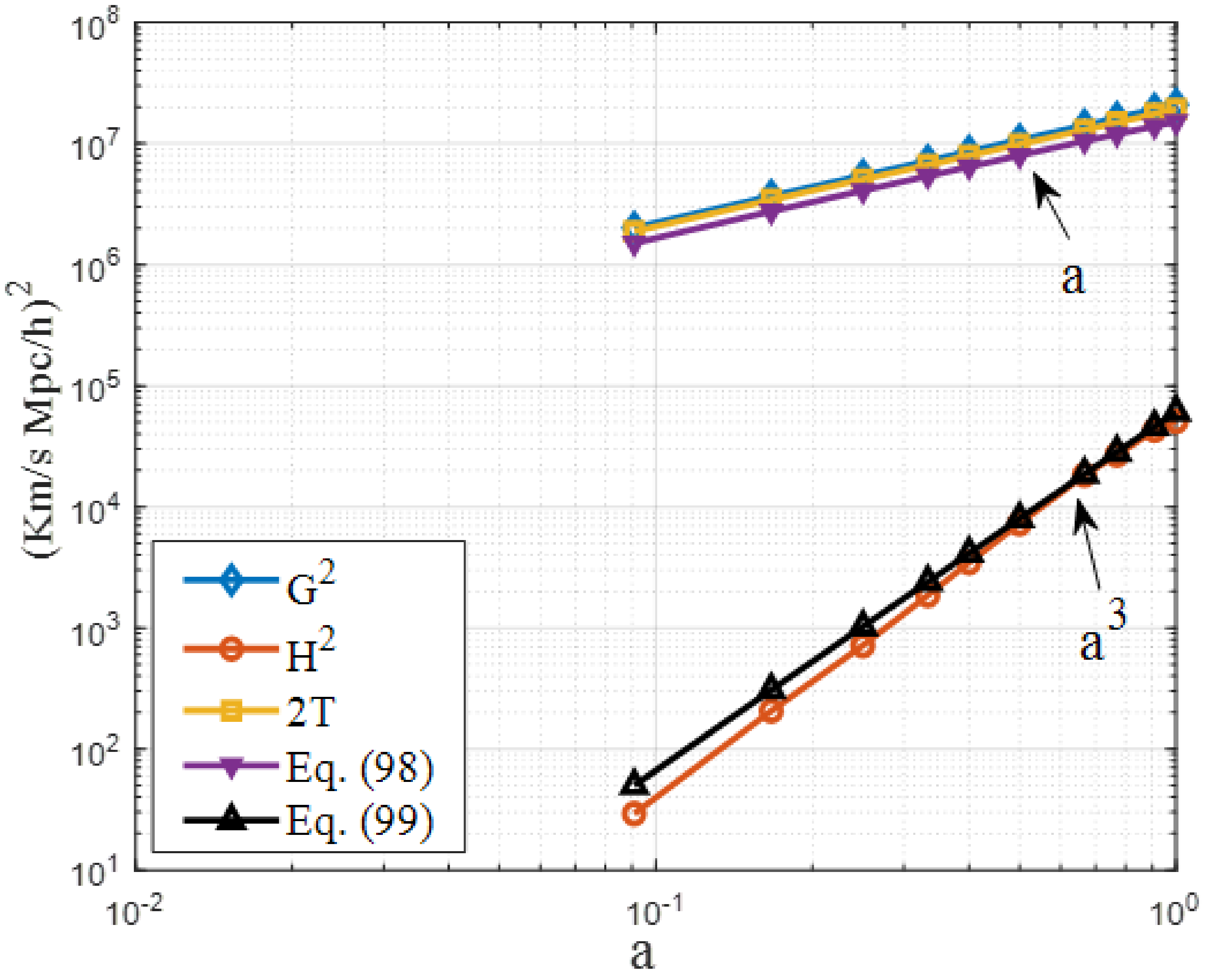
$$G = \frac{1}{V}\int_V x_j u_i \, d\mathbf{x}^3 \delta_{ij} = \mathbf{M}:\boldsymbol{\delta}$$

Virial quantity on large scale

$$G = \frac{\alpha_T}{24} L^2 \bar{u}_{k,k}$$

Virial quantity is related to divergence or density contrast

$$2T \approx \left\langle G^2\right\rangle = \frac{\alpha_T^2}{8} a_0 u^2 r_2^2 \propto a \quad \left\langle \left|\mathbf{H}\right|^2\right\rangle = 0.002 a_0 u^2 r_2^2 a^2 \propto a^3$$



Time variation of momentum with scale factor *a* from N-body simulation

Momentum and integration constants on halo scale

On small (halo) scale, velocity field is of constant divergence and matter density is non-uniform.

$$I_4 = -2\lim_{V\to\infty}\left(\langle T\rangle V\right) = \lim_{V\to\infty}\left[-|\mathbf{H}|^2 + (\mathbf{M}:\mathbf{I})\overline{u}_{k,k}\right]V$$

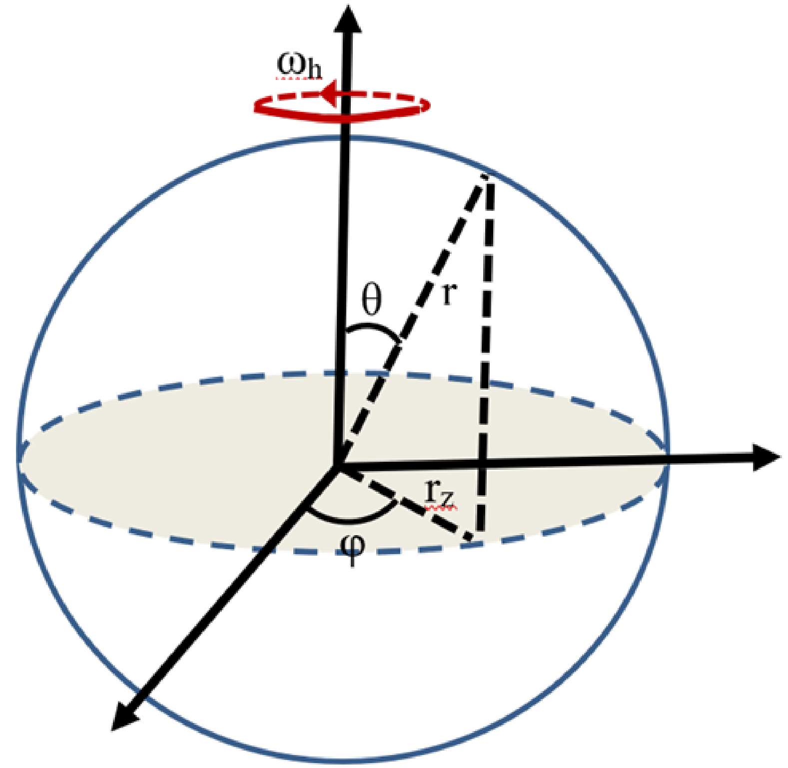
Momentum tensor:

$$\mathbf{M} = \frac{1}{m_h}\int_V \mathbf{x}\otimes\mathbf{u}\rho_h dx^3 = \begin{bmatrix} G/3 & -|\mathbf{H}|/2 & 0 \\ |\mathbf{H}|/2 & G/3 & 0 \\ 0 & 0 & G/3 \end{bmatrix}$$

Inertial tensor:

$$\mathbf{I} = \frac{1}{m_h}\int_V x_i x_j \rho_h d\mathbf{x}^3 = \frac{1}{3}r_g^2\delta_{ij}$$

$$T = \mathbf{M}:\mathbf{M} = \left(\frac{1}{3}G^2 + \frac{1}{2}|\mathbf{H}|^2\right)$$



Halo radial and angular momentum are equal

$$G = -|\mathbf{H}| = -Hr_g^2$$

$$\alpha_T = \frac{G^2}{T} = \frac{6}{2+3|\mathbf{H}|^2/G^2} = \frac{6}{5}$$

Summary and keywords

Large scale dynamics	Comoving/transformed system	Rate of energy cascade
Integration constants	Radial/angular momentum	Spin parameter
Velocity correlation function	Velocity spectrum function	Effective potential exponent

- The energy and momentum evolution of N-body system is analytically derived. This is made possible by introducing a new time scale *s*.
- The kinetic and potential energy of N-body system increase linearly with time with a constant rate of energy production ε_u.
- For entire N-body system, the radial momentum scales as $G_{py}\sim a^{3/2}$, while angular momentum $H_{py}\sim a^{5/2}$.
- The specific momentum (radial and angular) in halos scale as $\sim a^{3/2}$
- At same redshift, the analytically derived halo spin parameter decreases with halo mass, i.e. $\lambda_p=0.09$ for typical two-particle halos and $\lambda_p=0.031$ for large halos.
- The spin parameter of a given halo is a constant of time for early-stage halos with faster mass accretion and increases with time for late-stage halos with slower mass accretion.
- The radial/angular momentum are closely related to integral "constants" I_m that is defined as integral of velocity correlation or the *m*th derivative of energy spectrum at small *k*.