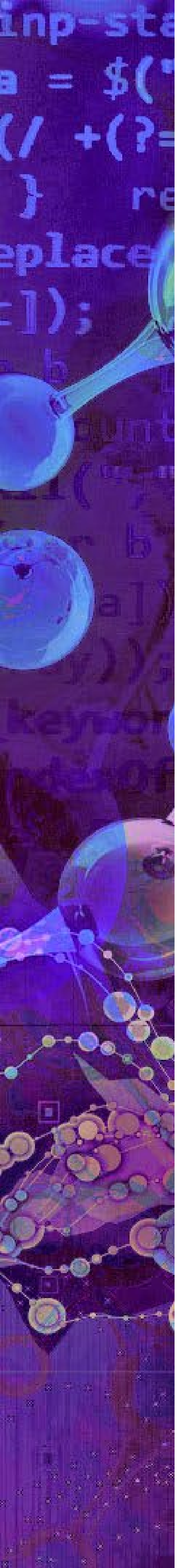

A comparative study of dark matter flow & hydrodynamic turbulence and its applications

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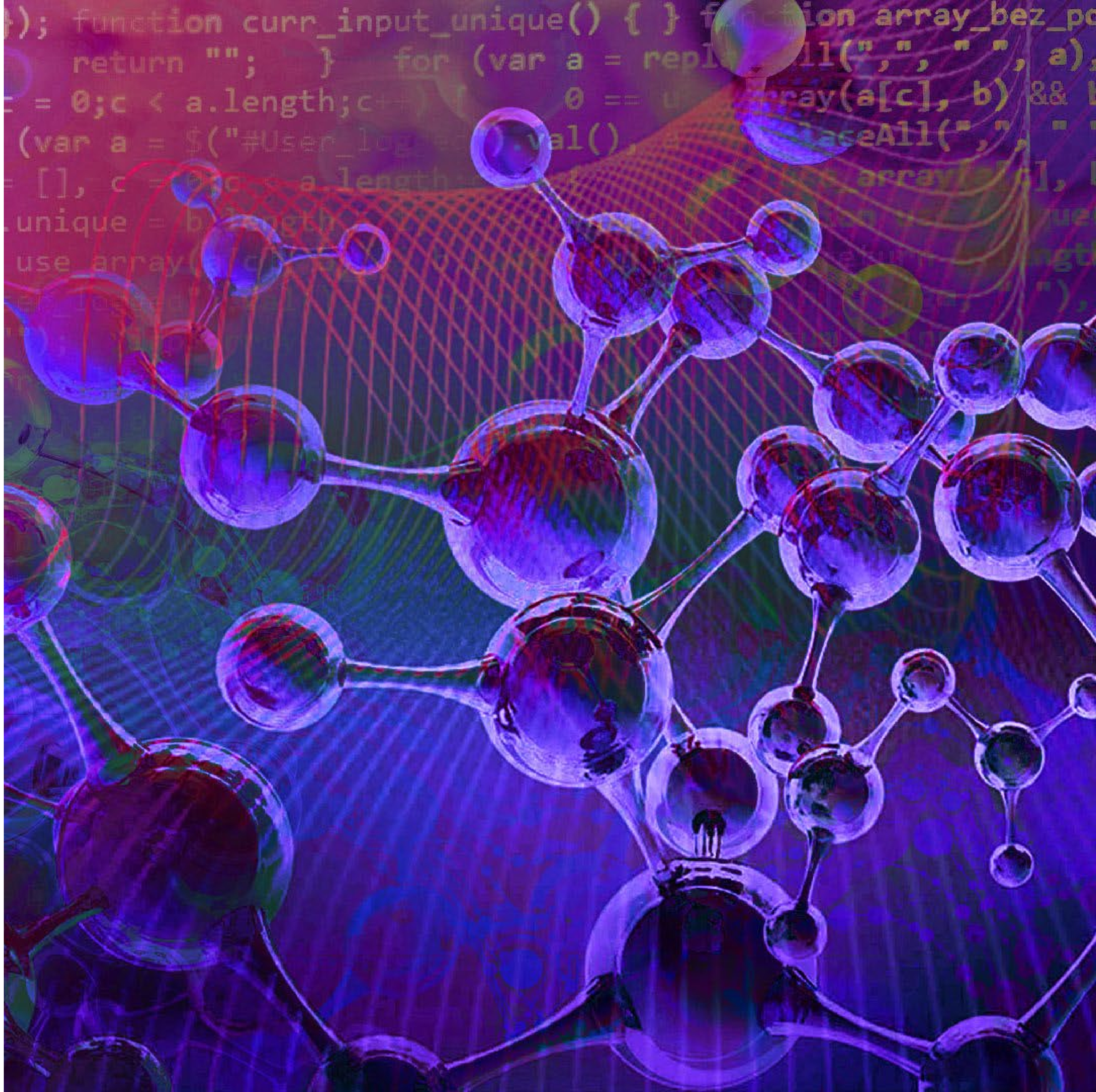
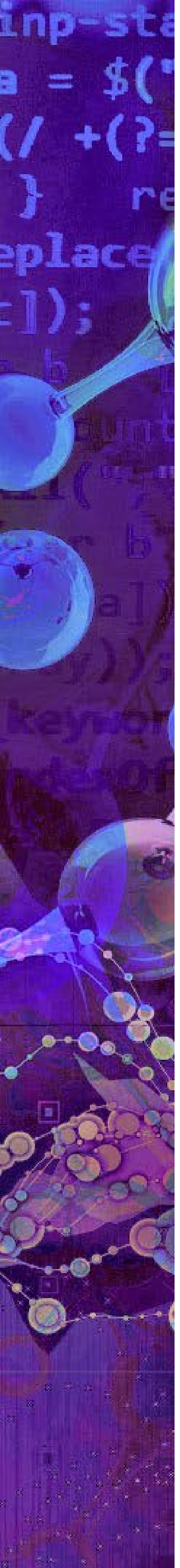


Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!



Data repository and relevant publications

Structural (halo-based) approach:

0.	**Data https://dx.doi.org/10.5281/zenodo.6541230**
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	**Data https://dx.doi.org/10.5281/zenodo.6569898**
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

Structural (halo-based) approach for dark matter flow

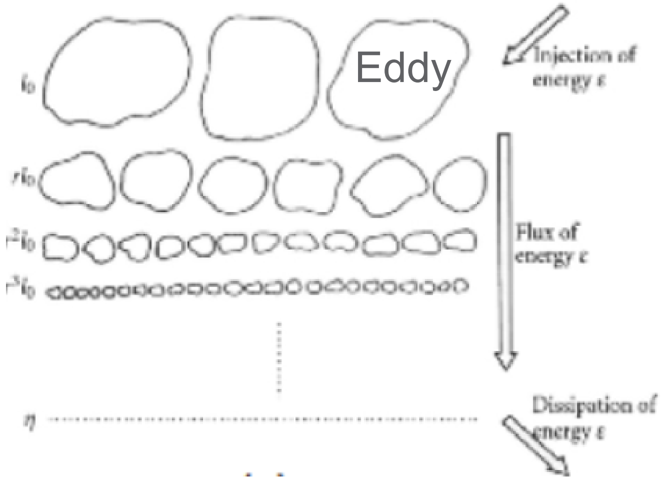
Energy cascade in dark matter flow

Xu Z., 2021, arXiv:2110.13885v1 [astro-ph.GA]
https://doi.org/10.48550/arXiv.2110.13885

Introduction

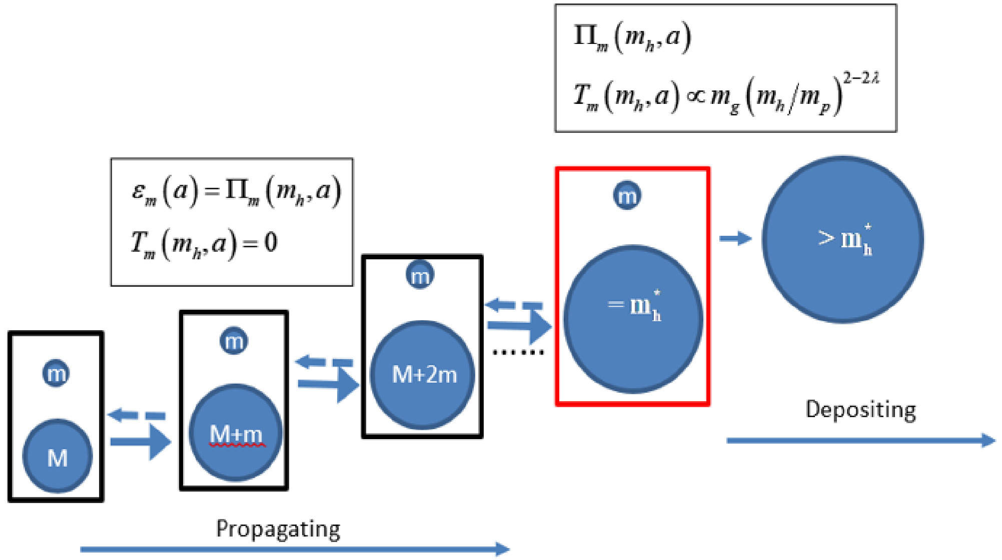
Review: In hydrodynamic turbulence, “Energy cascade” involves the energy transfer from large eddies to small eddies with a scale-independent rate of energy cascade (direct cascade). No mass cascade!
Vortex stretching is a major mechanism for energy cascade in turbulence.

“Big whorls have little whorls, That feed on their velocity;
And little whorls have lesser whorls, And so on to viscosity.”



“Eddy” is not a well-defined object in turbulence literature. However, “halo” are well-defined dynamically growing and rotating objects with nonuniform density, whose abundance and internal structure have been extensively studied over several decades.

“Little halos have big halos, That feed on their mass;
And big halos have greater halos, And so on to growth”



- Goal 1: Identify and formulate kinetic/potential energy cascade
- Goal 2: Identify a constant scale-independent rate of energy cascade
- Goal 3: Explore the effect of halo shape on energy cascade

Decomposition of kinetic energy

Decompose particle velocity into halo velocity and velocity fluctuation ("Reynolds decomposition")

$$\mathbf{v}_p = \mathbf{v}_h + \mathbf{v}_p^{'}$$

Similarly, decompose velocity dispersion into halo velocity dispersion and halo virial dispersion

$$\sigma^2 = \sigma_h^2 + \sigma_v^2$$

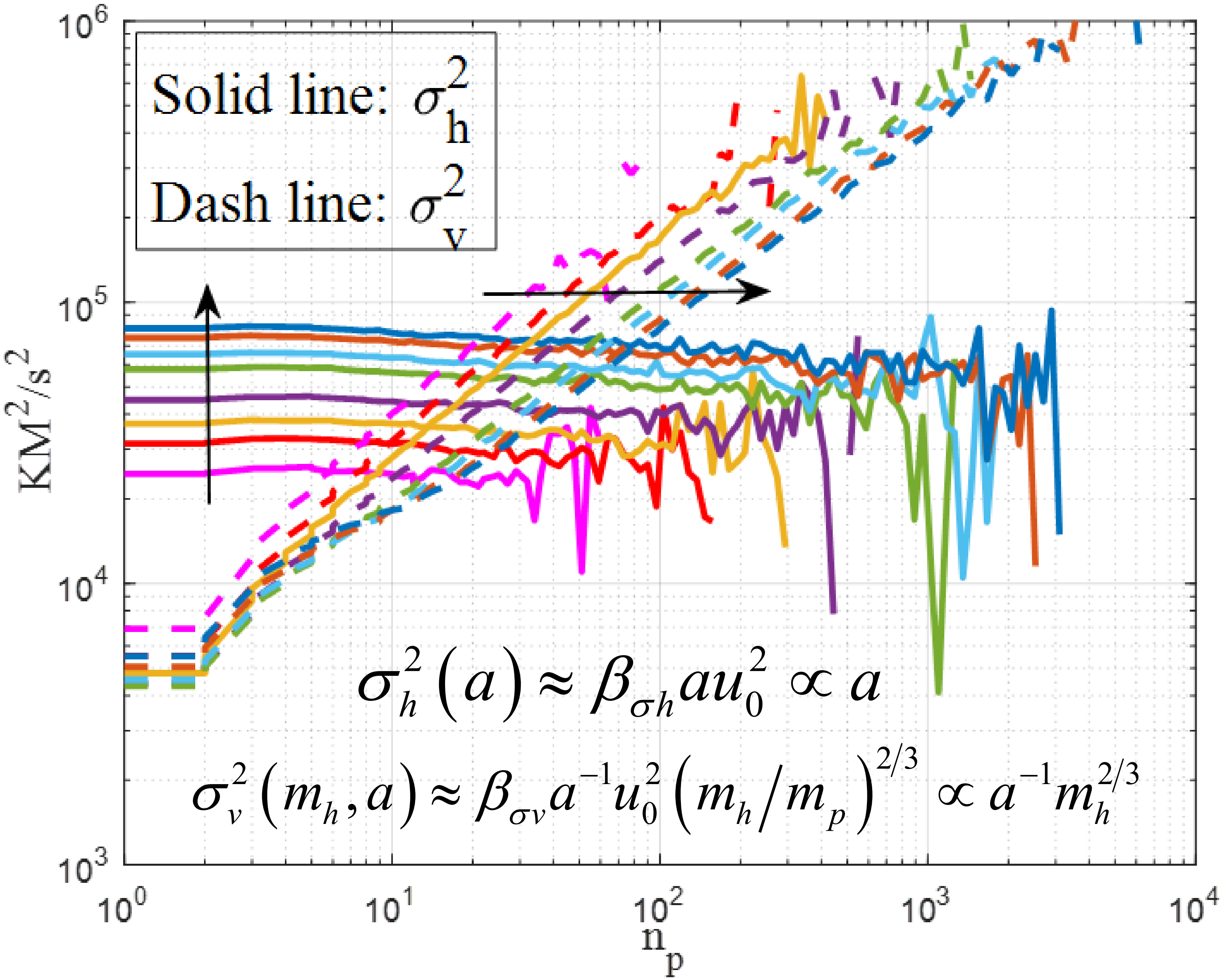
Halo group temperature

Halo temperature

$$\sigma_h^2 = \mathrm{var}\left(\mathbf{v_h}\right)$$

Halo group temperature is independent of halo size

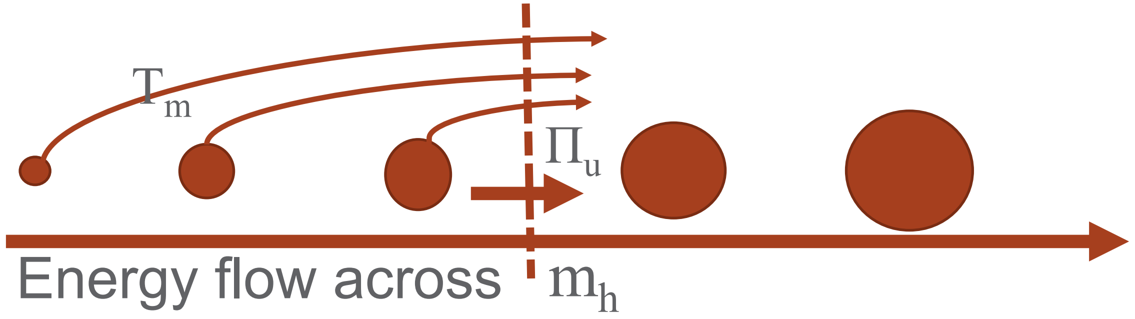
$$\sigma_v^2 = \mathrm{var}\left(\mathbf{v}_p^{'}\right) \propto m_h^{2/3}$$



Variation with halo size for redshifts z = 0, 0.1, 0.3, 0.5, 1.0, 1.5, 2.0, and 3.0

(Kinetic) energy flux functions

Mass flux function: total mass flux from all halos below m_h



Mass transfer function: rate of mass transfer for halos of mass m_h

$$\Pi_m(m_h,a) = -\frac{\partial}{\partial t}\left[M_h(a)\int_{m_h}^{\infty} f_M(m,a)\,dm\right] = \int_0^{m_h} T_m(m,a)\,dm$$

$$T_m(m_h,a) = \frac{\partial \Pi_m(m_h,a)}{\partial m_h} = \frac{\partial m_g(m_h,a)}{m_p \partial t}$$

Energy flux function for halo kinetic energy σ_h^2:

$$\Pi_{kh}(m_h,a) = -\int_{m_h}^{\infty} T_m(m,a)\sigma_h^2(m,a)\,dm \approx \Pi_m(m_h,a)\left\langle \sigma_h^2 \right\rangle$$

Energy flux function for virial kinetic energy σ_v^2:

$$\Pi_{kv}(m_h,a) = -\int_{m_h}^{\infty} T_m(m,a)\sigma_v^2(m,a)\,dm \neq \Pi_m(m_h,a)\left\langle \sigma_v^2 \right\rangle$$

Mean (specific) halo kinetic energy:

$$\left\langle \sigma_h^2 \right\rangle = \int_0^{\infty} f_M(m_h,m_h^*)\sigma_h^2(m_h,a)\,dm_h \propto a$$

Mean (specific) virial kinetic energy:

$$\left\langle \sigma_v^2 \right\rangle = \int_0^{\infty} f_M(m_h,m_h^*)\sigma_v^2(m_h,a)\,dm_h \propto a$$

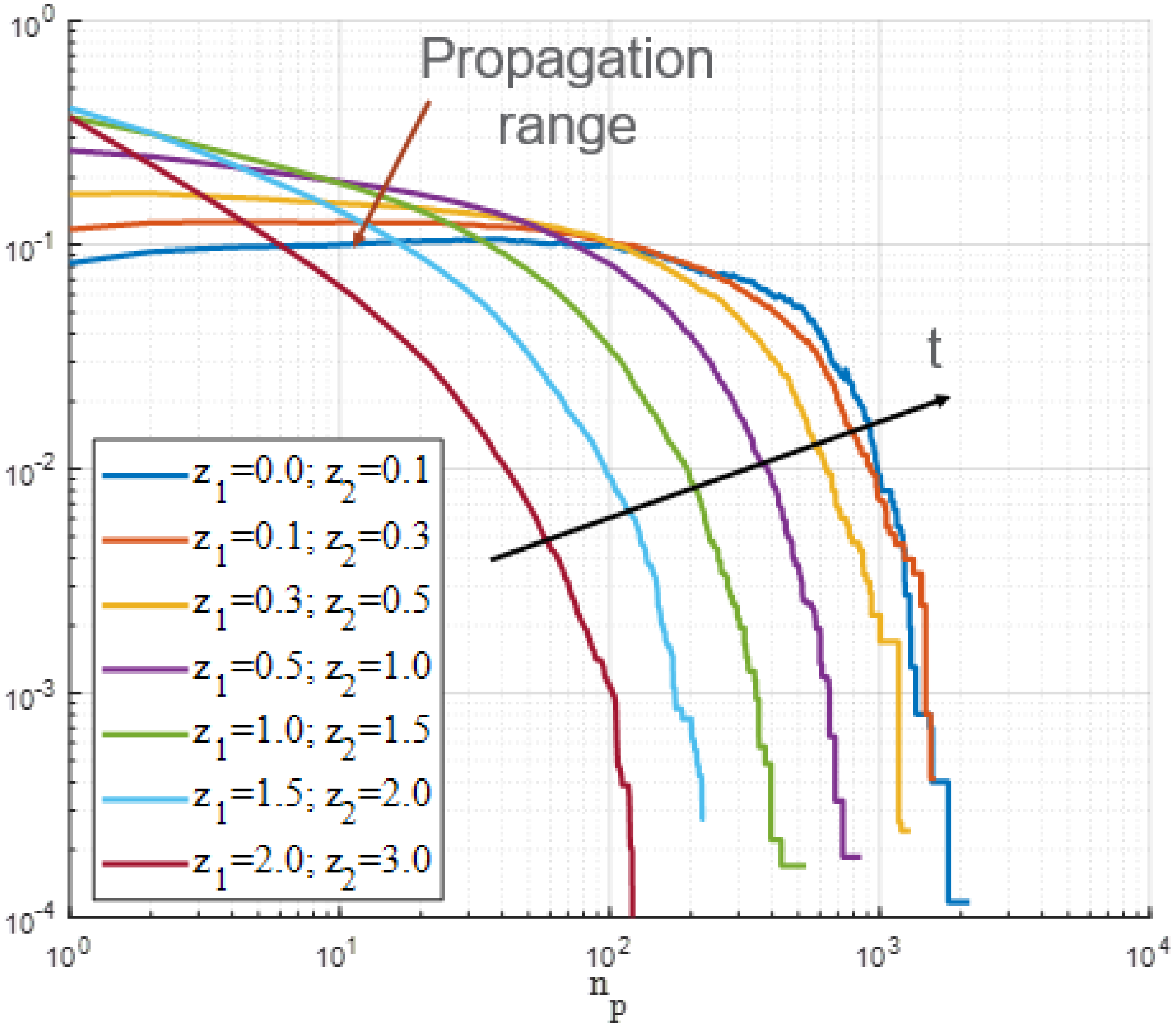
Equipartition requires: $\left\langle \sigma_h^2 \right\rangle \approx \left\langle \sigma_v^2 \right\rangle = \frac{1}{2}\sigma^2$

Total mass of all halos: $M_h(a)$
Halo mass: m_h
Halo mass function: $f_M(m,a)$
Dispersion of all particles: u^2

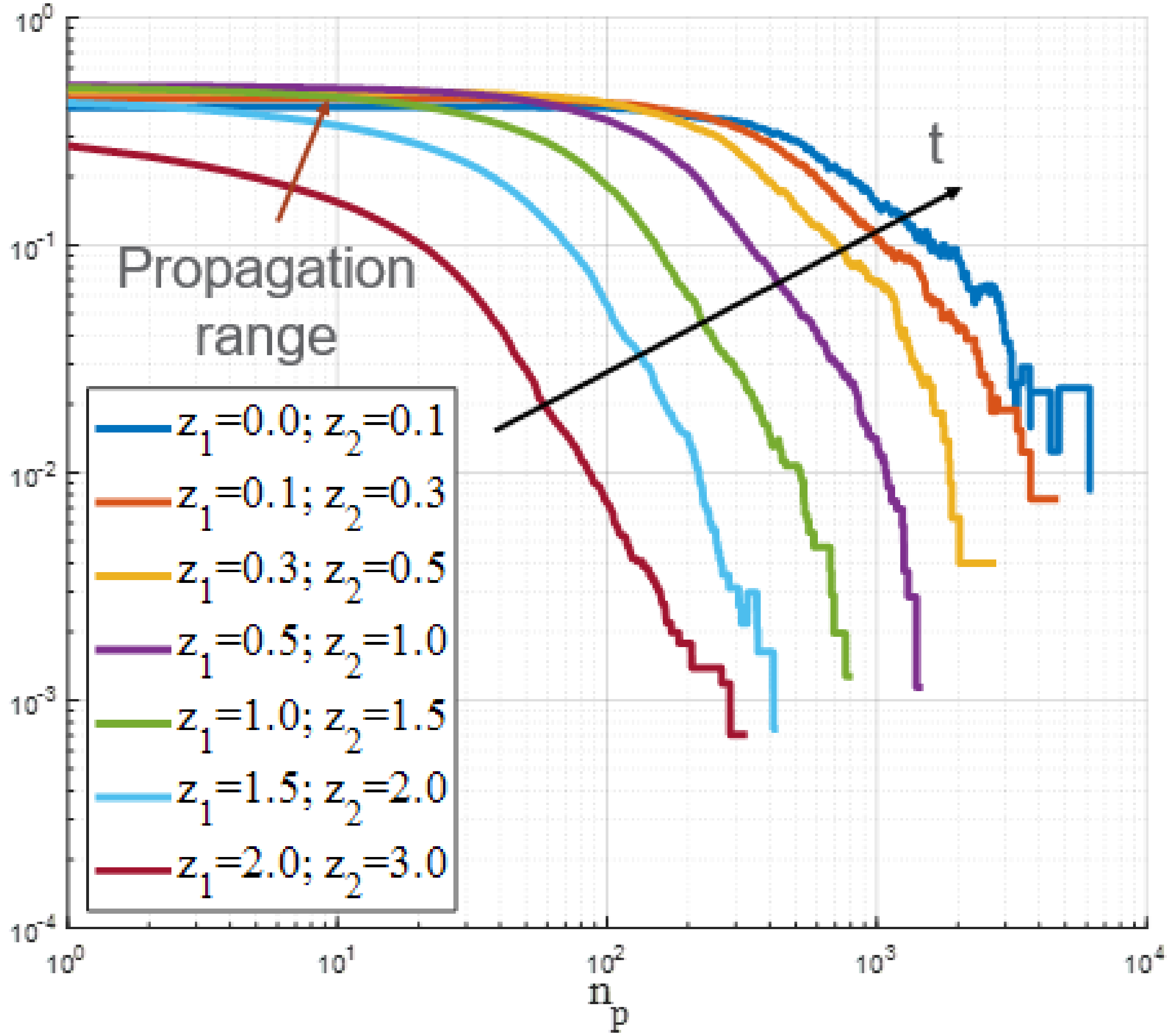
- **Direct** energy cascade in hydrodynamic turbulence through the change of vortex shape

- In dark matter flow, **inverse** energy cascade is facilitated by the inverse mass cascade through mass transfer function T_m

(Kinetic) energy flux functions π_{kh} and π_{kv}



The variation of energy flux function π_{kv} with the size of halo groups.



The variation of energy flux function π_{kh} with the size of halo groups.

(Potential) energy flux functions

Decompose particle potential into inter-halo potential (due to interaction with particles from other halos) and intra-halo potential (due to interaction with particles in the same halo):

$$\phi = \phi_h + \phi_v$$

Inter-halo potential

Intra-halo potential

Inter-halo potential is relatively independent of halo size

The virial ratios:

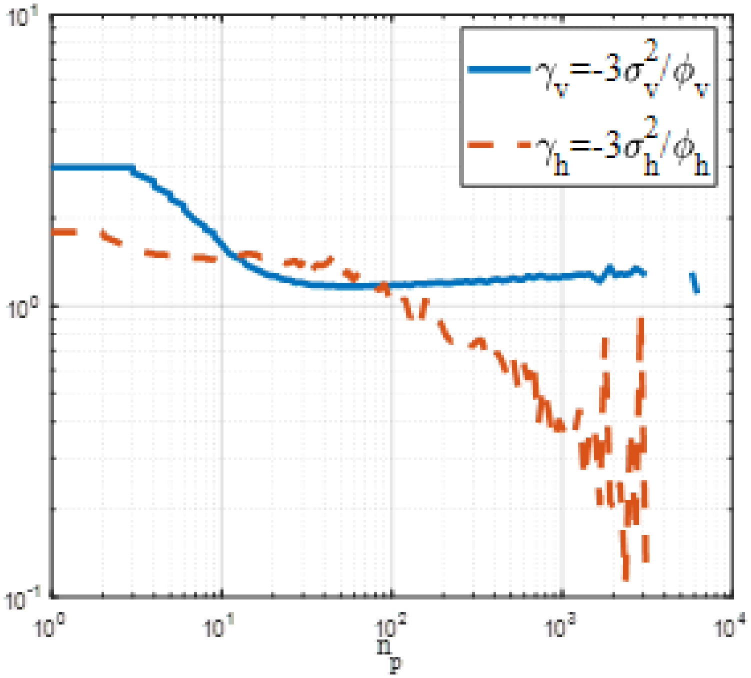
Intra-halo: $\gamma_v = -3\sigma_v^2 / \phi_v$

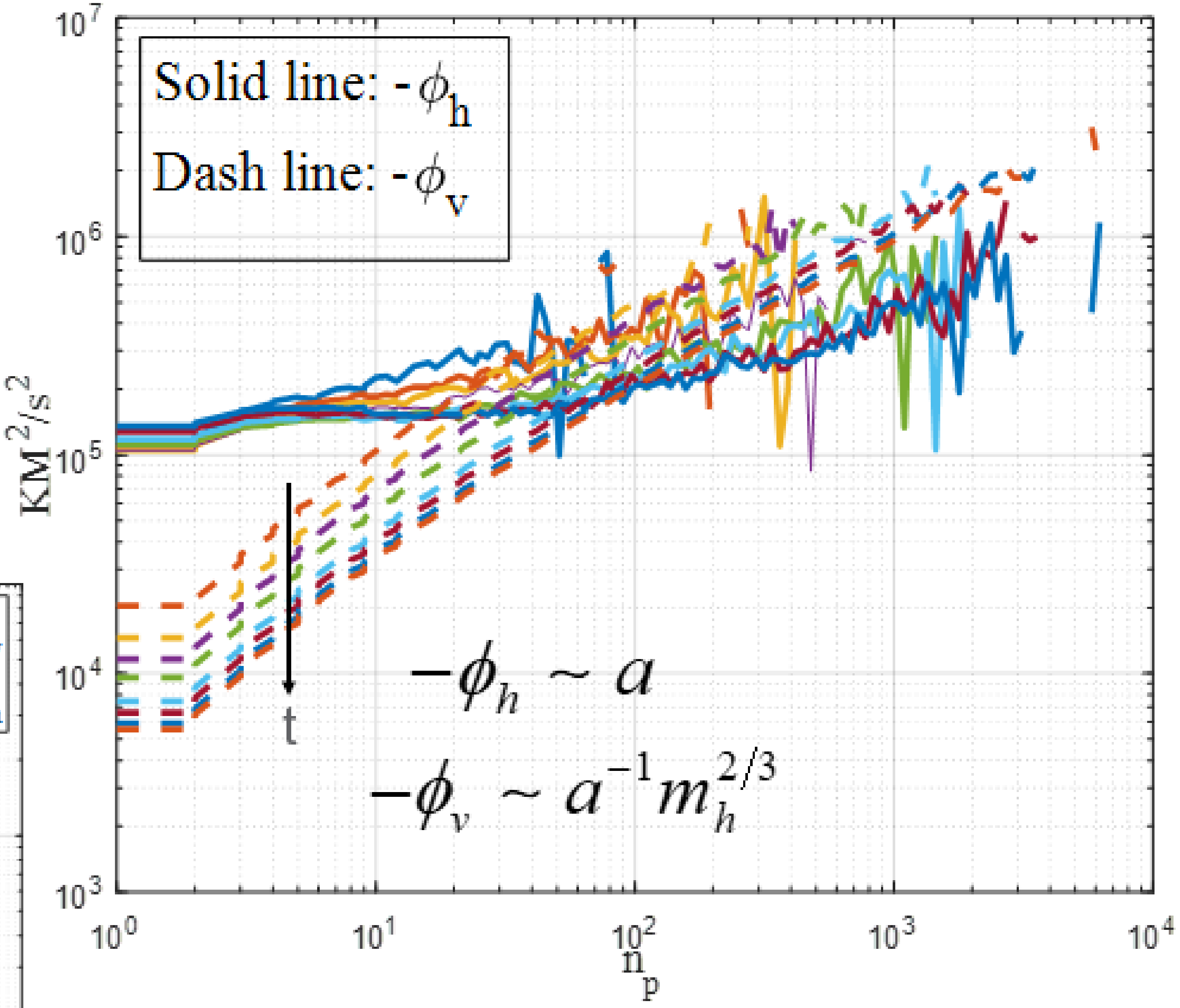
Inter-halo: $\gamma_h = -3\sigma_h^2 / \phi_h$

For large halos: $\gamma_v \approx 1.3$

due to halo surface energy

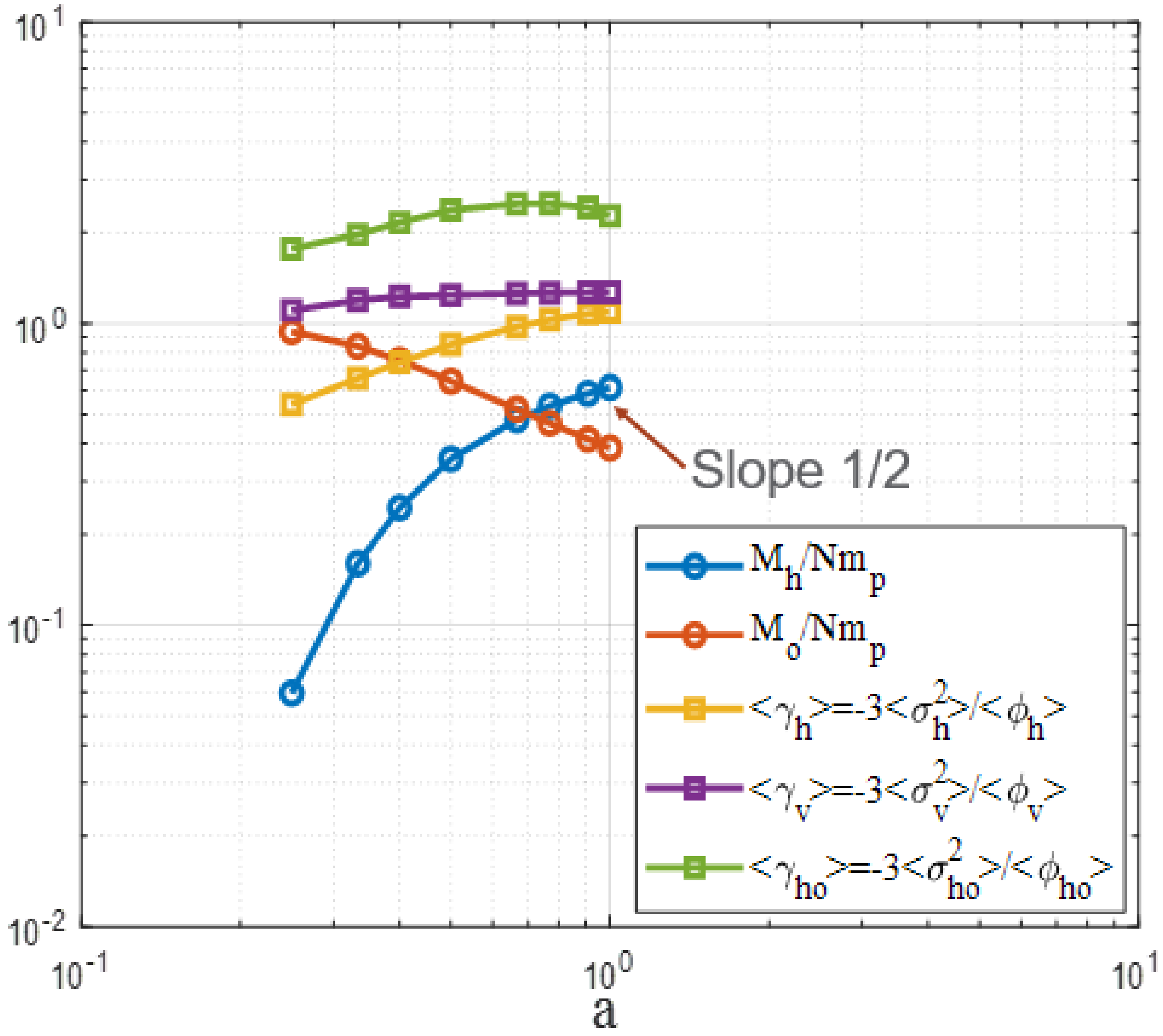
Direct cascade for potential energy from large to small





Variation with halo size for redshifts z = 0, 0.1, 0.3, 0.5, 1.0, 1.5, 2.0, and 3.0

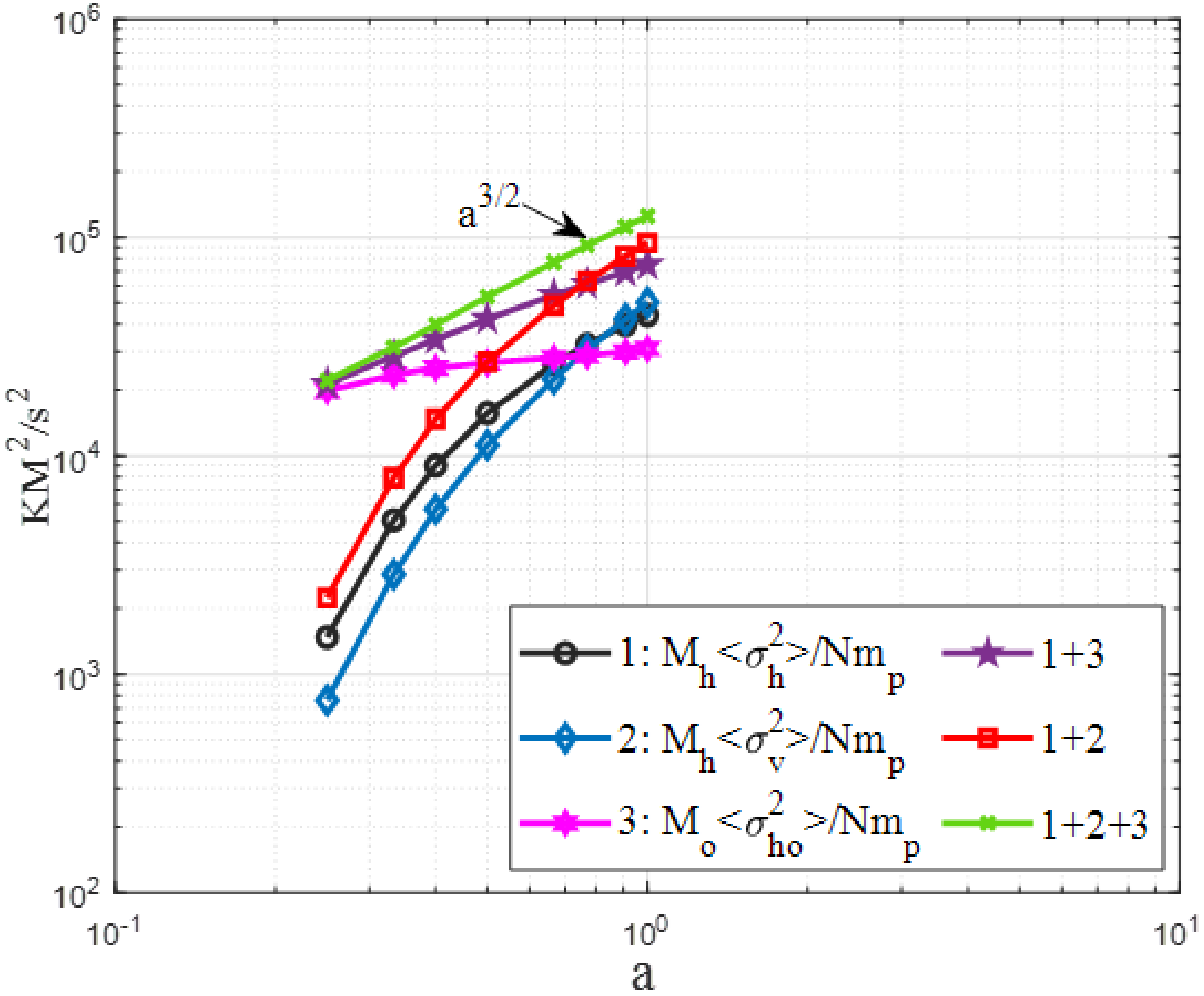
Redshift evolution of halo mass and virial ratio



The variation of total halo mass M_h, out-of-halo mass M_o and virial ratios with scale factor *a*.

- Mass flux from out-of-halo to halos sustains the total halo mass growing as $M_h(a) \sim a^{1/2}$, as predicted from mass cascade.
- ~ 60% of total mass are in halos and ~40% in out-of-halo (single merges)
- For the motion of halos, virial ratio (yellow) takes longer time to reach equilibrium due to weak gravity between halos.
- For motion in halos, virial equilibrium is established much faster with virial ratio ≈1.3 (yellow).
- Virial ratio≈2 (green) for out-of-halo particles(single mergers). The out-of-halo sub-system is energy conserved (no virilization), i.e. KE+PE = 0.

Redshift evolution of kinetic energies



Variation of three kinetic energies for halo and out-of-halo particles with scale factor *a*

- Total total kinetic energy of entire N-body system (green line: 1+2+3) grows ∝ t.
- Total kinetic energy in out-of-halo sub-system (magenta: 3) is time-invariant.
- The total kinetic energy of halo sub-system (red: 1+2) becomes dominant over out-of-halo sub-system grows ∝ t.

$$\left\langle\sigma_h^2\right\rangle \approx \left\langle\sigma_v^2\right\rangle = \frac{1}{2}\sigma^2$$

- A cross-over can be found at around a=0.5.
- A constant and scale-independent rate of energy cascade can be identified:

$$\varepsilon_u = -\frac{3}{2}\frac{u^2}{t} = -\frac{3}{2}\frac{u_0^2}{t_0} = -\frac{9}{4}H_0 u_0^2 \approx -4.6\times10^{-7}\,\frac{m^2}{s^3}$$

Rate of mass and kinetic energy cascade

The rate of mass cascade:

$$\varepsilon_m = \Pi_m\left(m_h \to 0, a\right) = -\frac{1}{2} M_h\left(a\right) H \propto a^{-1}$$

The rate of cascade of halo kinetic energy σ_h^2:

$$\varepsilon_{kh} = \Pi_{kh}\left(m_h \to 0, a\right) = \varepsilon_m \left\langle \sigma_h^2 \right\rangle = -\frac{1}{2} M_h\left(a\right) H \left\langle \sigma_h^2 \right\rangle \propto a^0$$

The rate of cascade of virial kinetic energy σ_v^2 :

$$\varepsilon_{kv} = \Pi_{kv}\left(m_h \to 0, a\right) = -\frac{5}{2} M_h\left(a\right) H \left\langle \sigma_v^2 \right\rangle \propto a^0$$

The rate of cascade of total kinetic energy:

$$\varepsilon_u = \frac{3}{2} \frac{\left(\varepsilon_{kh} + \varepsilon_{kv}\right)}{M_h\left(a\right)} \frac{M_h\left(a\right)}{M_{tot}} = -\frac{9}{4} H \sigma^2 \frac{M_h\left(a\right)}{M_{tot}} \approx \frac{3}{2} \frac{u^2}{t}$$

$$\left\langle \sigma_h^2 \right\rangle \approx \left\langle \sigma_v^2 \right\rangle = \frac{1}{2} \left\langle \sigma^2 \right\rangle \propto a^1$$

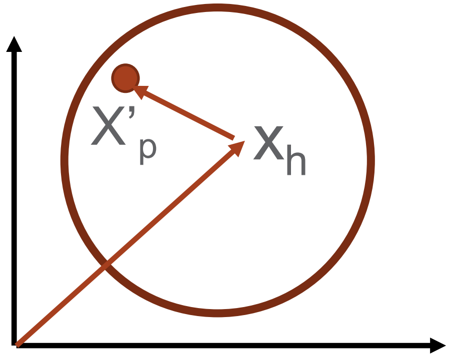
- Total mass in N-body system: M_{tot}
- Total halo mass in all halos: M_h
- Total mass in out-of-halo: M_{oh}
- One-dimensional velocity dispersion in N-body system: u^2
- One-dimensional velocity dispersion in all halos: $<\sigma^2>$
- One-dimensional halo velocity dispersion in all halos: $<\sigma_h^2>$
- One-dimensional halo virial dispersion in all halos: $<\sigma_v^2>$
- Hubble parameter: H
- Physical time: t

Inverse cascade of halo radial and rotational kinetic energy

Decompose halo particle position and velocity

$$\mathbf{x}_p = \mathbf{x}_h + \mathbf{x}'_p$$

$$\mathbf{u}_p = \mathbf{u}_h + \mathbf{u}'_p$$



Define the mean square radius r_g:

$$r_g = \sqrt{\sum_{p=1}^{n_p} \left|\mathbf{x}'_p\right|^2 \Big/ n_p}$$

Angular momentum:

$$\mathbf{H}_h = \frac{1}{n_p} \sum_{i=1}^{n_p} \left(\mathbf{x}'_p \times \mathbf{u}'_p\right)$$

(peculiar) virial quantity (radial momentum):

$$G_{hp} = \frac{1}{n_p} \sum_{i=1}^{n_p} \left(\mathbf{x}'_p \cdot \mathbf{u}'_p\right)$$

$$\gamma_G = \frac{-G_{hp}(m_h, a)\, a^{3/2}}{\left|\mathbf{H}_h(m_h, a)\right|} = \frac{f_G(m_h)}{f_H(m_h)}$$

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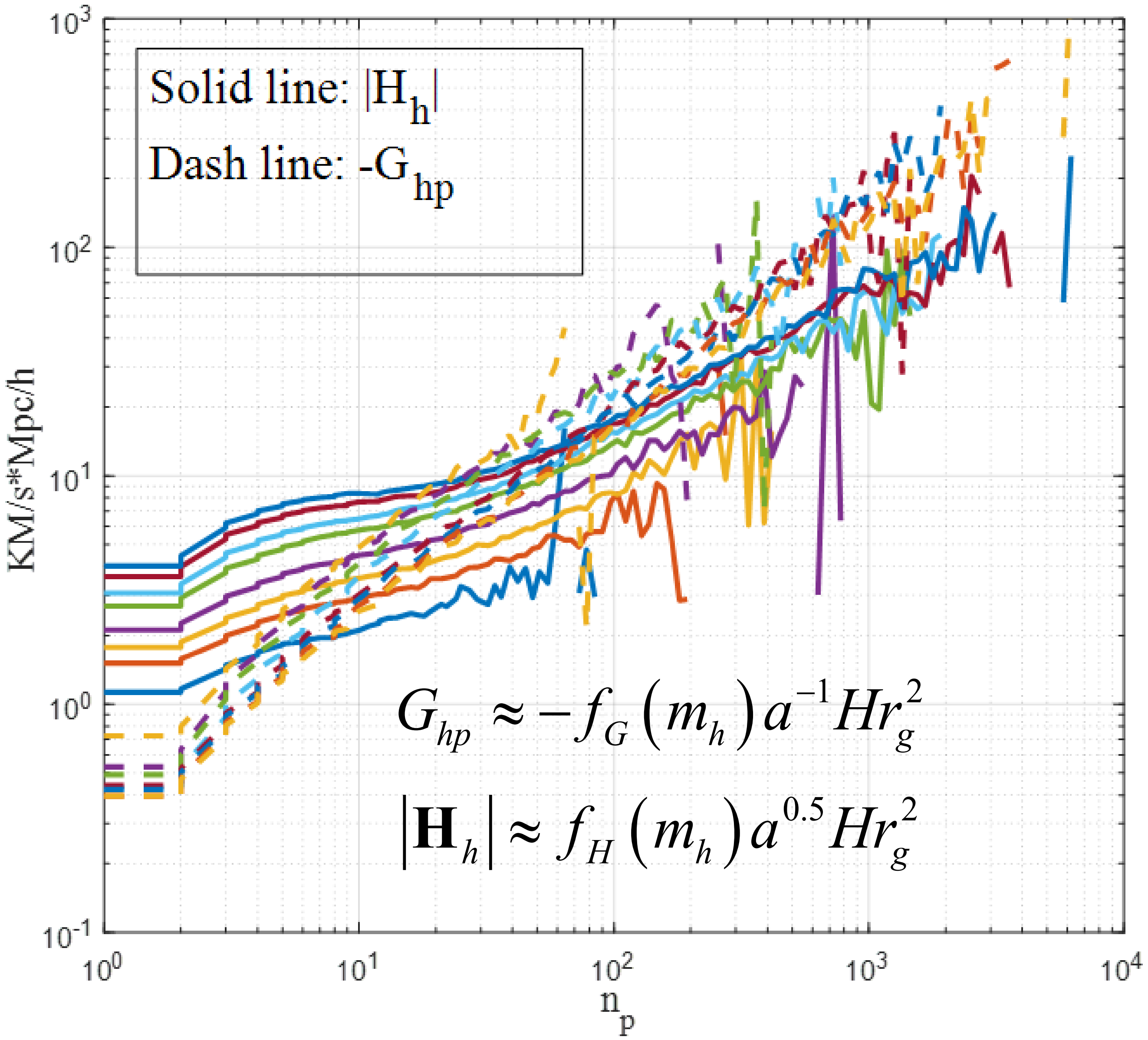
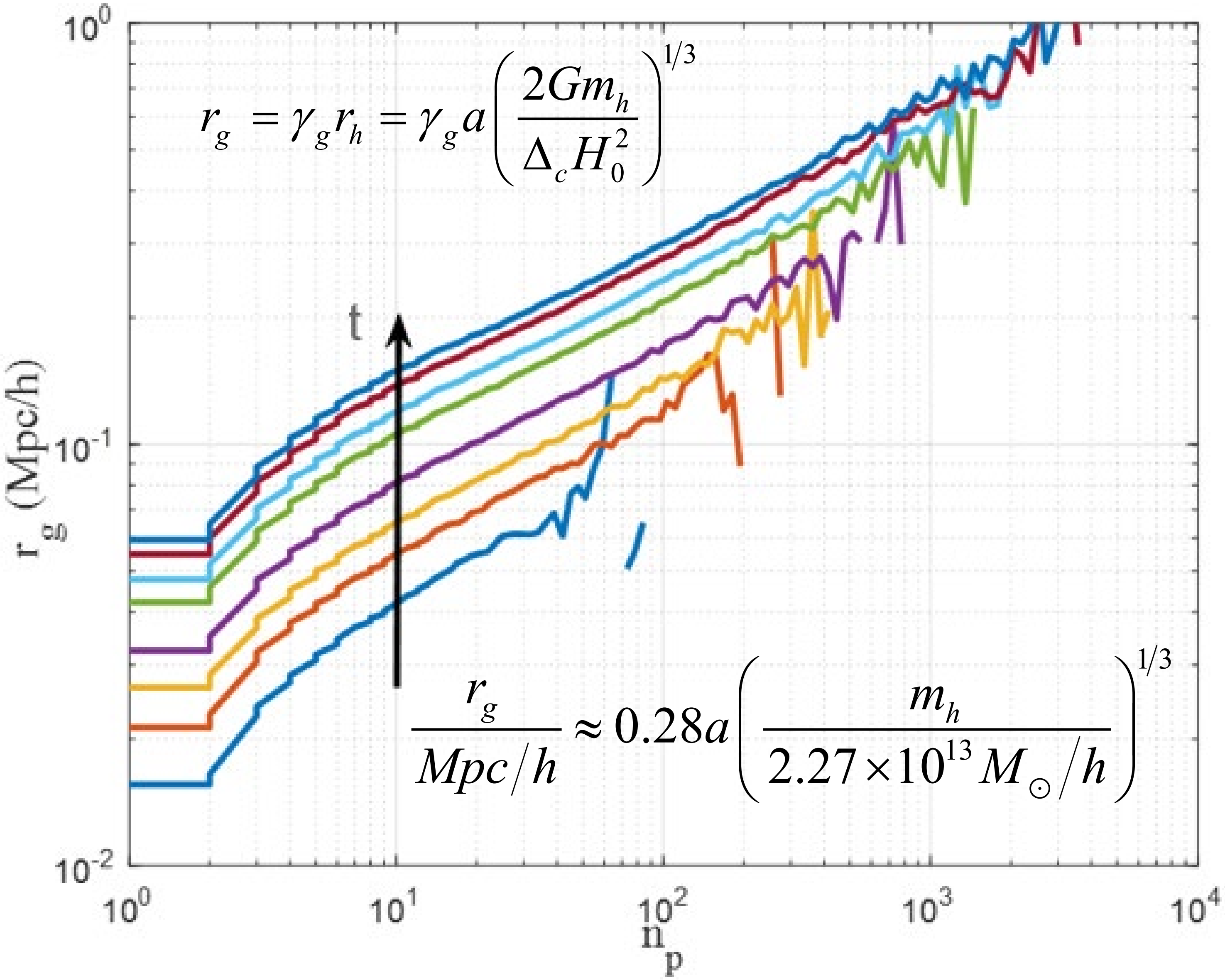


$$G_{hp} \approx -f_G(m_h)\, a^{-1} H r_g^2$$

$$\left|\mathbf{H}_h\right| \approx f_H(m_h)\, a^{0.5} H r_g^2$$

Variation with halo size for different redshifts z = 0, 0.1, 0.3, 0.5, 1.0, 1.5, 2.0, and 3.0.

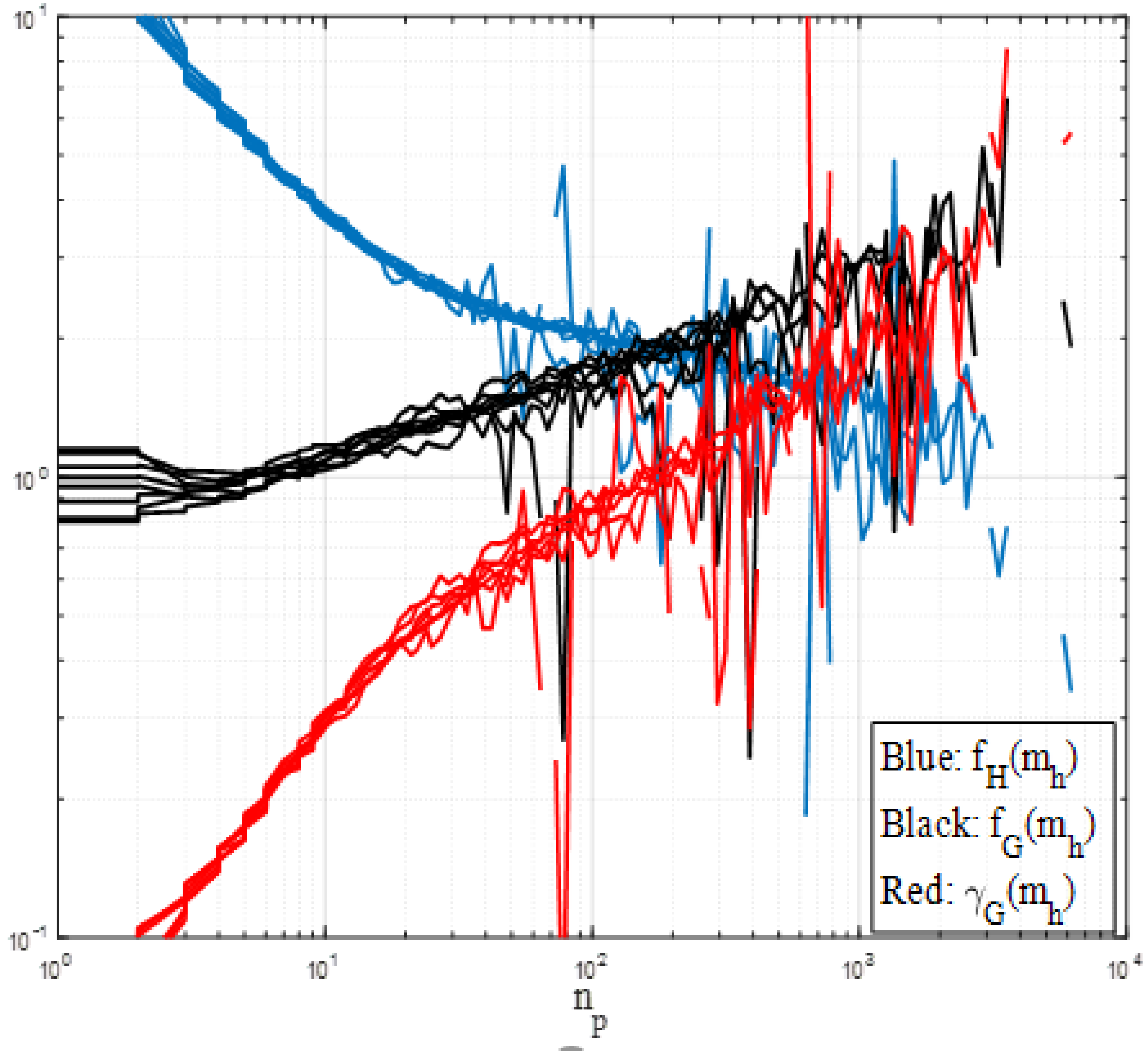
Modeling halo angular and radial momentum



$$r_g = \gamma_g r_h = \gamma_g a\left(\frac{2Gm_h}{\Delta_c H_0^2}\right)^{1/3}$$

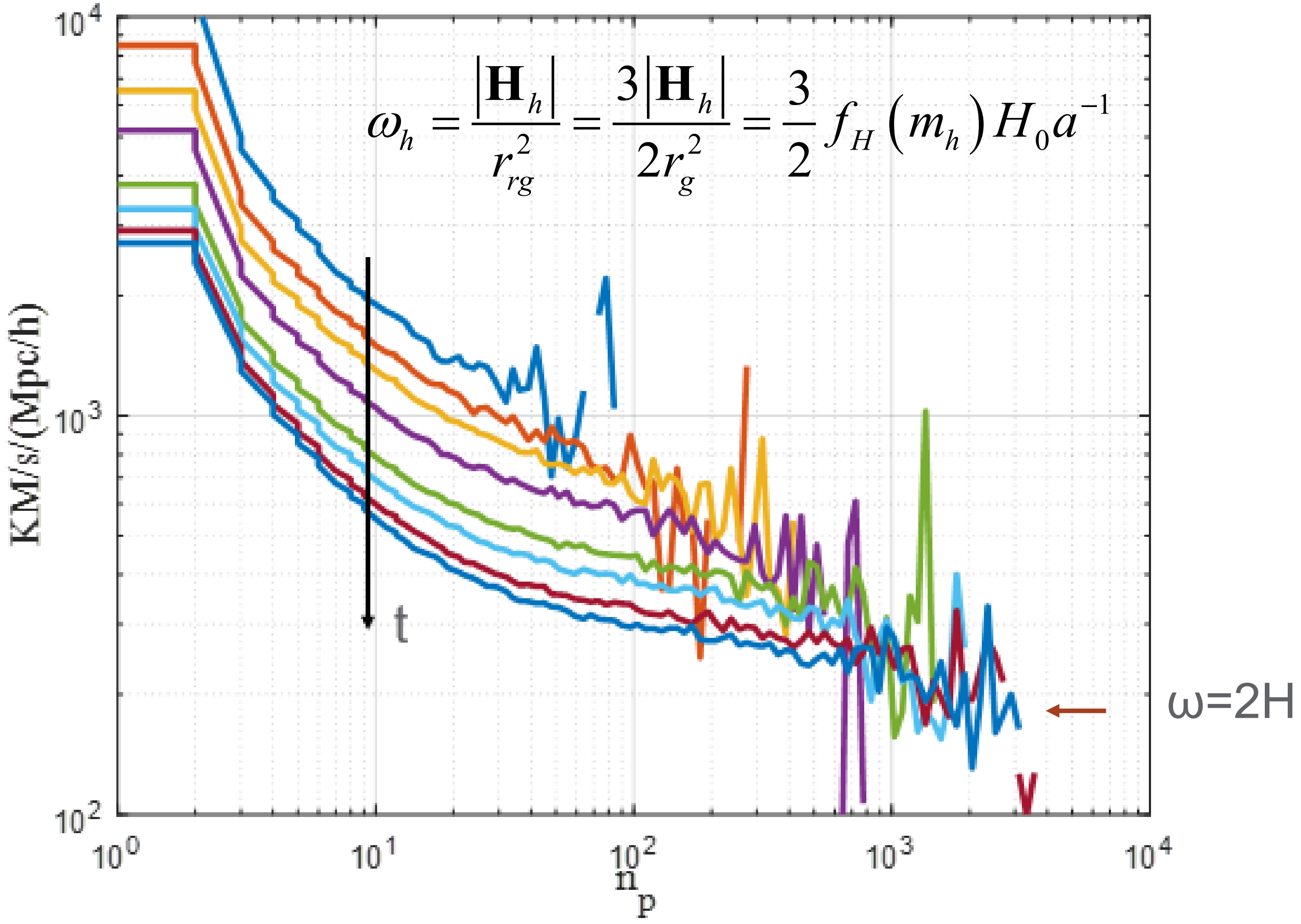
$$\frac{r_g}{Mpc/h} \approx 0.28a\left(\frac{m_h}{2.27\times10^{13} M_\odot/h}\right)^{1/3}$$

The variation of mean square radius r_g

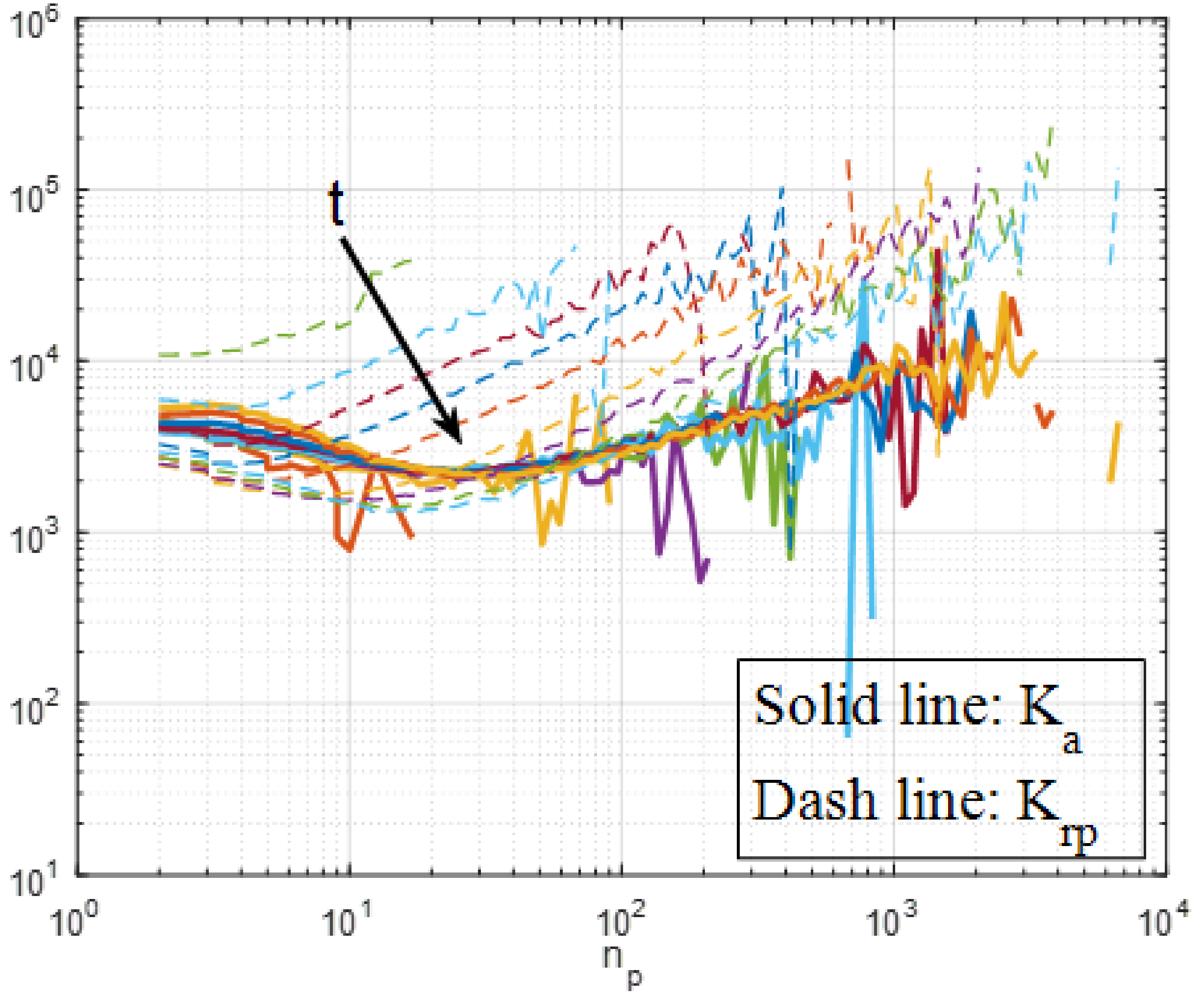


The variation of two coefficients f_G, f_H and ratio γ_G

Halo angular velocity and kinetic energy from coherent motion (mean flow)



$$\omega_h=\frac{|\mathbf{H}_h|}{r_{rg}^2}=\frac{3|\mathbf{H}_h|}{2r_g^2}=\frac{3}{2}f_H(m_h)H_0a^{-1}$$



The variation of halo angular velocity, rotational kinetic energy and radial kinetic energy

$$K_a=\frac{1}{2}|\mathbf{H}_h|\omega_h=\frac{3}{4}\left(|\mathbf{H}_h|/r_g\right)^2=\frac{3}{4}\gamma_g^2\left[f_H(m_h)\right]^2\left[\frac{2Gm_hH_0}{\Delta_c}\right]^{2/3}$$

$$K_{rp}=\frac{1}{2}\left(G_{hp}/r_g\right)^2=\frac{1}{2}\gamma_g^2a^{-3}\left[f_G(m_h)\right]^2\left[\frac{2Gm_hH_0}{\Delta_c}\right]^{2/3}$$

The effect of halo shape on energy cascade

Vortex Stretching (shape changing) responsible for energy cascade in turbulence.

What about the shape change of halo?

Assuming ellipsoid shape, 3x3 inertia tensor for every halo:

$$I_{ij} = \sum_{p=1}^{n_p} x'_{p,i} x'_{p,j}$$

➡ Three eigenvalues (length of semimajor axis)

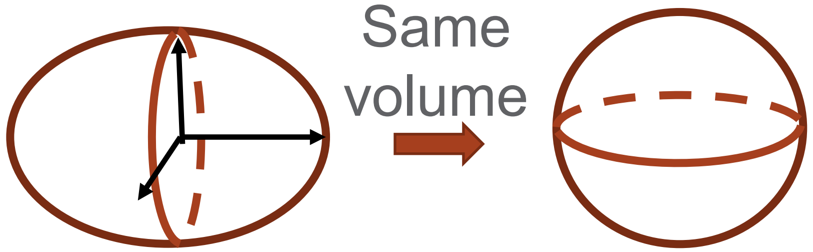
$$r_{\lambda 1} \le r_{\lambda 2} \le r_{\lambda 3}$$

Mean square radius:

$$r_g^2 = r_{\lambda 1}^2 + r_{\lambda 2}^2 + r_{\lambda 3}^2$$

Define two critical ratios:

$$\lambda_{r1} = \frac{r_{\lambda 2} - r_{\lambda 1}}{r_{\lambda 3} - r_{\lambda 2}}$$



Same volume

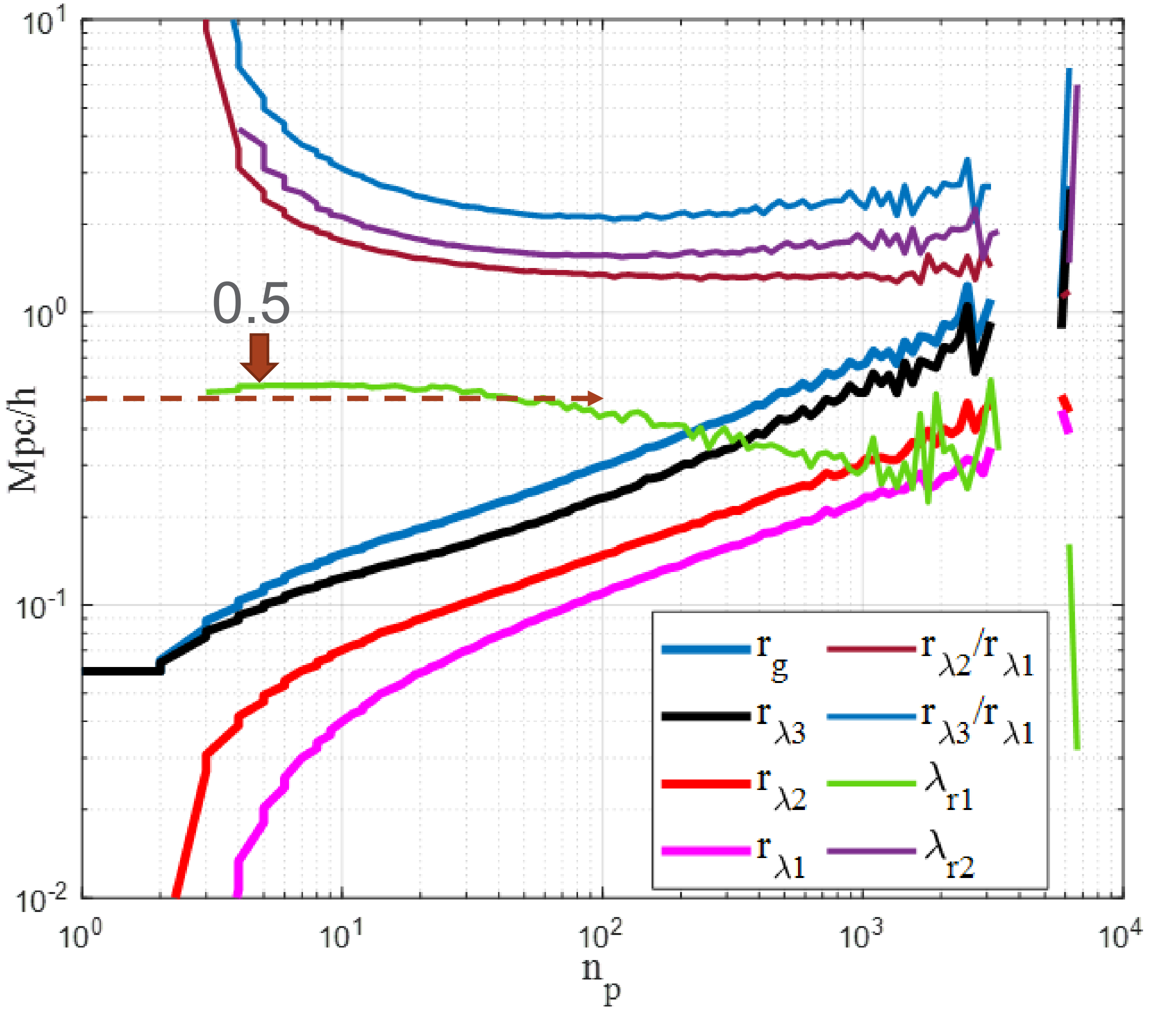
Moment of inertia for ellipsoid

$$\lambda_{r2} = \frac{r_{\lambda 3}^2 + r_{\lambda 2}^2}{2\left(r_{\lambda 1} r_{\lambda 2} r_{\lambda 3}\right)^{2/3}}$$

Moment of inertia for sphere

$\lambda_{r1} \approx 0.5$ for small halos, a unique path of shape evolution (green);

$\lambda_{r2} = 1$ for sphere;

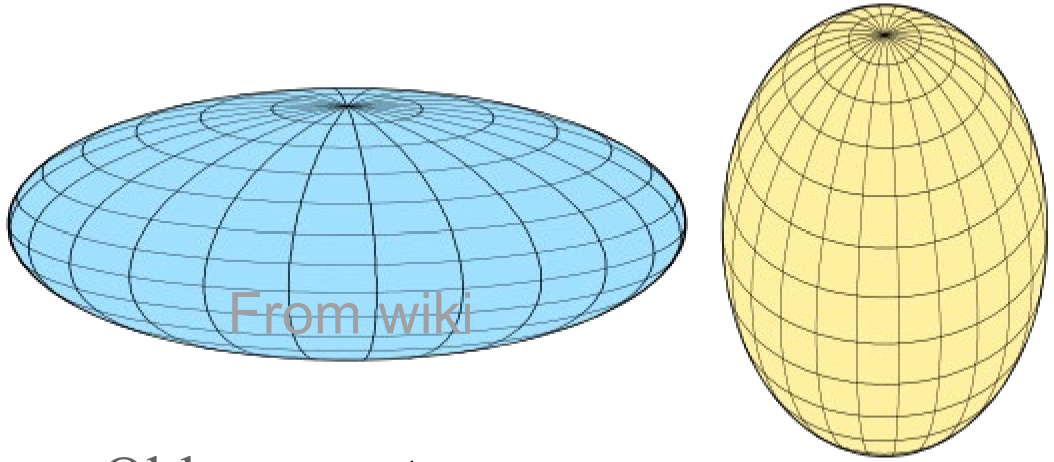


Simulated halos:

$$\lambda_{r2} = [1.55, 2]$$

Change of halo shape should not play a significant role in energy cascade.

Various halo shape parameters



Oblate: $r_{\lambda 1} < r_{\lambda 2} = r_{\lambda 3}$ Prolate: $r_{\lambda 1} = r_{\lambda 2} < r_{\lambda 3}$

Triaxiality parameter:

$$h_t = \frac{r_{\lambda 3}^2 - r_{\lambda 2}^2}{r_{\lambda 3}^2 - r_{\lambda 1}^2}$$

$h_t = 1$ prolate

$h_t = 0$ oblate

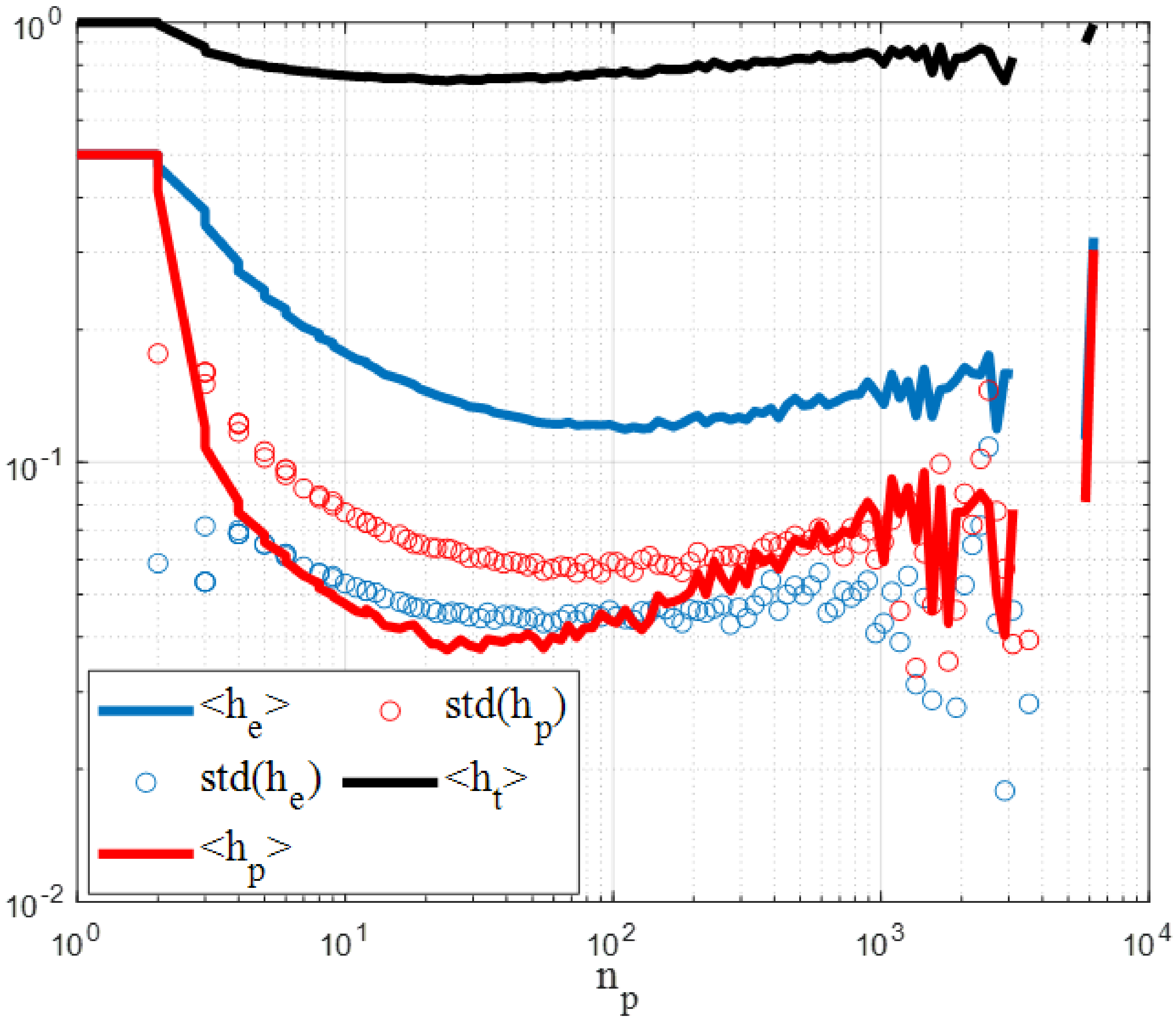
Ellipticity & prolateness parameters:

$$h_e = \frac{r_{\lambda 3} - r_{\lambda 1}}{2\left(r_{\lambda 1} + r_{\lambda 2} + r_{\lambda 3}\right)}$$

$h_p = -h_e$ oblate

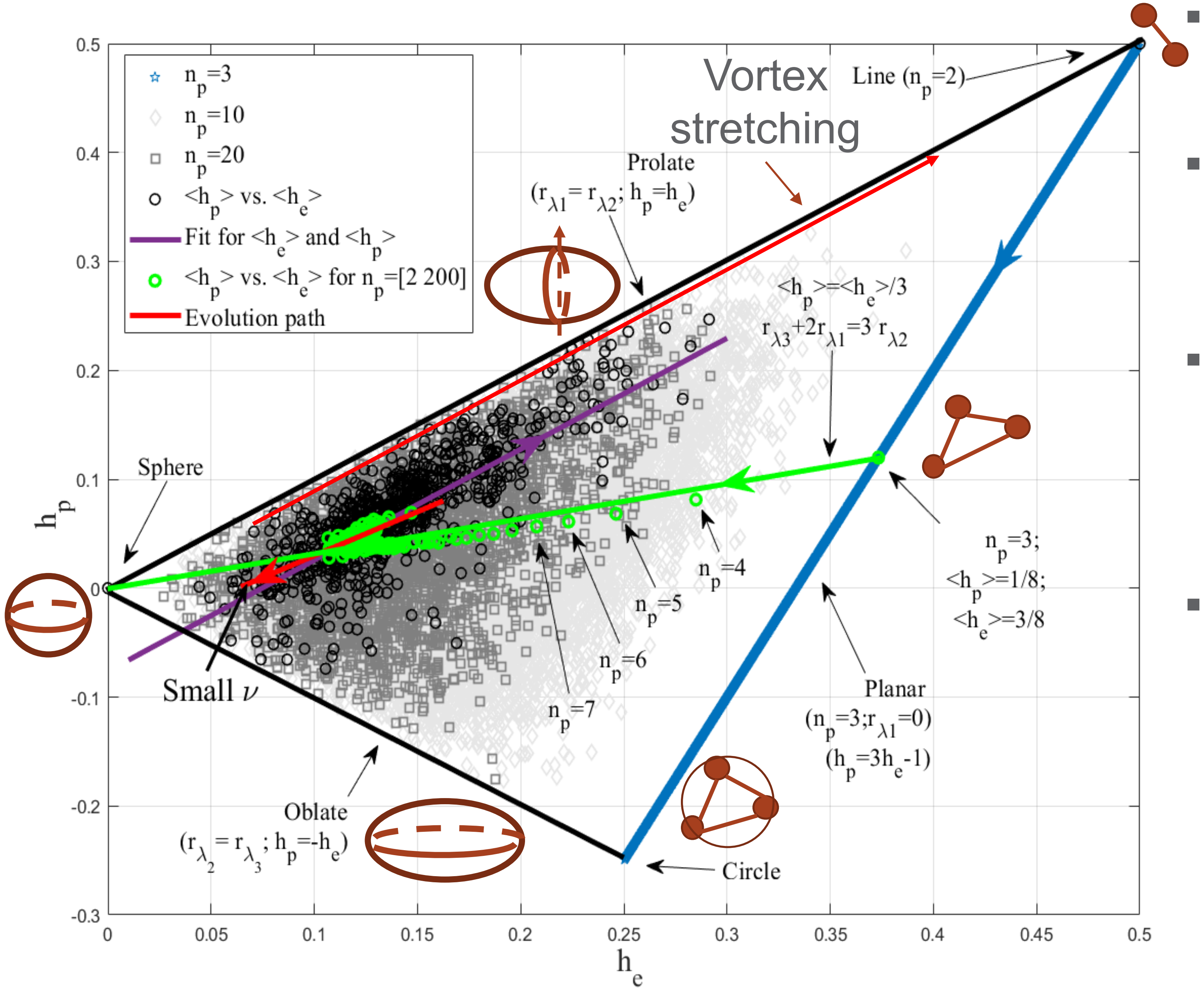
$$h_p = \frac{r_{\lambda 3} - 2r_{\lambda 2} + r_{\lambda 1}}{2\left(r_{\lambda 1} + r_{\lambda 2} + r_{\lambda 3}\right)}$$

$h_p = h_e$ prolate



The variation of halo shape parameters with halo size at z=0

Two-dimension h_e-h_p mapping of halo shape



- All three-body halos have planar structure (blue line) with mean values of 1/8 and 3/8.
- The mean shape parameters for all halo groups (black circles). Green circles highlight the halos in range of n_p=[3 200]. Halos are more prolate.
- With increasing size, the shape of halos evolves consistently toward sphere along a unique path (green line) before a “V” turn. Path required λ_{r1}=0.5.
- Red line with arrow pointing to low peak height indicates the evolution path of simulated halo shape from early stage (ν=5) to late stage (ν=0.5).

$$h_e = 0.098\log_{10}\nu + 0.094$$

$$h_p = 0.079\log_{10}\nu + 0.025$$

Peak height: $\nu = \delta_{cr}/\sigma(m_h, z)$ [5 to 0.5]

Summary and keywords

Inverse energy cascade	Direct energy cascade	Halo inertia tensor
Energy flux function	Energy transfer function	Halo mean square radius
Prolate & oblate	Ellipticity & prolateness	Halo moment of inertia
Halo virial/velocity dispersion	Intra- and inter-halo potential	Halo radial & angular momentum

- Establish connections of energy cascade in turbulence and dark matter flow
- Direct energy cascade in hydrodynamic turbulence is facilitated by the vortex stretching (shape changing) along its axis of rotation
- Inverse cascade of kinetic energy from small to large mass scales in dark matter flow
- Direct cascade of potential energy from large to small mass scales
- A constant scale-independent rate of energy cascade $\varepsilon_u \sim a^0$ and *a* is scale factor
- Energy cascade in dark matter flow is mostly facilitated by the mass cascade of halos
- The shape change of halos does not play the major role.
- A unique evolution path of halo shape that gradually approaches spherical shape with increasing halo size