A comparative study of dark matter flow & hydrodynamic turbulence and its applications

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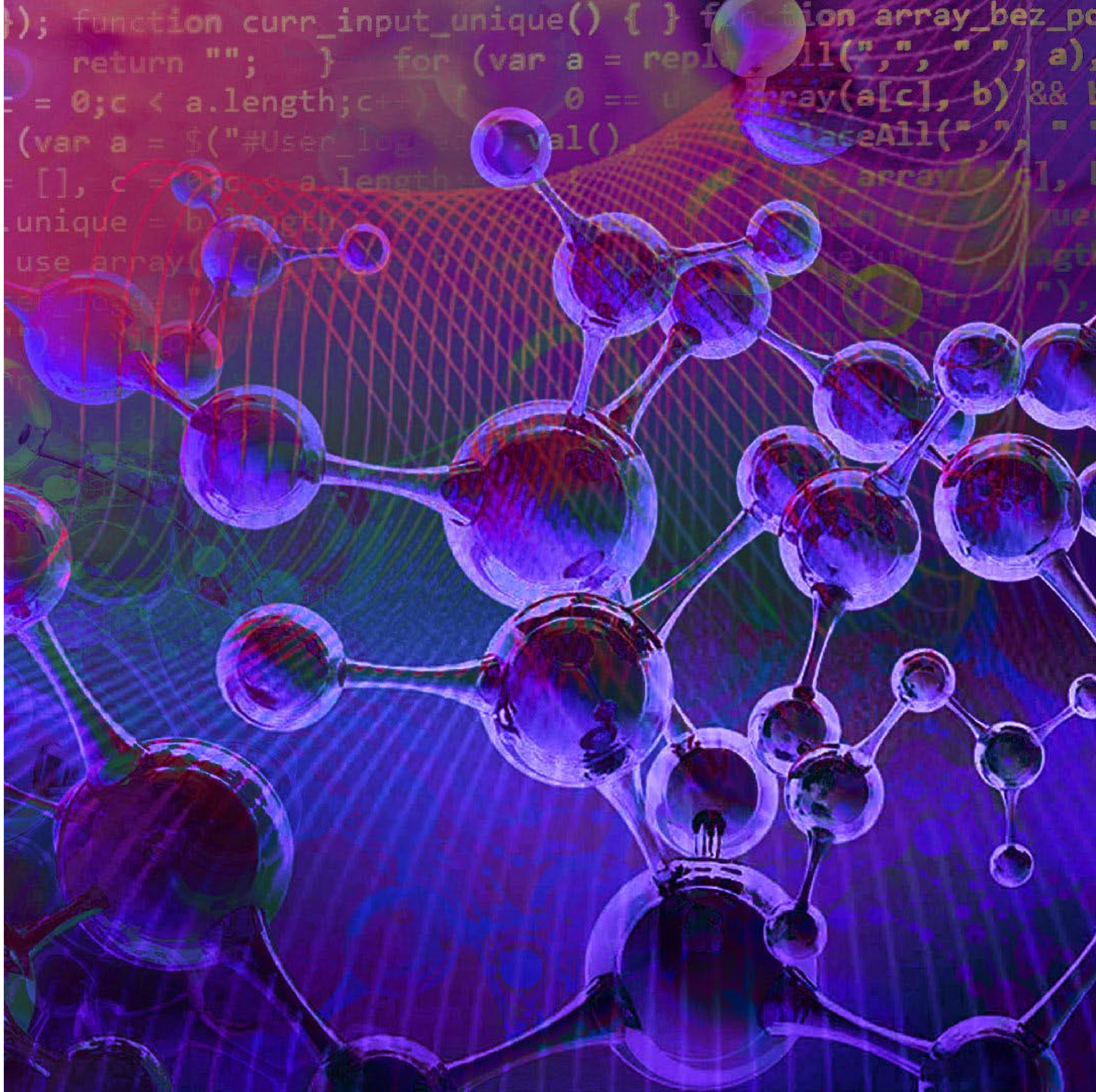

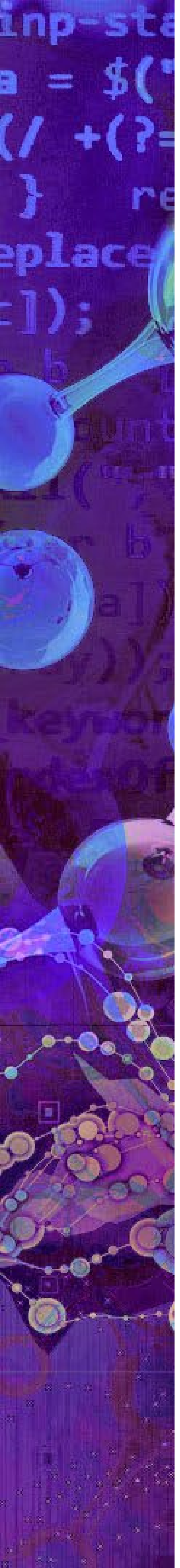

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Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!

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Data repository and relevant publications

Structural (halo-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6541230
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6569898
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

Structural (halo-based) approach for dark matter flow

Maximum entropy distributions in dark matter flow

Xu Z., 2021, arXiv:2110.03126v1 [astro-ph.CO]

https://doi.org/10.48550/arXiv.2110.03126

Maximum entropy distributions in kinetic theory of gases

Review on how to derive maximum entropy distributions (Boltzmann distribution)

Assume the distribution of one-dimensional gas molecule velocity is some unknown function X(v)

Two constraints on X(v), normalization and fixed mean kinetic energy:

$$\int_{-\infty}^{\infty} X(v)\,dv = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} X(v)\,\varepsilon(v)\,dv = \frac{3}{2}k_B T = \frac{3}{2}\sigma_0^2$$

Particle energy:

$$\varepsilon(v) = 3v^2/2$$

Write down the entropy functional with Lagrangian multiplier:

$$S[X(v)] = -\int_{-\infty}^{\infty} X(v)\ln X(v)\,dv + \lambda_1\left(\int_{-\infty}^{\infty} X(v)\,dv - 1\right) + \lambda_2\left(\int_{-\infty}^{\infty} X(v)\,\varepsilon(v)\,dv - \frac{3}{2}\sigma_0^2\right)$$

This is the key to be identified for dark matter flow

Taking the variation of the entropy functional with respect to distribution X:

$$\frac{\delta S(X(v))}{\delta X} = -\ln X(v) - 1 + \lambda_1 + \lambda_2\varepsilon(v) = 0 \;\Rightarrow\; X(v) \propto \exp\left(\lambda_2 v^2\right)$$

Boltzmann distribution

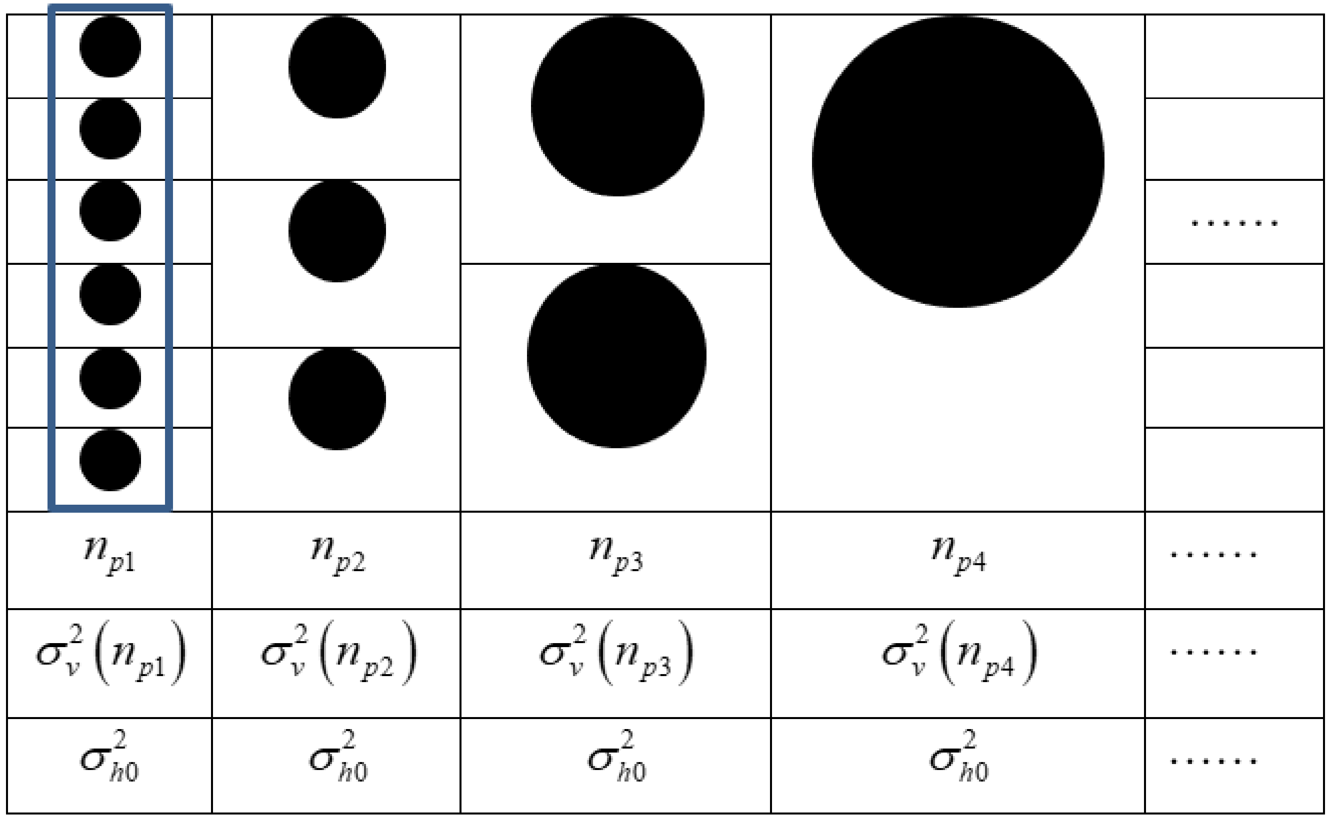
Distribution for particle energy:

$$E(\varepsilon) = 2\sqrt{\frac{\varepsilon}{\pi\sigma_0^2}}\frac{1}{\sigma_0^2}e^{-\varepsilon/\sigma_0^2}$$

Maxwell-Boltzmann distribution for speed:

$$Z(v) = \sqrt{\frac{2}{\pi}}\frac{v^2}{\sigma_0^3}e^{-v^2/2\sigma_0^2}$$

Maximum entropy distributions in dark matter flow



- Long-range and collisionless nature
- Identify all halos of different sizes at given *z*
- Group halos according to halo size n_p

Goal: maximum entropy distributions in DMF

Symbol	Physical meaning
X(v)	Distribution of one-dimensional particle velocity *v*
Z(v)	Distribution of particle speed
E(ε)	Distribution of particle energy ε
$H(\sigma_v^2)$	Distribution of particle virial dispersion σ_v^2 (halo mass function)

A general power-law for two-body potential:

$$V(r) \propto r^n$$

n=-1 for standard gravity

Velocity and dispersion decomposition

Decompose particle velocity into halo velocity and velocity fluctuation ("Reynolds decomposition")

$$\mathbf{v}_p = \mathbf{v}_h + \mathbf{v}'_p$$

Similarly, decompose velocity dispersion into halo velocity dispersion and halo virial dispersion

$$\sigma^2 = \sigma_h^2 + \sigma_v^2$$

Halo group temperature

Halo temperature

$$\sigma_h^2 = \mathrm{var}\left(\mathbf{v_h}\right)$$

Halo group temperature is independent of halo size

$$\sigma_v^2 = \mathrm{var}\left(\mathbf{v}'_p\right) \propto m_h^{1+n/3}$$

Gaussian velocity distribution (Maxwell-Boltzmann statistics) is expected for all particles in the same halo group.

$$X(v) = \int_0^\infty \boxed{\frac{1}{\sqrt{2\pi\sigma}} e^{-v^2/2\sigma^2}} H\left(\sigma_v^2\right) d\sigma_v^2$$

weighted average

Boltzmann distribution

\# of particles in halo group

$$Z(v) = \int_0^\infty \boxed{\sqrt{\frac{2}{\pi}} \frac{v^2}{\sigma^3} e^{-v^2/2\sigma^2}} H\left(\sigma_v^2\right) d\sigma_v^2$$

weighted average

Maxwell-Boltzmann distribution

\# of particles in halo group

Relation between X and Z:

$$Z(v) = -2v \frac{\partial X}{\partial v}$$

Particle energy in dark matter flow

In a given halo group, from Virial Theorem:

$$2\langle KE\rangle_g - n\langle PE\rangle_g = 0$$

The specific kinetic energy of particle in that group:

$$\langle KE\rangle_g = (3/2)\sigma^2$$

The total specific energy of particle in group:

$$\varepsilon_h = \langle KE\rangle_g + \langle PE\rangle_g = \left(\frac{3}{2}+\frac{3}{n}\right)\sigma^2$$

Energy distribution with respect to particle speed v:

$$\langle \varepsilon_h\rangle = \int_0^\infty \varepsilon_v(v)\,dv$$

Mean particle energy for entire system

Energy distribution with respect to particle speed

$$N\varepsilon_v(v)\,dv = \left(\underbrace{\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{v^2}{\sigma^3}e^{-v^2/2\sigma^2}dv}_{2}\,\underbrace{N\varepsilon_h\left(\sigma_v^2\right)H\left(\sigma_v^2\right)d\sigma_v^2}_{1}\right)$$

For entire system, energy of all particles with a speed of v:

$$\varepsilon_v(v)\,dv = 2\left(\frac{3}{2}+\frac{3}{n}\right)v^2X(v)\,dv$$

Energy per particle with a speed of v:

$$\varepsilon(v) = \frac{\varepsilon_v(v)\,dv}{Z(v)\,dv} = \frac{X(v)v^2}{Z(v)}\left(3+\frac{6}{n}\right)$$

$$Z(v) = -2v\frac{\partial X}{\partial v}$$

# of particles with speed v

Energy per particle with a speed of v:

$$\varepsilon(v) = -\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

Maximum entropy distributions in dark matter flow

Deriving maximum entropy distributions in dark matter flow (X distribution)

Two constraints on X(v):

$$\int_{-\infty}^{\infty} X(v)\,dv = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} X(v)\varepsilon(v)\,dv = \frac{3}{2}k_B T = \frac{3}{2}\sigma_0^2$$

Particle energy:

$$\varepsilon(v) = -\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

This is the key

Write down the entropy functional with Lagrangian multiplier:

$$S[X(v)] = -\int_{-\infty}^{\infty} X(v)\ln X(v)\,dv + \lambda_1\left(\int_{-\infty}^{\infty} X(v)\,dv - 1\right) + \lambda_2\left(\int_{-\infty}^{\infty} X(v)\varepsilon(v)\,dv - \frac{3}{2}\sigma_0^2\right)$$

Taking the variation of the entropy functional with respect to X:

$$\frac{\delta S(X(v))}{\delta X} = -\ln X(v) - 1 + \lambda_1 + \lambda_2\varepsilon(v) = 0 \;\Rightarrow\; X(v) = \frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)}$$

The X distribution

E distribution for particle energy:

$$E(\varepsilon) = -\frac{2n}{3(n+2)}\frac{e^{-\gamma}\sqrt{\gamma^2-\alpha^2}}{\alpha K_1(\alpha)v_0^2}$$

Z distribution for speed:

$$Z(v) = \frac{1}{\alpha K_1(\alpha)}\cdot\frac{v^2}{v_0^3}\cdot\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{\sqrt{\alpha^2+(v/v_0)^2}}$$

Maximum entropy distributions in dark matter flow

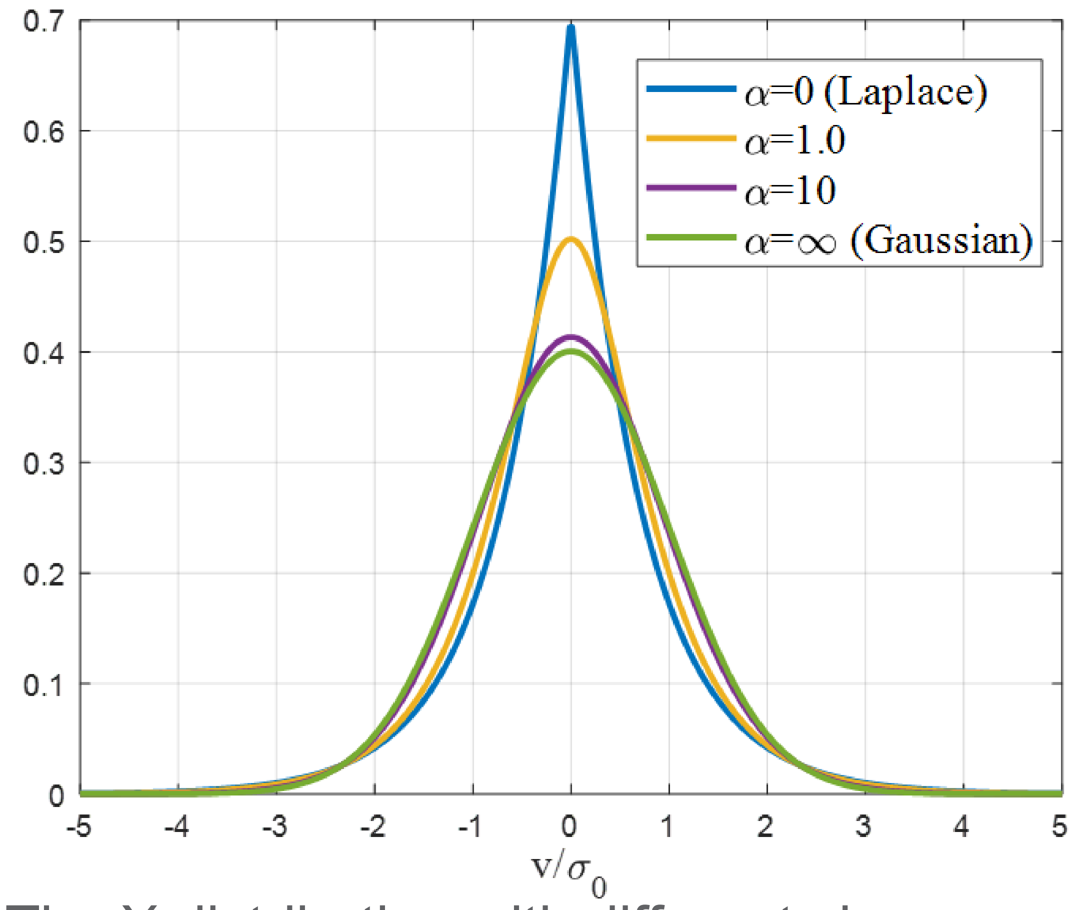
Gaussian core for $|v| \ll v_0$

$$X(v) = \frac{e^{-\alpha}}{2\alpha v_0 K_1(\alpha)} \exp\left(-\frac{v^2}{2\alpha v_0^2}\right)$$

Exponential wings for $|v| \gg v_0$

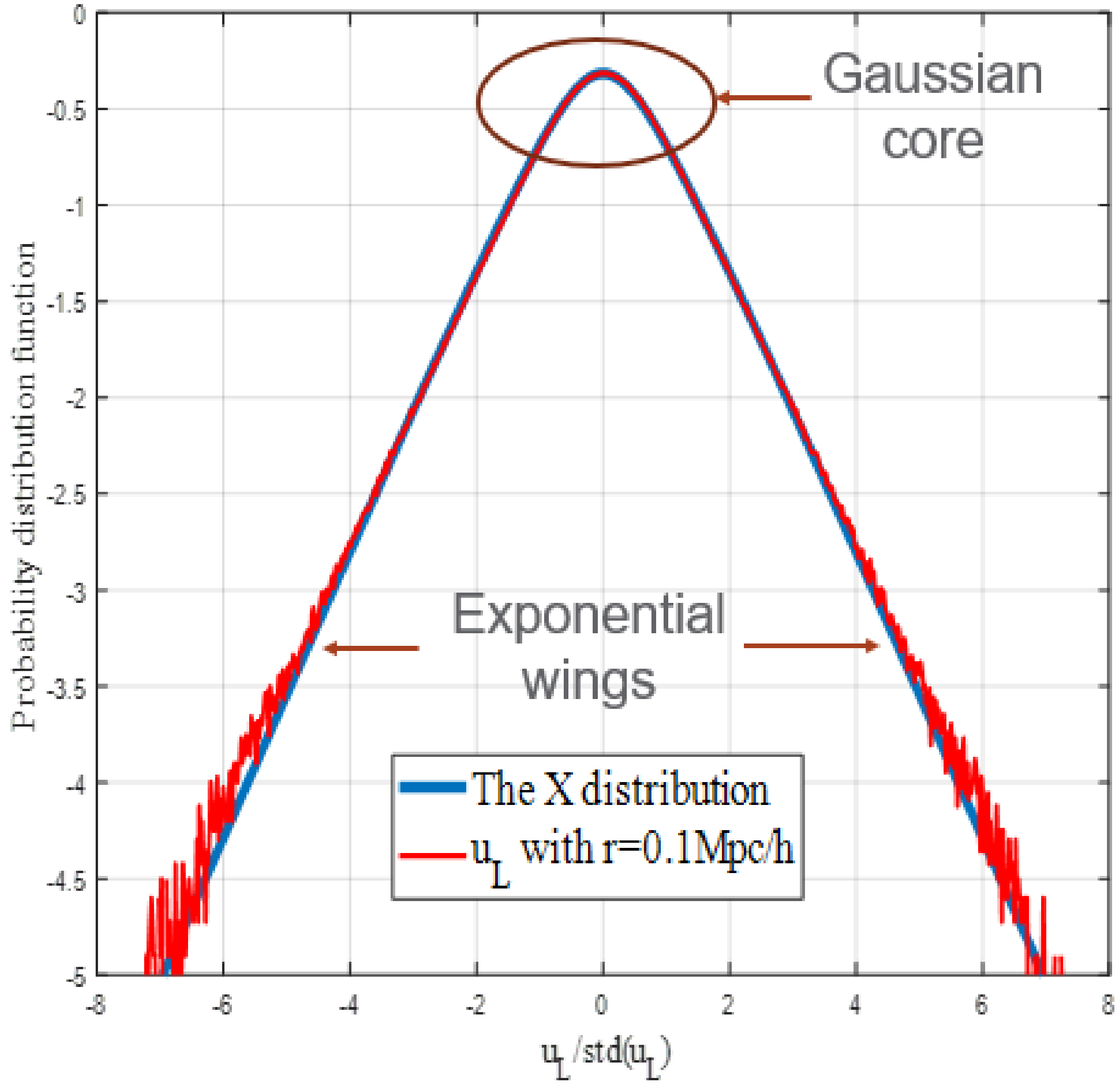
$$X(v) = \frac{1}{2\alpha v_0 K_1(\alpha)} \exp\left(-\frac{v}{v_0}\right)$$

Bessel function



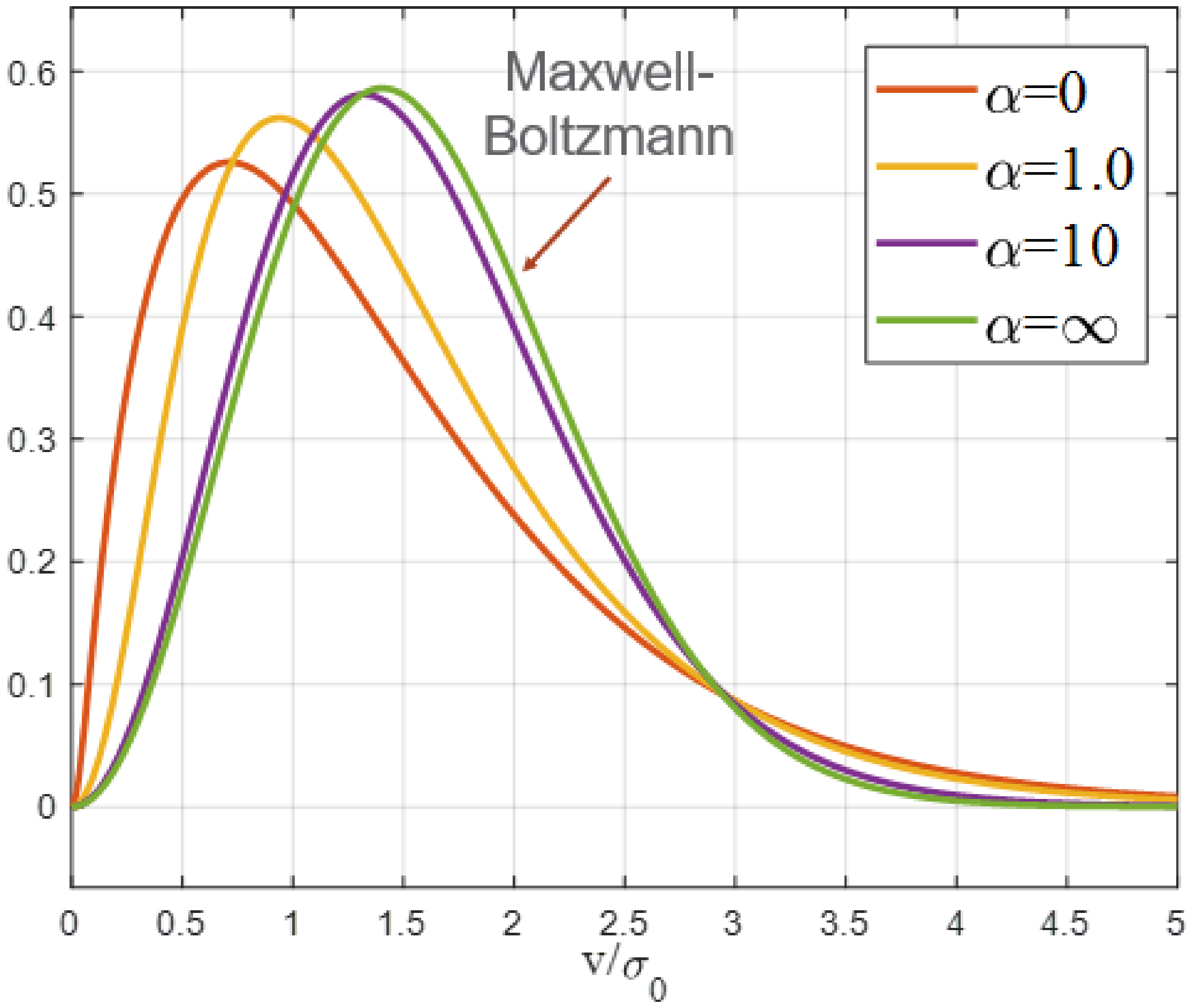
The X distribution with different shape parameter α

X is a two-parameter distribution with shape parameter α and velocity scale v_0

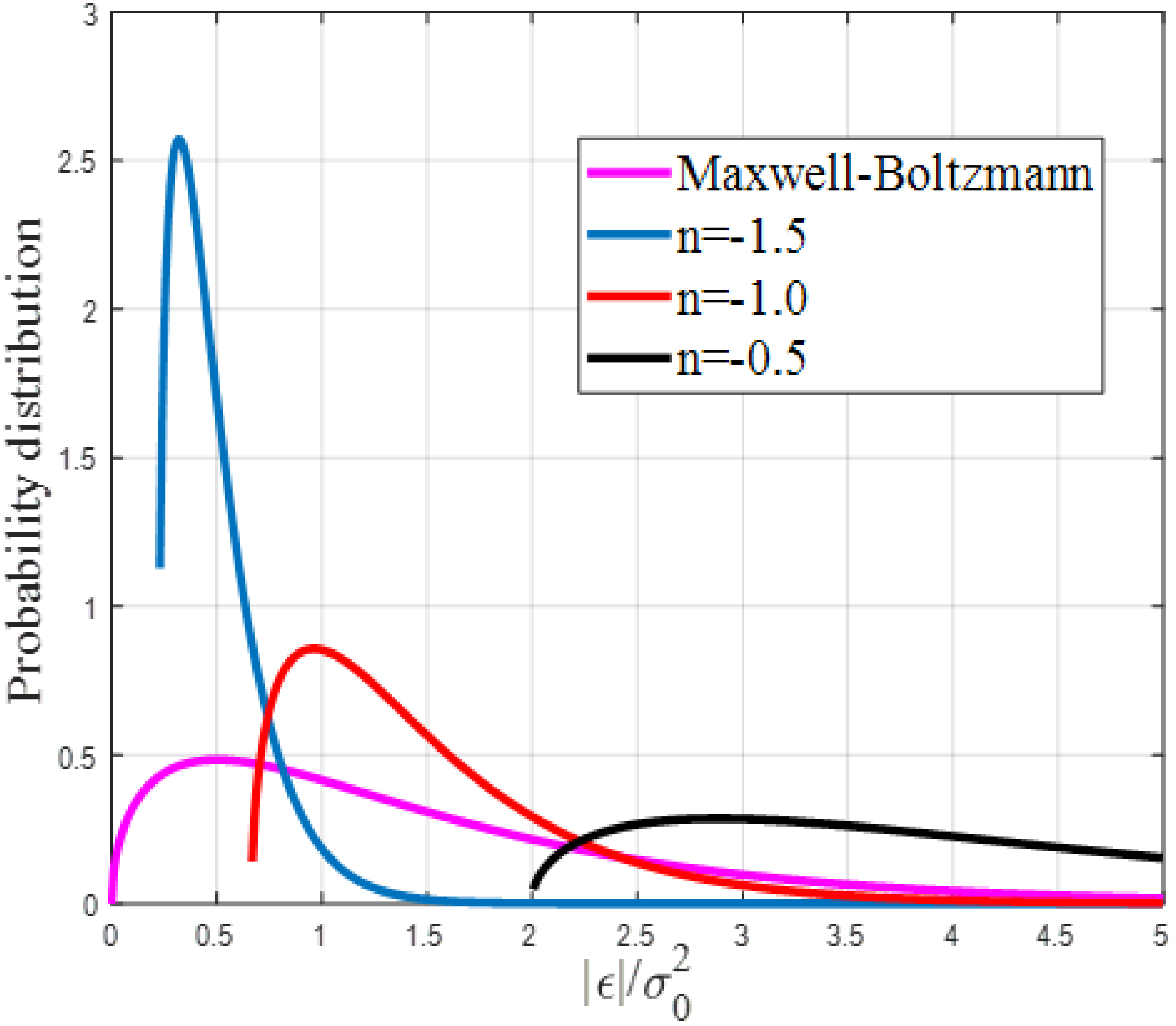


Comparison with N-body simulation

Maximum entropy distributions in dark matter flow



The Z distribution for particle speed with different shape parameter α



The E distribution for particle energy with different potential exponent n

Particle energy in dark matter flow

$$X(v)=\frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)} \qquad \varepsilon(v)=-\frac{X(v)v}{\partial X/\partial v}\left(\frac{3}{2}+\frac{3}{n}\right)$$

Particle energy:

$$\varepsilon(v)=\frac{3}{2}\left(1+\frac{2}{n}\right)v_0^2\sqrt{\alpha^2+\left(\frac{v}{v_0}\right)^2}$$

Gaussian core for $|v|\ll v_0$

$$\varepsilon(v)\approx\frac{3}{2}\left(1+\frac{2}{n}\right)\left(\alpha v_0^2+\frac{v^2}{2\alpha}\right)\propto v^2$$

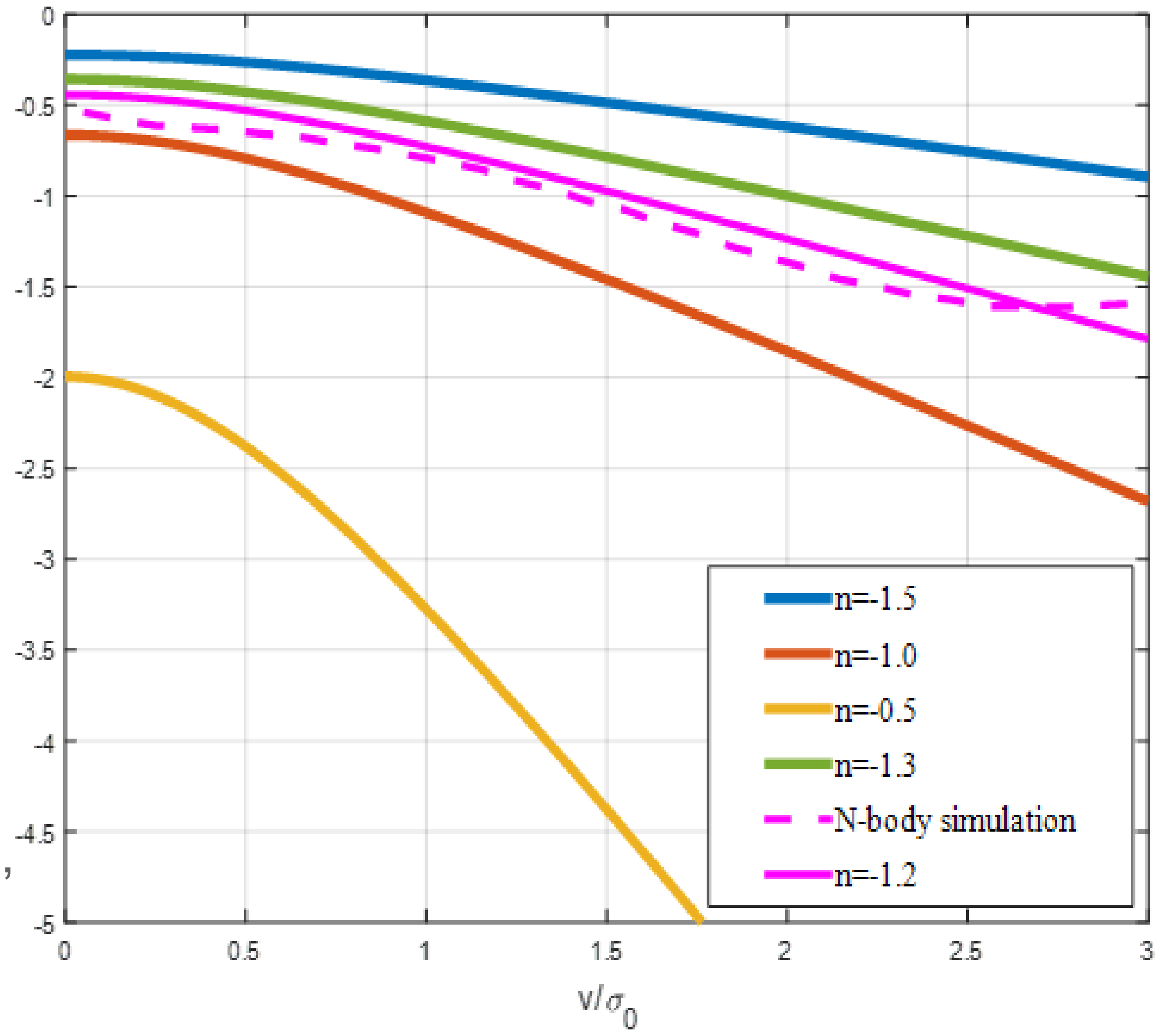
Inner halo, Newtonian behavior

Exponential wings for $|v|\gg v_0$

$$\varepsilon(v)\approx\frac{3}{2}\left(1+\frac{2}{n}\right)v_0 v\propto v$$

Outer region of halo, non-Newtonian behavior

External field effects and MOND??



Comparison with N-body simulation

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Summary and key words

Maximum entropy	Velocity distribution	Entropy functional
Speed distribution	Energy distribution	Particle energy
Gaussian core & exponential wings	Shape parameter	Velocity scale

- Statistical theory for maximum entropy distributions of velocity, speed, and energy in dark matter flow
- Halo mass function can be a direct result to maximizing system entropy
- Maximum entropy velocity distribution (X distribution) naturally exhibits a Gaussian core at small velocity and exponential wings at large velocity (as observed from N-body simulations)
- Kinetic energy of dark matter particles follows a parabolic scaling for small speed ($\varepsilon \sim v^2$, Newtonian) and linear scaling ($\varepsilon \sim v$, non-Newtonian) for large speed. This might be relevant for "deep-MOND" behavior.