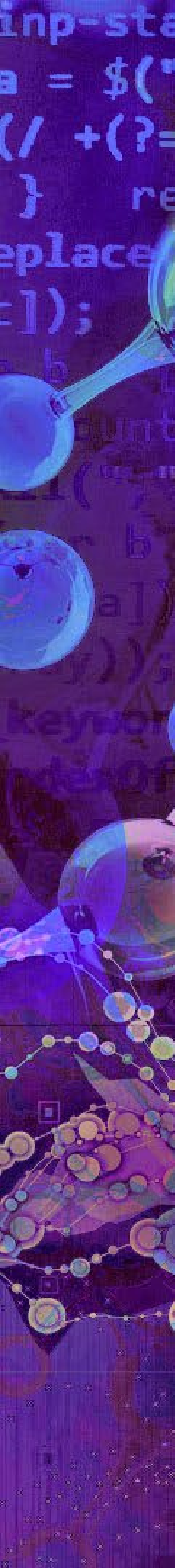

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A comparative study of dark matter flow & hydrodynamic turbulence and its applications

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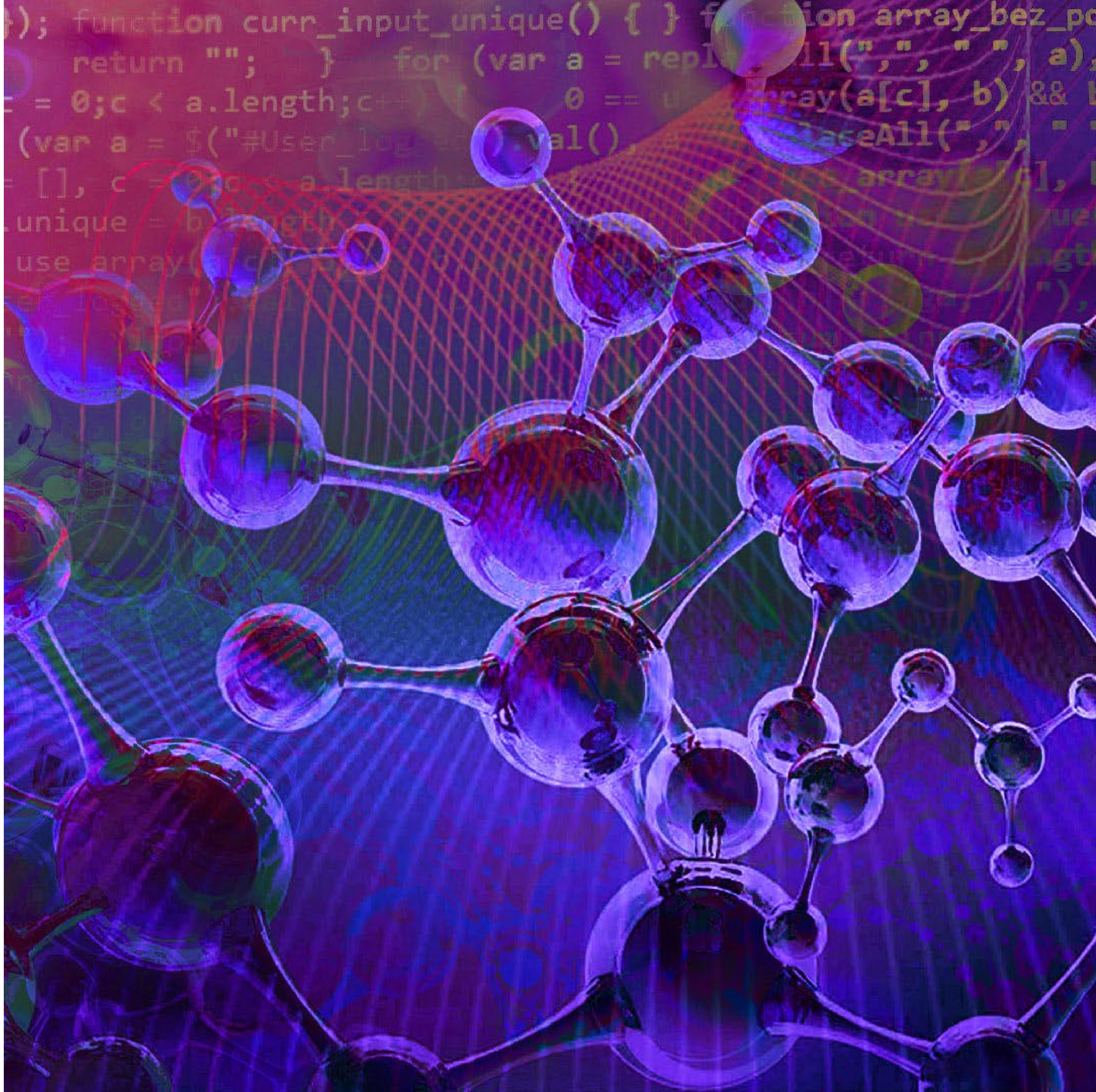


Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!

Data repository and relevant publications

Structural (halo-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6541230
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6569898
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

Structural (halo-based) approach for dark matter flow

Effect of mass cascade on halo energy, size, and density profile

Xu Z., 2021, arXiv:2109.12244v1 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2109.12244

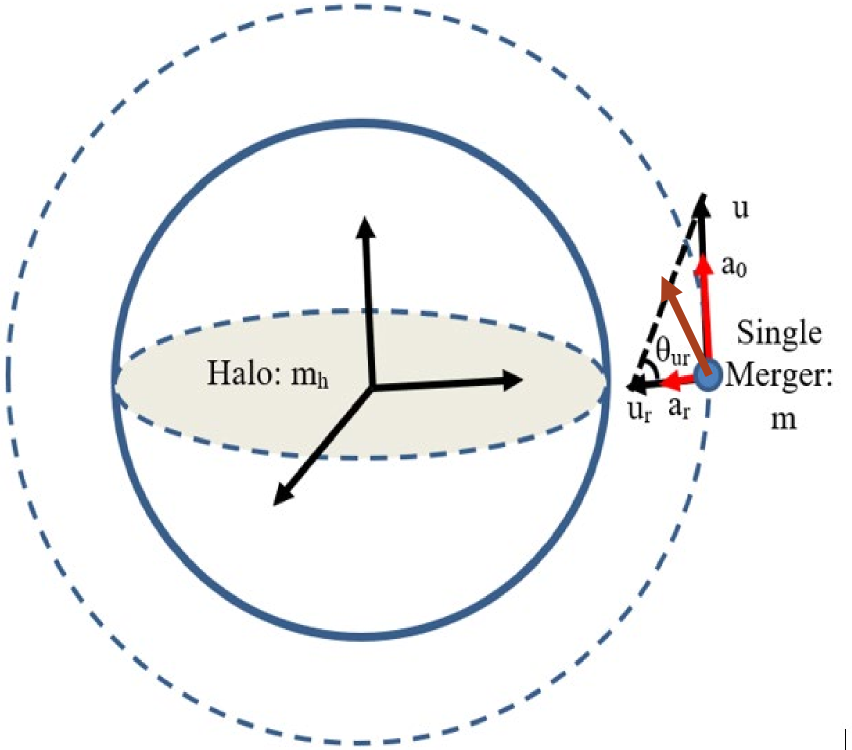
Introduction

Review: In hydrodynamic turbulence, "Energy cascade" involves the energy transfer from large eddies to small eddies with a scale-independent rate of energy cascade. No mass cascade!

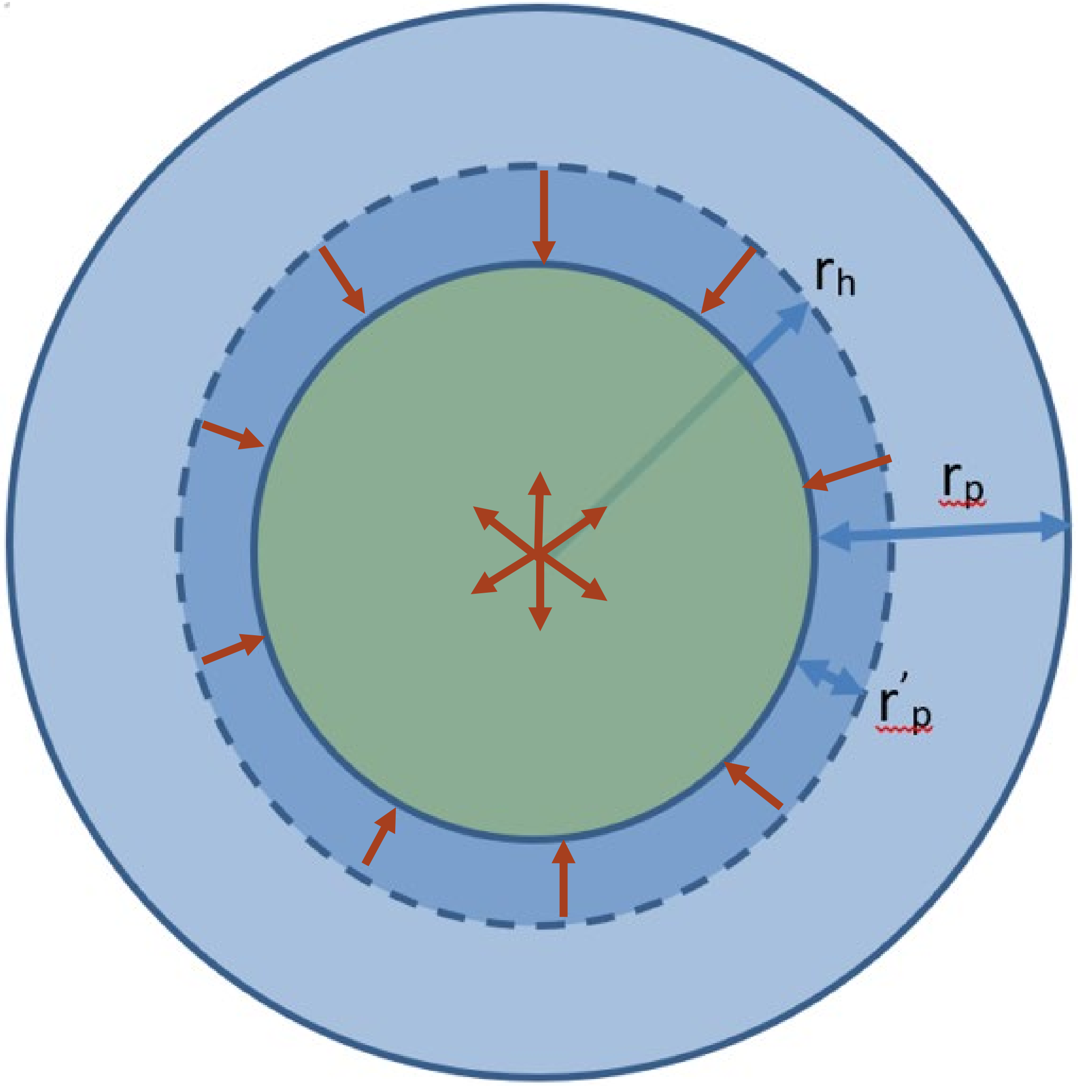
"Little halos have big halos, That feed on their mass; And big halos have greater halos, And so on to growth"

"Eddy" is not a well-defined object in turbulence literature. However, "halo" are well-defined dynamical objects, whose abundance and internal structure have been extensively studied over several decades.

- Goal 1: Explore effects of inverse mass cascade on halo energy, momentum, halo size and internal structure (density) evolution.
- Goal 2: Explore the dynamic evolution of halo size (geometric Brownian motion)
- Goal 3: Explore the random walk of particle in halos with a randomly evolution size. This leads to a universal halo density profile.



Halo mass accretion, deformation, and radial flow



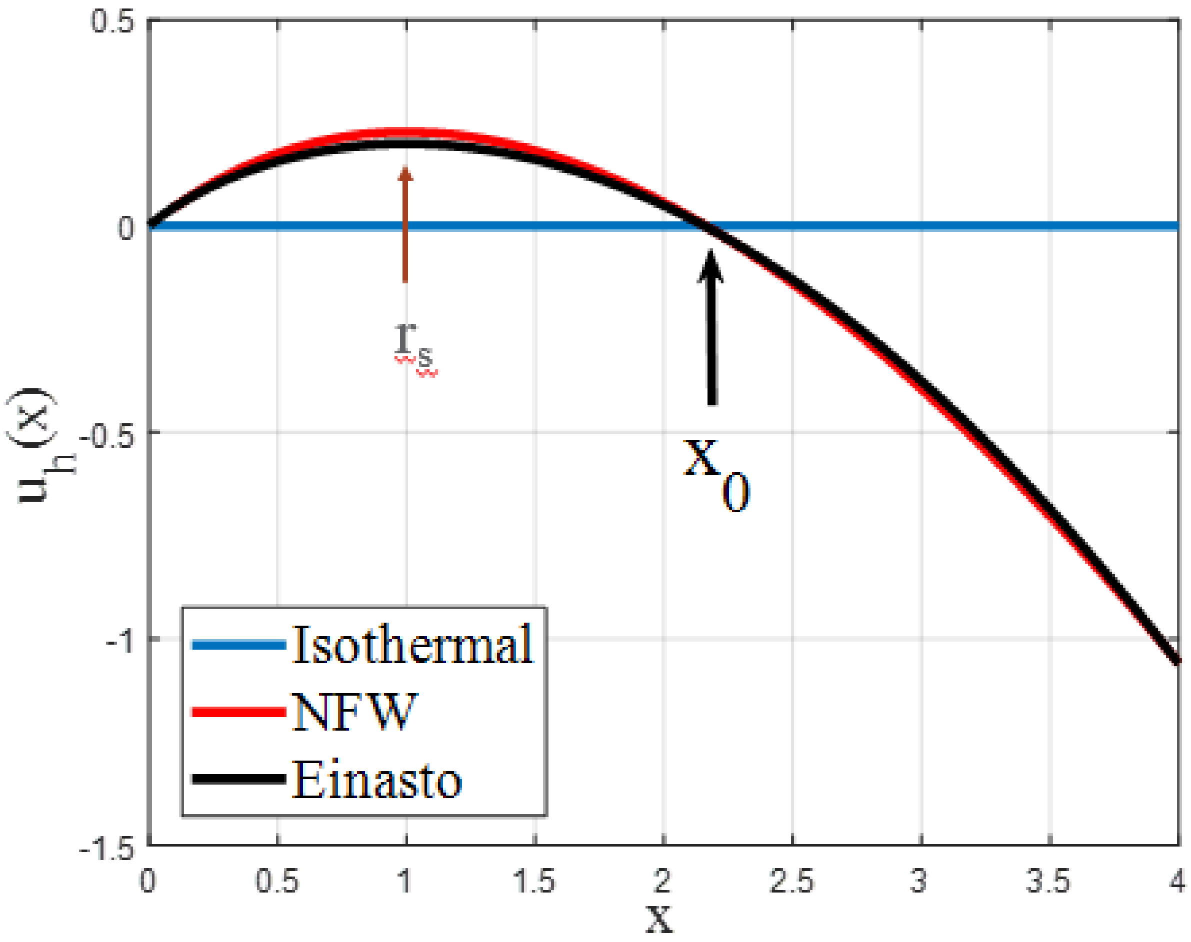
Schematic plot of halo mass accretion and deformation

- Halo grows with a new layer of particles of thickness r_p formed due to halo mass accretion (mass cascade)
- Original halo (dash line) deforms in size (shrinks to green) by r_p' due to gravity of new layer
- The net change in halo size is r_p - r_p'
- Halo deformation at halo surface induces a non-zero inward radial flow u_r
- What about the radial flow at halo center??
  - Must be outwards if no blackhole considered

Halo deformation parameter $$\alpha_h = 1 - r_p' / r_p$$

Isothermal profile (vanishing radial flow, no time to relax or deform due to extremely fast mass accretion): $$\alpha_h = 1$$

Effect of radial flow on halo density profile



- Outward flow in core and inward flow in outer region
- Radial flow creates a new length scale for any halo density: the scale radius r_s
- Vanishing radial flow for isothermal: extremely fast mass accretion and no time for halo to deform

Reduced spatial/ temporal coordinate: $$x(r,a)=\frac{r}{r_s(a)}=\frac{cr}{r_h(a)}$$

Function $F(x)$ for enclosed mass at given r: $$m_r(r,a)=m_h(a)\frac{F(x)}{F(c)}$$

Halo density: $$\rho_h(r,a)=\frac{1}{4\pi r^2}\frac{\partial m_r(r,a)}{\partial r}=\frac{m_h(a)}{4\pi r_h^3}\frac{c^3F'(x)}{x^2F(c)}$$

Radial continuity equation: $$\frac{\partial\rho_h(r,a)}{\partial t}+\frac{1}{r^2}\frac{\partial\left[r^2\rho_h(r,a)u_r(r,a)\right]}{\partial r}=0$$

Radial flow equation: $$u_h(x)=\left[x-\frac{F(x)}{F'(x)}\right]\frac{\partial\ln r_h}{\partial\ln t}$$ ← Mass cascade

Density ρ_h ⟷ F(x) ⟷ Radial flow $u_h(x)$

NFW: $F(x)=\ln(1+x)-x/(1+x)$

Einasto: $F(x)=\Gamma(3/\alpha)-\Gamma(3/\alpha,2x^\alpha/\alpha)$

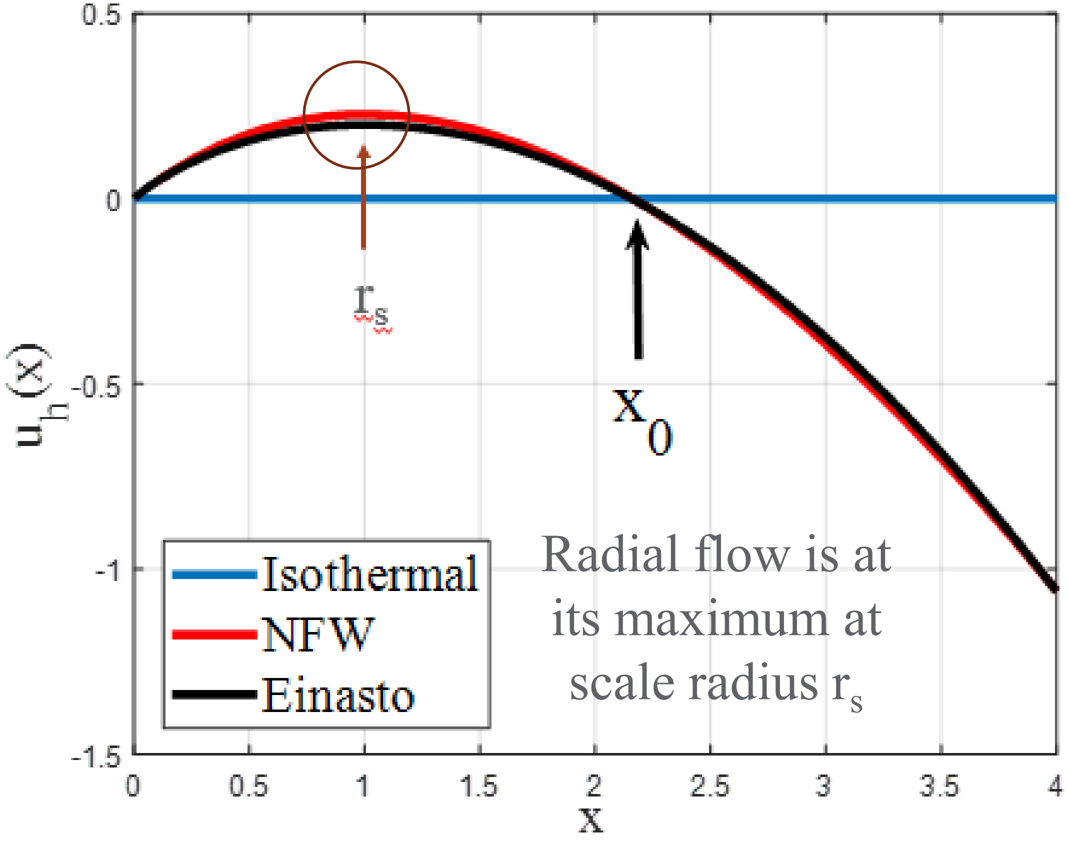
Isothermal: $F(x)=x/c$

Radial flow and angle of incidence

Logarithmic slope of density:

$$\frac{\partial \ln \rho_h}{\partial \ln x} = \frac{\partial \ln F'(x)}{\partial \ln x} - 2 = \frac{\partial u_h / \partial x}{1 - u_h / x} - 2$$

At r=r$_s$ $\quad \frac{\partial \ln \rho_h}{\partial \ln x} = -2$ and $\frac{\partial u_h}{\partial x} = 0$



- Single mergers merging with halo at an angle: angle of incidence
- Neither perpendicular nor tangential
- Angle of incidence determined by peculiar radial flow u_p and circular velocity v_{cir}

$$\cot(\theta_{vr}) = \frac{u_p}{v_{cir}} = \frac{1}{2\pi}\left(\frac{1}{\alpha_h} - \frac{1}{3}\right)$$

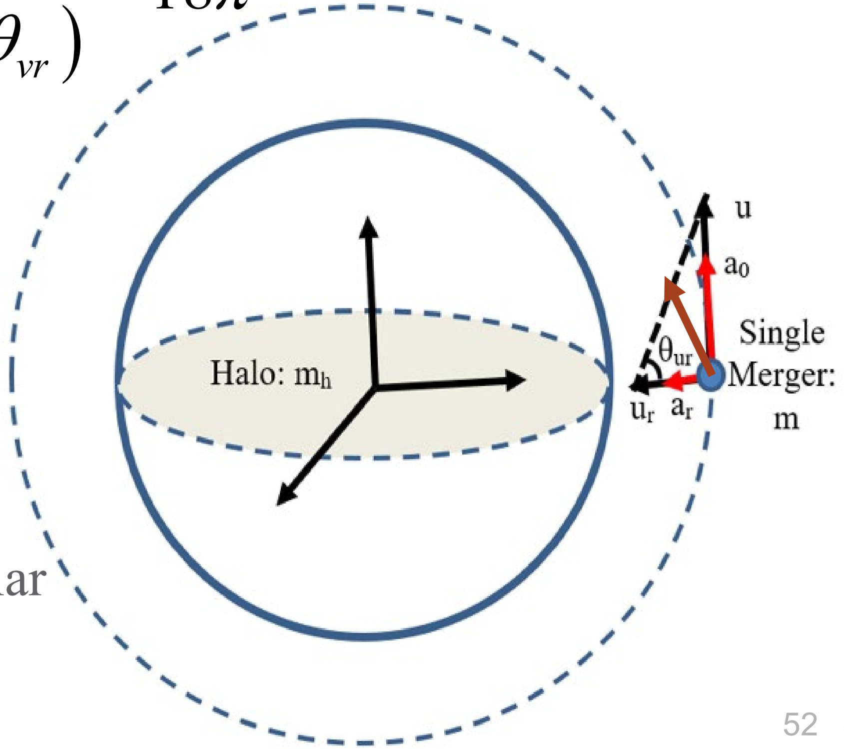
Deformation parameter for Isothermal profile: $\alpha_h = 1$

$$\cot(\theta_{vr}) = \frac{1}{3\pi} = \sqrt{\frac{2}{\Delta_c}}$$

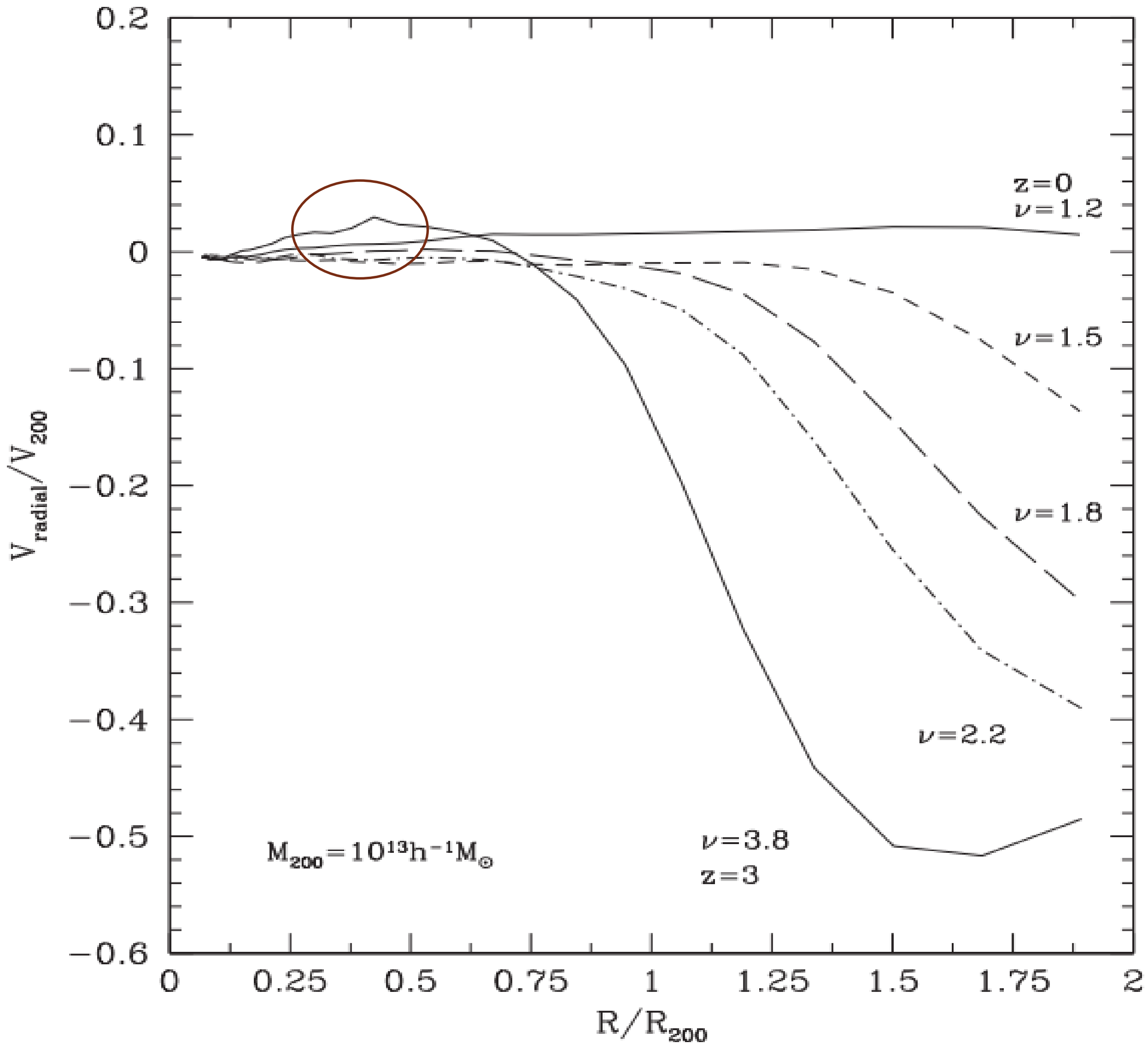
and

$$\Delta_c = \frac{2}{\cot^2(\theta_{vr})} = 18\pi^2$$

- Determine critical halo density Δ_c , (two-body collapse model)
- Determine the rate of energy cascade
  - No energy cascade if tangential
  - Maximum cascade if perpendicular
- Understand the critical MOND acceleration a_0



Radial flow from simulation



Radial flow from simulation

Klypin A. etc.., 2016, Mon. Not. R. Astron. Soc., 457, 4340

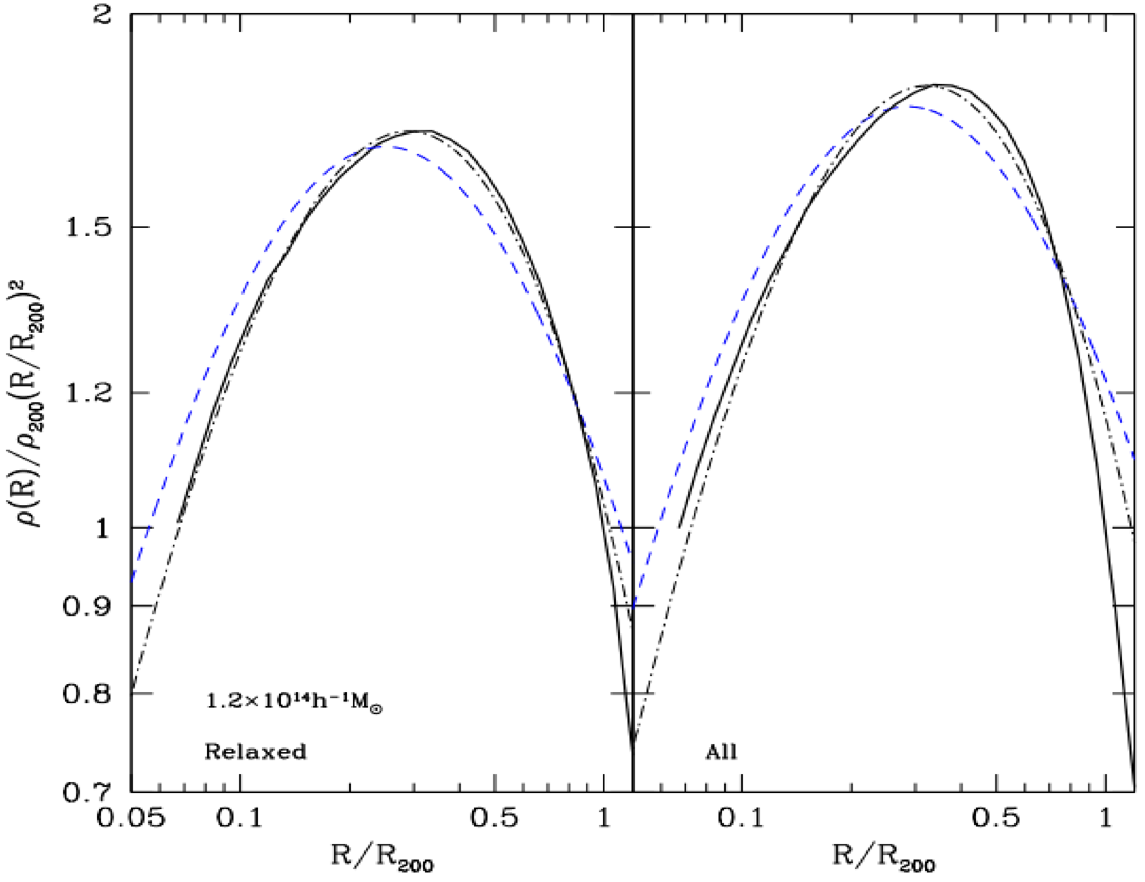


Figure 8. Density profiles of haloes with mass $M_{200} \approx 1.2 \times 10^{14}\, h^{-1}\, M_\odot$ at $z = 1.5$ (full curves). Left (right) panels show relaxed (all) haloes. Dot–dashed curves show Einasto fits, which have the same virial mass as haloes in the simulation. The NFW profiles (dashed curves) do not provide good fits to the profiles and significantly depend on what part of the density profile is chosen for fits.

Einasto profile is better than NFW for massive halos (high peak height ν), why?

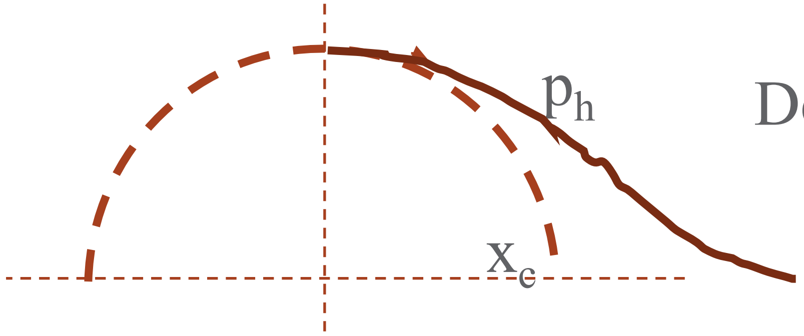
Radial flow u_r and pressure around halo center

Radial flow at halo center:

- Term 1 from mass cascade usually neglected
- The radial flow should vanish for virialized small halos with extremely slow mass accretion (late stage); gravity exactly balances pressure; stable clustering hypothesis (SCH)
- The radial flow should be the Hubble flow for large halos with extremely fast mass accretion (early stage).
- In spherical collapse model, the initial velocity of mass shells is simply the Hubble flow

Define a halo deformation rate:

$$\gamma_h = \partial u_h / \partial x\big|_{x=0}$$



Jeans' equation:

$$\underbrace{\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}}_{1}+\frac{1}{\rho_h}\frac{\partial\left(\rho_h\sigma_r^2\right)}{\partial r}=-\frac{\partial\phi_h(r,a)}{\partial r}=-\frac{Gm_r(r,a)}{r^2}$$

$$\underbrace{\sigma_r^2\frac{\partial\ln\left(\rho_h\sigma_r^2\right)}{\partial\ln x}}_{1}=\underbrace{x\frac{r_s^2}{t^2}\left[\frac{\partial u_h}{\partial x}\left(x\frac{\partial\ln r_s}{\partial\ln t}-u_h\right)+u_h\left(1-\frac{\partial\ln r_s}{\partial\ln t}\right)\right]}_{2}-\underbrace{v_c^2}_{3}$$

1: from pressure; 2: from radial flow; 3: from gravity

Parabolic pressure around halo center:

$$p_h(x)\equiv\rho_h(x)\sigma_r^2(x)=p_h(x=0)-\frac{\rho_h^2(x=0)\,v_{cir}^2}{2\bar{\rho}_h(a)\,c^2}x^2$$

Define a halo core size x_c :

$$p_h(x_c)=0 \Rightarrow x_c=\sqrt{\frac{2\bar{\rho}_h(a)}{\rho_h(0)}}\,\frac{c\sigma_r(0)}{v_{cir}}$$

Double power-law for halo density

Density profiles	Concentration c	Deformation parameter α_h	Deformation rate parameter γ_h	$\rho_h(r < r_s)$
Isothermal	3.5	1	0	r^{-2}
NFW	3.5	0.8329	1/2	r^{-1}
Einasto (α=0.2)	3.5	0.8371	2/3	r^{0}
			3/4	r^{1}

Density ρ_h ⟺ F(x) ⟺ Radial flow $u_h(x)$

$$\frac{\partial \ln \rho_h}{\partial \ln x} = \frac{\partial \ln F'(x)}{\partial \ln x} - 2 = \frac{\partial u_h/\partial x}{1 - u_h/x} - 2 \qquad \frac{\partial u_h(x)}{\partial x} = \frac{F(x)F''(x)}{F'^2(x)}$$

Double power-law:

$$\rho_h(r < r_s) \propto r^{(3\gamma_h - 2)/(1-\gamma_h)}$$

$$\rho_h(r > r_s) \propto r^{\frac{c(\alpha_h - 1)}{c - x_0} - 2}$$

$$\gamma_h = \partial u_h/\partial x\big|_{x=0}$$

$$\alpha_h = c\frac{F'(c)}{F(c)}$$

- Double power-law is a natural result due to radial flow in outer and inner regions
- Halo deformation parameter from mass cascade controls density in outer region
- Halo deformation parameter controls density in inner region
- The larger deformation rate at center, the larger logarithmic slope (baryonic feedback for core-cusp?)

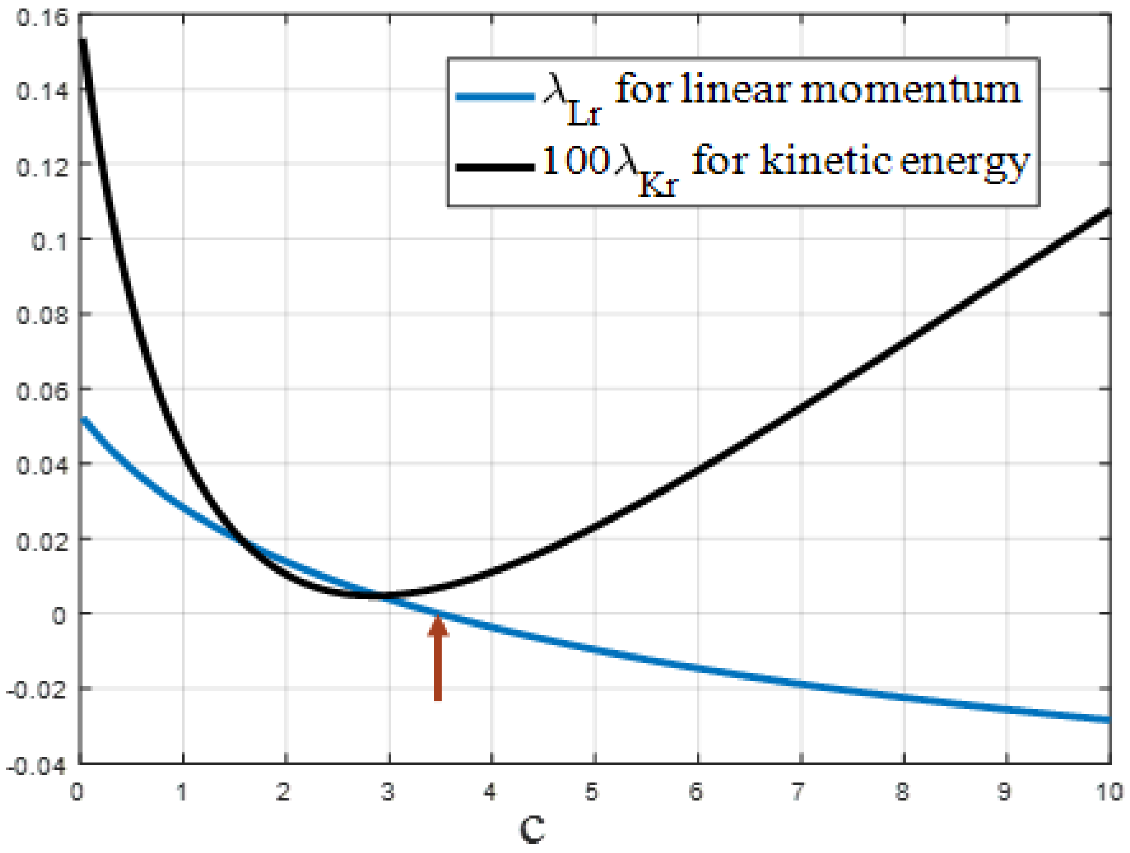
The limiting concentration c for large halos and radial momentum and kinetic energy

Vanishing radial Linear momentum (halos at turn-around):

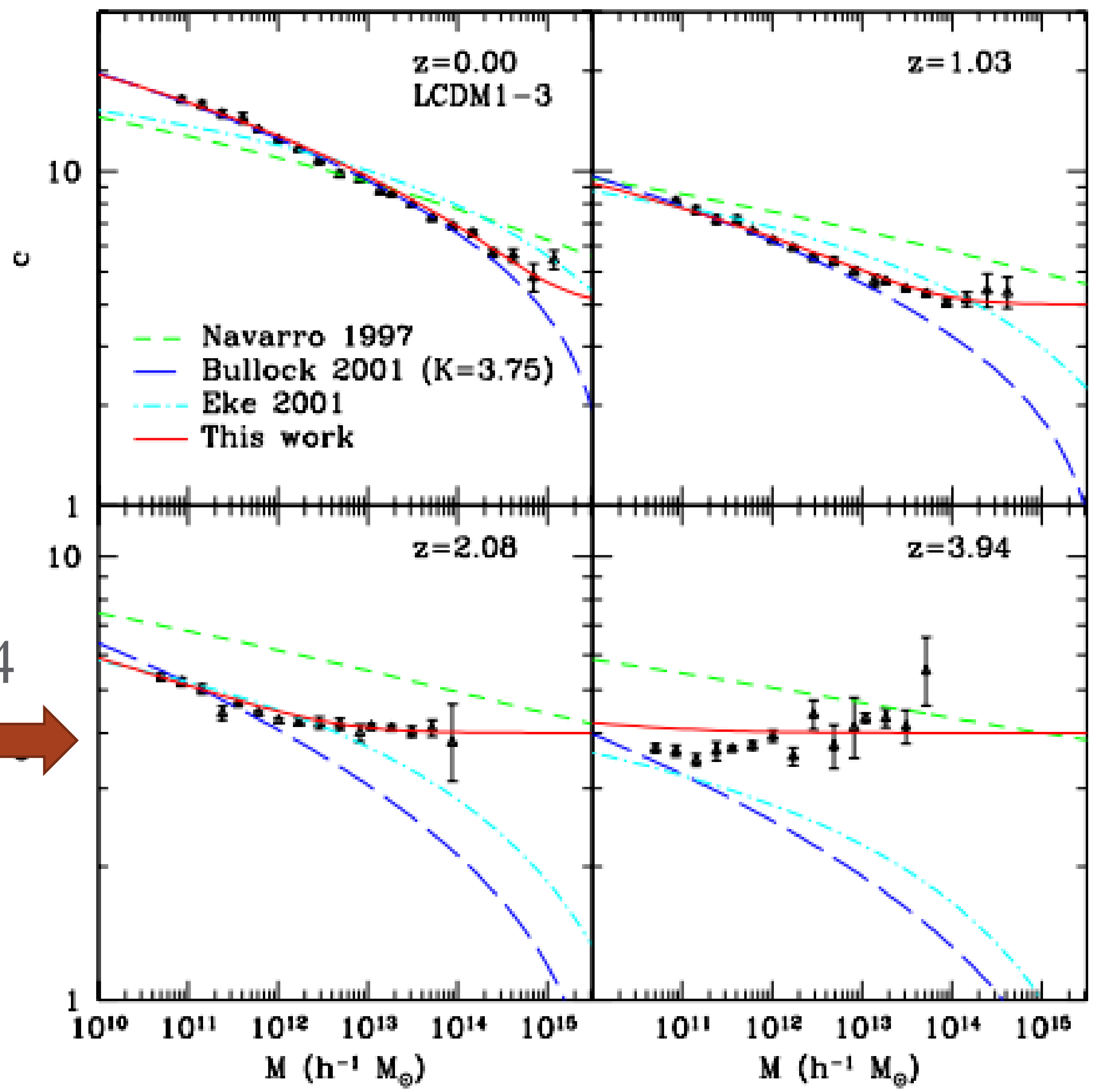
$$L_{hr}(a)=\int_0^{r_h} u_r(r,a)4\pi r^2\rho_h(r,a)dr=\frac{m_h v_{cir}}{2\pi cF(c)}\left(cF(c)-2\int_0^c F(x)dx\right)$$

$$cF(c)=2\int_0^c F(x)dx$$

Limiting concentration for large halos $c=3.5$ for NFW profile

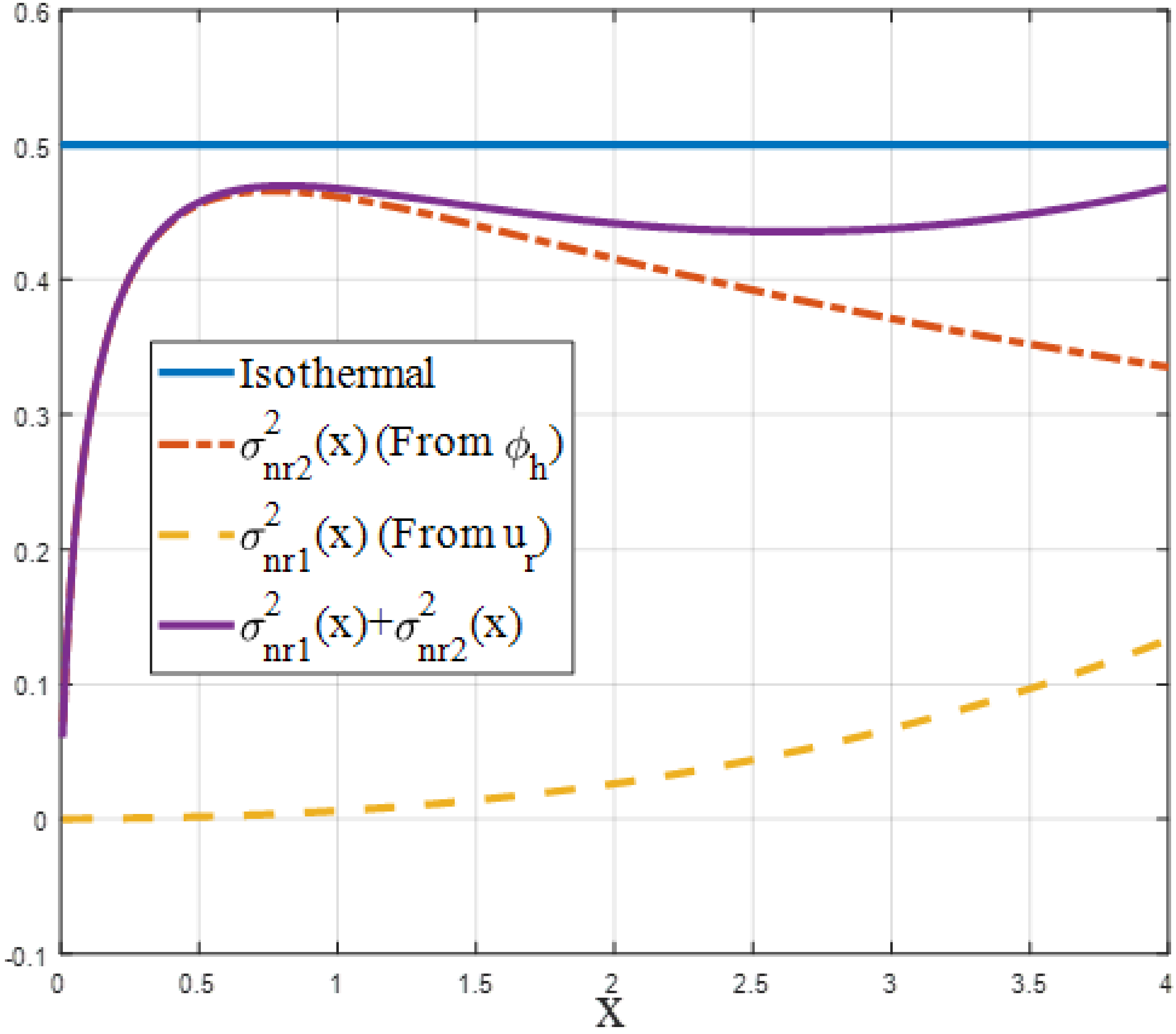


Limiting c~4 from simulation



Zhao etc.., 2009, Astrophys. J., 707, 354

Effect of radial flow on velocity dispersion



- Radial flow usually neglected for virialized halos;
- Effect of radial flow can be significant for halos in their early life before fully virialized (high peak height *v*);
- The radial flow tends to enhance the radial random motion and is only significant in the halo outer region.

Mass cascade induced halo surface energy

Standard virial theorem for static halos with a vanishing radial flow (K_σ is 1D kinetic energy):

$$6K_\sigma - n\Phi_h = 0 \quad \text{Potential exponent} \quad n = -1$$

Jeans' equation for isotropic growing halos with non-zero radial flow:

$$\frac{\partial(\rho_h u_r)}{\partial t} + \frac{1}{r^2}\frac{\partial(\rho_h r^2 u_r^2)}{\partial r} + \frac{\partial(\rho_h \sigma_r^2)}{\partial r} + \rho_h \frac{G m_r(r,a)}{r^2} = 0$$

Integrating Jeans' Equation leads to a generalized virial theorem for growing halos with fast mass accretion:

mean flow

$$6K_\sigma + \Phi_h = I_h - 2K_u + S_u + S_\sigma$$

Rewrite to introduce effective exponent n_e:

$$6K_\sigma - n_e\Phi_h = 0 \text{ and } n_e = -1 + \frac{\cancel{I_h - 2K_u} + S_u + S_\sigma}{\Phi}$$

Halo surface energy:

$$S_{eh} = (S_u + S_\sigma), \quad n_e \approx -1 + \frac{S_{eh}}{\Phi} \approx -1.3 \neq -1$$

Halo surface tension:

$$S_{th} = S_{eh}/(2A_h) \quad \text{Surface area: } A_h = 4\pi r_h^2$$

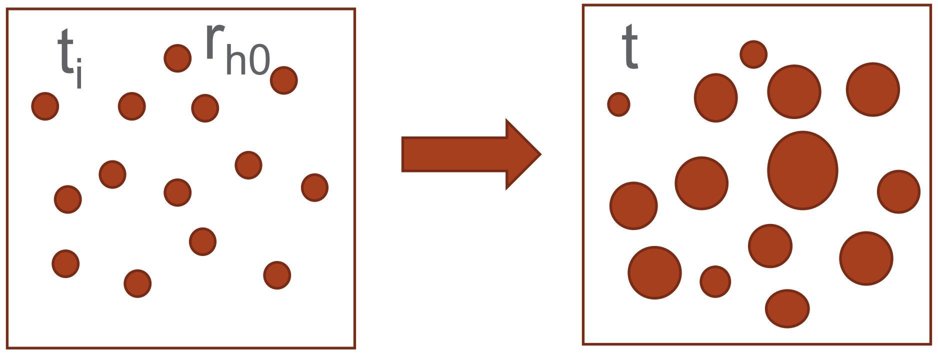
Young–Laplace equation relates the pressure jump across halo surface to halo radius or curvature;

$$\Delta P_h = \frac{2S_{th}}{r_h} = \frac{S_{eh}}{A_h r_h} \approx 0.1\bar{\rho}_h v_{cir}^2$$

$$S_{th} = \alpha_{st} G \rho_{sur}^2 r_h \propto r_h^{-1}$$ ➡ Halo surface mass density: $\rho_{sur} \sim r_h^{-1}$

Mass cascade (fast mass accretion) leads to finite halo surface energy, surface tension, surface mass density, and an effective potential exponent $n_e \sim -1.3$, confirmed by N-body simulation.

Halo size evolution from theory of mass cascade



Solution leads to a **lognormal** probability distribution of halo size:

$$P_{rh}\left(r_h,t\right)=\frac{1}{r_h\sqrt{8\pi D_{rh}\ln\left(t/t_i\right)/3}}\exp\left\{-\frac{\left(\ln\left(r_h/r_{h0}\right)-\left(1-2D_{rh}/3\right)\ln\left(t/t_i\right)\right)^2}{8D_{rh}\ln\left(t/t_i\right)/3}\right\}$$

1D Random walk of halos in mass space:

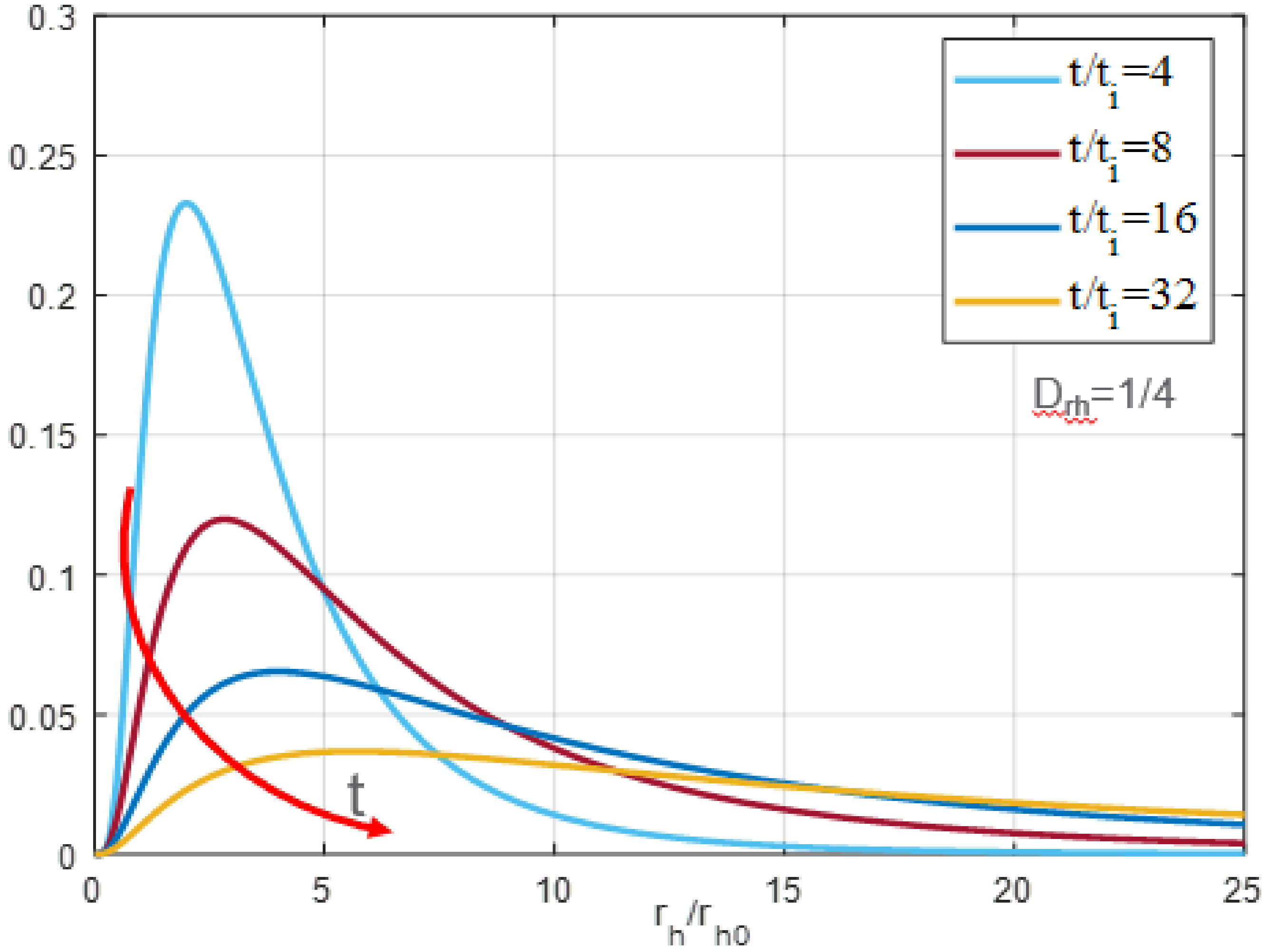
$$\frac{\partial m_h\left(t\right)}{\partial t}=\frac{m_p\xi\left(t\right)}{\tau_g\left(m_h\right)}=\sqrt{2D_p\left(m_h\right)}\varsigma\left(t\right)$$

1D Random walk of halos in size space (**Geometric Brownian** motion):

$$\frac{dr_h\left(t\right)}{dt}=\frac{3}{2}Hr_h\left(t\right)+Hr_h\left(t\right)\xi_{rh}\left(t\right)$$

Covariance:

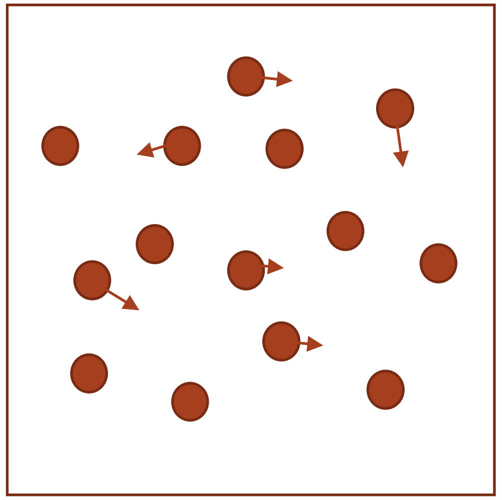
$$\left\langle\xi_{rh}\left(t\right)\xi_{rh}\left(t'\right)\right\rangle=2D_{rh}\,\delta\left(t-t'\right)/H$$



Particle distribution in halos: a review of Brownian motion

Quick review of standard Brownian motion in viscous liquid:

A spherical particle of radius a_B moving at a constant velocity u_h in a fluid of viscosity η_B subject to a force F_B. Local steady-state velocity u_h can be determined by the driving force F_B, i.e., the gradient of the osmotic pressure $\Pi_B = \rho_B k_B T$, which is a localized short-range force.



Current velocity from stokes law:

$$u_h = \frac{F_B}{6\pi\eta_B a_B} = -\frac{1}{6\pi\eta_B a_B}\cdot\frac{1}{\rho_B}\frac{\partial \Pi_B}{\partial x} = -\frac{\mu_B}{\rho_B}\frac{\partial(\rho_B k_B T)}{\partial x}$$

Osmotic velocity from diffusion flux:

$$u_h^* = D_B \frac{\partial \ln \rho_B}{\partial x}$$

A simple closure:

$$u_h = -u_h^*$$

The Einstein relation:

$$D_B = \mu_B k_B T$$

Stochastic equations for Brownian motion (forward and backward):

$$\frac{dr_t}{dt} = \left[u_h(x_t) + u_h^*(x_t)\right] + \sqrt{2D_B}\,\xi(t)$$

$$\frac{dr_t}{dt} = \left[u_h(x_t) - u_h^*(x_t)\right] + \sqrt{2D_B}\,\xi^*(t)$$

Fokker-Planck equations (forward and backward):

$$\frac{\partial P_r(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\left(u_h(x) + u_h^*(x)\right)P_r\right] + D_B\frac{\partial^2 P_r}{\partial x^2}$$

$$\frac{\partial P_r(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[\left(u_h(x) - u_h^*(x)\right)P_r\right] - D_B\frac{\partial^2 P_r}{\partial x^2}$$

Diffusion equation for density distribution:

$$\frac{\partial P_r(x,t)}{\partial t} = D_B\frac{\partial^2 P_r}{\partial x^2}$$

$$u_h^* = -u_h = D_B\frac{\partial \ln P_r}{\partial x}$$

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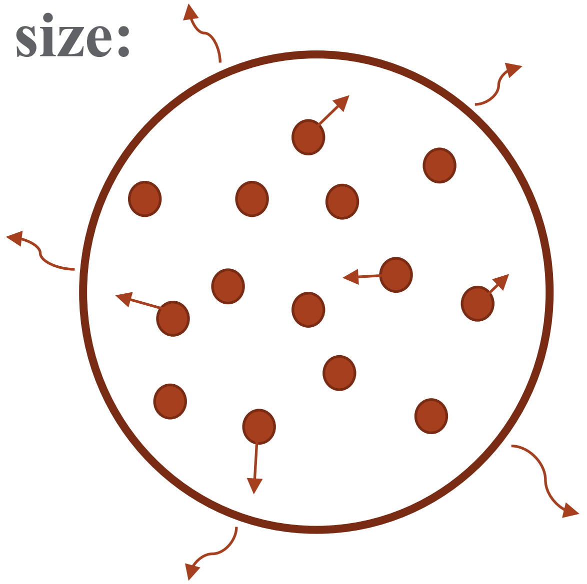
Particle distribution in halos: formulation

Brownian motion of particle in halos with stochastically (lognormal) growing size:



Stochastic equations for Brownian motion (forward and backward):

$$\frac{dr_t}{dt} = \underbrace{\frac{r_s(t)}{t}\left[u_h(x_t)+u_h^*(x_t)\right]}_{1} + \underbrace{\sigma(x_t)\, r_s(t)\, H\xi_{rh}(t)}_{2}$$

$$\frac{dr_t}{dt} = \frac{r_s(t)}{t}\left[u_h(x_t)-u_h^*(x_t)\right] + \sigma(x_t)\, r_s(t)\, H\xi_{rh}^*(t)$$

Multiplicative noise (dependent on r_t itself) due to random varying halo size!!

Radial flow

Osmotic flow

- Due to long-range interaction, $u_h \neq -u_h^*$
- Key is to find a simple closure to close equation! (an example in ref.)

Fokker-Planck equations (forward and backward):

$$\frac{\partial P_r(r,t)}{\partial t} = -\frac{r_s(t)}{t}\frac{\partial}{\partial r}\left[\left(u_h(x)+u_h^*(x)\right)P_r\right] + r_s^2(t)\, HD_{rh}\frac{\partial^2}{\partial r^2}\left(\sigma^2(x)P_r\right)$$

$$\frac{\partial P_r(r,t)}{\partial t} = -\frac{r_s(t)}{t}\frac{\partial}{\partial r}\left[\left(u_h(x)-u_h^*(x)\right)P_r\right] - r_s^2(t)\, HD_{rh}\frac{\partial^2}{\partial r^2}\left(\sigma^2(x)P_r\right)$$

$$x\frac{\partial P_r}{\partial x} = \frac{\partial}{\partial x}\left[u_h(x)P_r\right]$$

$$u_h^*(x) = d_r\sigma^2(x)\frac{\partial}{\partial x}\ln\left[\sigma^2(x)P_r(x)\right]$$

Exact relation between current and osmotic velocities:

$$u_h^*(x) = \frac{d_r\sigma^2(x)}{x-u_h(x)}\frac{\partial u_h}{\partial x} + d_r\frac{\partial \sigma^2(x)}{\partial x}$$

With $\sigma(x_t) \sim x_t$ expected

Particle distribution in halos: halo density profile

To derive halo density, adopting a simple model of osmotic velocity :

$$\boxed{u_h^*(x) = \gamma_r x - \beta_r x^{1+\alpha_r}}$$

Two-parameter particle distribution function:

$$P_r(x) = \frac{b_r^{\ a_r}}{\Gamma(a_r)(a_r - b_r)} \exp\left(-b_r x^{\frac{1}{a_r - b_r}}\right) x^{\frac{b_r}{a_r - b_r}}$$

Three-parameter halo density profile:

$$\rho_h(x) = \frac{m_h P_r(x)}{4\pi r_s^3 x^2} = \rho_s e^{b_r} \exp\left(-b_r x^{\frac{1}{a_r - b_r}}\right) x^{\frac{3b_r - 2a_r}{a_r - b_r}}$$

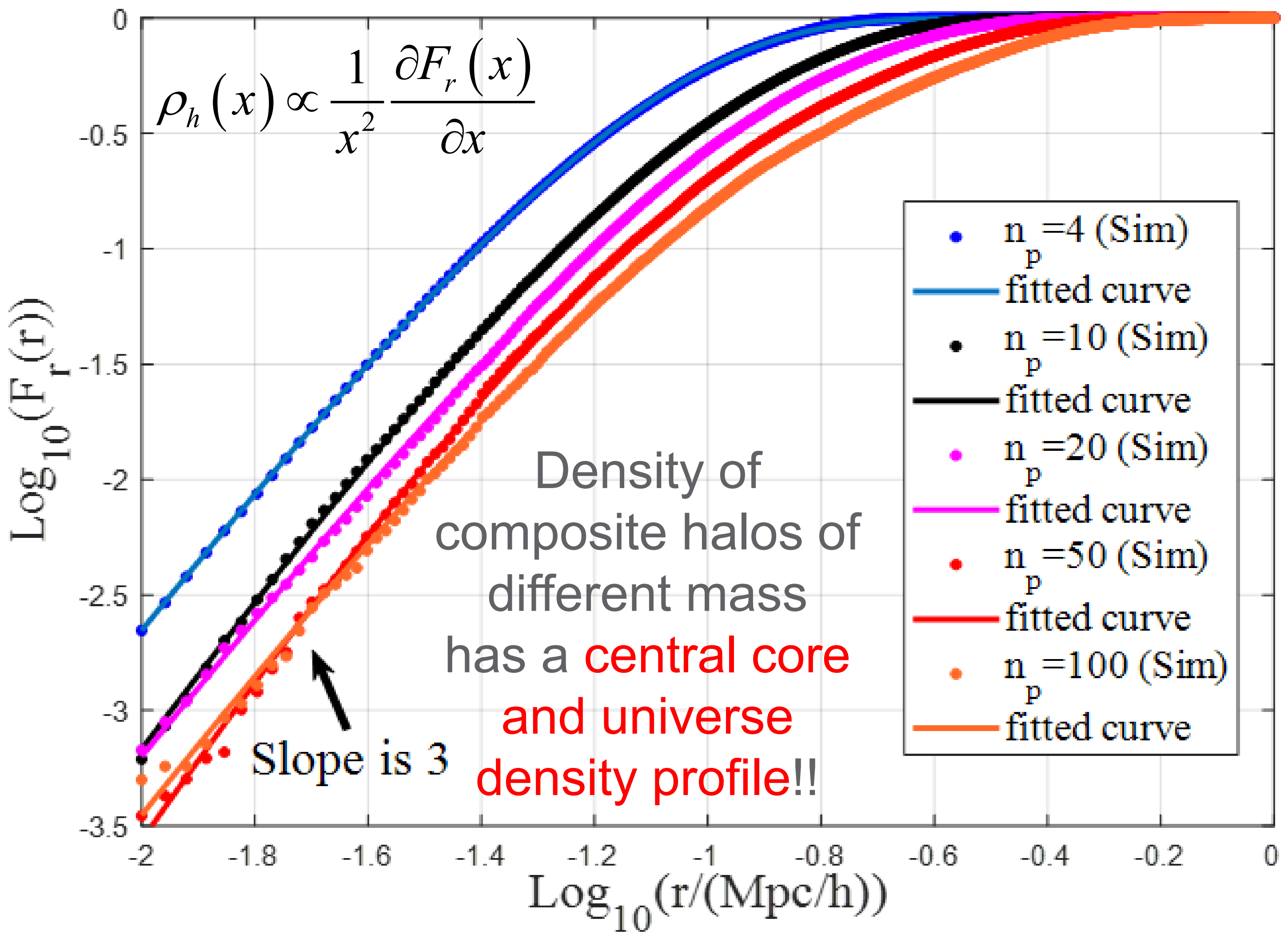
$$a_r / b_r = 3/2$$

Two-parameter Einasto:

$$\rho_h(r) = \rho_s e^{2/\alpha} \exp\left[-\frac{2}{\alpha} x^{\alpha}\right] = \rho_s e^{2/\alpha} \exp\left[-\frac{2}{\alpha}\left(\frac{r}{r_s}\right)^{\alpha}\right]$$

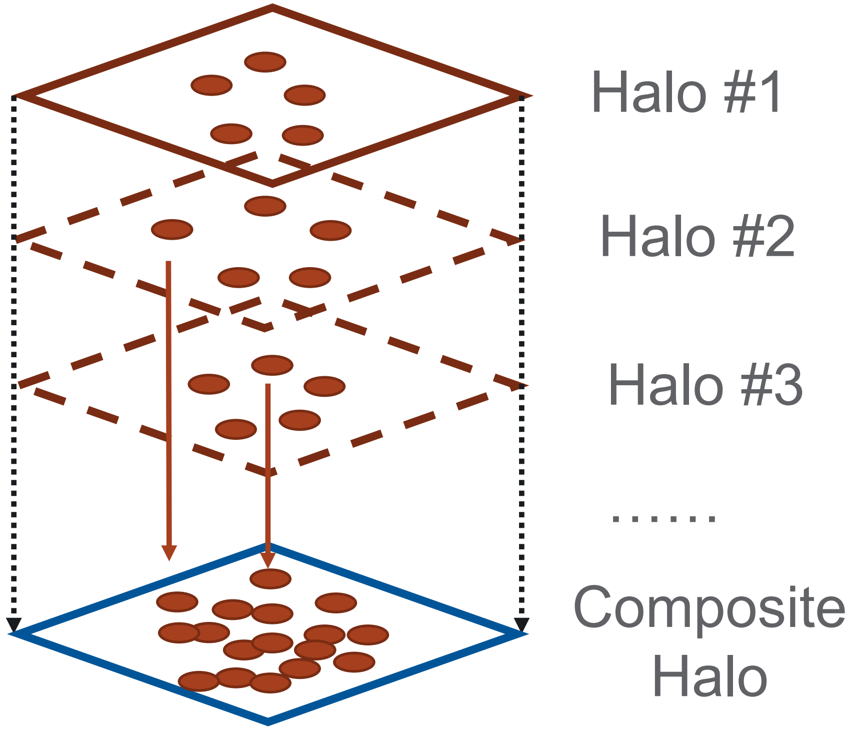
Two-parameter cumulative distribution function:

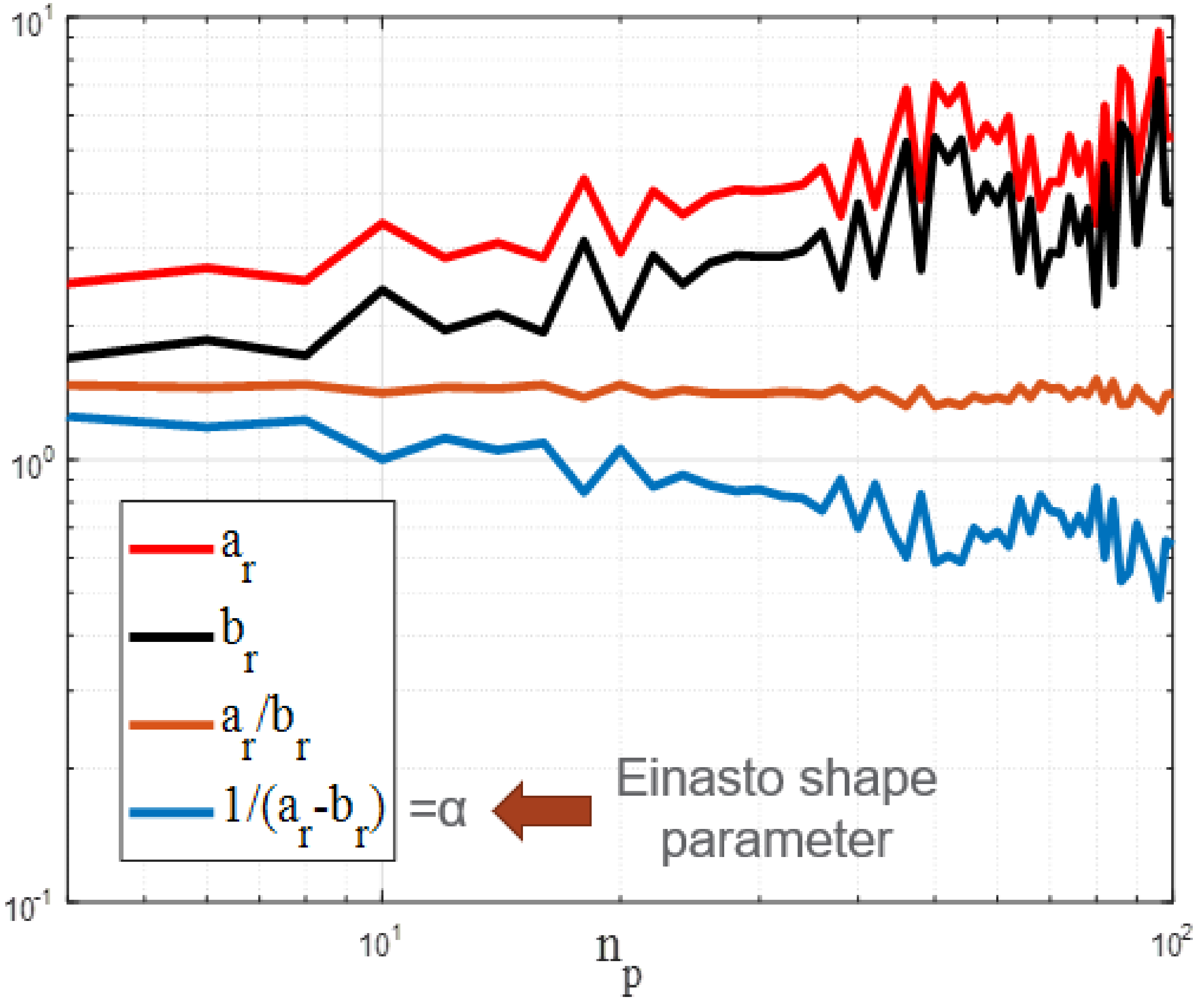
$$F_r\left(x = \frac{r}{r_s}\right) = \int_0^x P_r(y)\,dy = \frac{m_r}{m_h} = 1 - \frac{\Gamma\left(a_r, b_r x^{1/(a_r - b_r)}\right)}{\Gamma(a_r)} = \frac{\gamma\left(a_r, b_r x^{1/(a_r - b_r)}\right)}{\Gamma(a_r)}$$



Particle distribution in halos and halo density profile

Constructing composite halo for a halo group including all halos of the same mass:





- Composite halo reflects complete statistics of particle distribution resulting from particle random-walk in dynamic halos;
- All composite halos have a central core (no cusp)
- The density profile of composition halo (α=[1.2 0.7]) can be different from individual halo (α≈0.2);

- Fitted a_r/b_r=3/2 for all size of halo groups (implies an Einasto profile)

Equation of state for relative pressure and density

$$\rho_h(r) = \rho_s e^{2/\alpha} \exp\left(-b_r x^{\frac{2}{b_r}}\right) \quad \text{with} \quad b_r = 2/\alpha$$

For small x (halo center)

$$\rho_h(x) \approx \rho_h(0)\left(1 - b_r x^{\frac{2}{b_r}}\right)$$

Parabolic pressure at halo center:

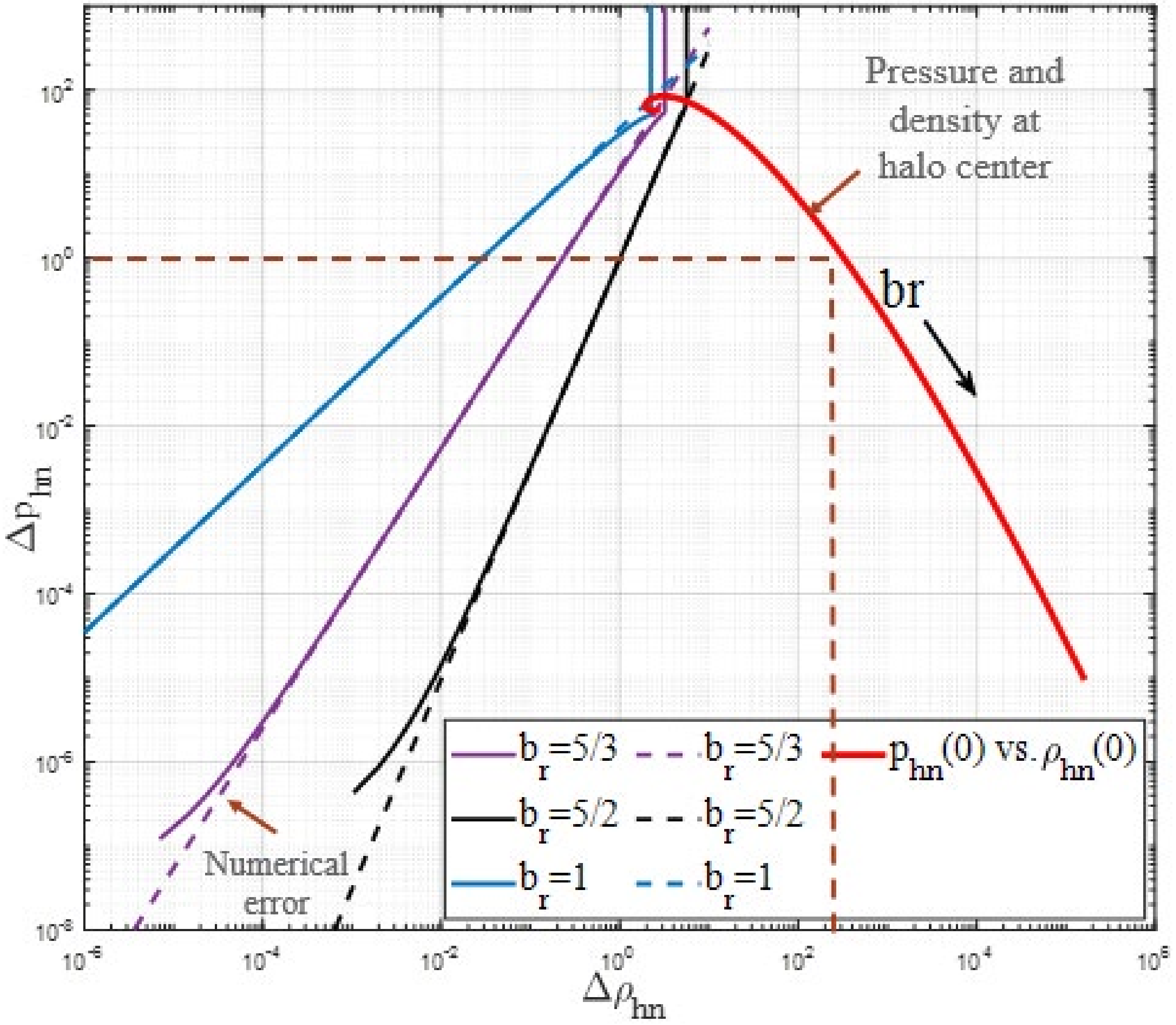
$$p_h(x) = \rho_h(x)\sigma_r^2(x) = p_h(x=0) - \frac{1}{2}\frac{\rho_h^2(0)v_{cir}^2}{\bar{\rho}_h c^2}x^2$$

Cancel x in both Equations:

$$\left[p_h(0) - p_h(x)\right] = \frac{\left[\rho_h(0)\right]^{2-b_r} v_{cir}^2}{2(b_r)^{b_r}\bar{\rho}_h c^2}\left[\rho_h(0) - \rho_h(x)\right]^{b_r}$$

Equation of state (EoS) for relative pressure and relative density (relative to the center of halo):

$$\Delta p_h = K_s(\Delta\rho_h)^{b_r}$$



- EoS is good for entire range of relative P and ρ
- Why? might because of halo grows from center

Summary and key words

Radial flow & scale radius	Halo surface energy/tension	Current velocity	Mean flow& random motion
Deformation parameter α_h	Deformation rate parameter γ_h	Osmotic velocity	Limiting concentration
Angle of incidence	Random walk	Fokker-Planck	Equation of state

- Mass cascade induced nonzero radial flow (outwards and inwards).
- Self-similar solution to relate halo density profile with radial flow.
- Radial flow leads to an extra length scale (the scale radius r_s).
- Limiting halo concentration c=3.5 for fast growing halos at their early stage, with a Hubble flow at halo center leading to a central core.
- Composite halos from N-body simulation always have a central core.
- Radial flow enhances velocity dispersion in outer region.
- Radial flow leads to a nonzero halo surface energy/tension.
- Random walk of halo size is a geometric Brownian process with log-normal distribution
- Random walk of particles in halo with varying size leads to analytical particle probability distribution (i.e. the halo density profile).
- Equation of state for relative pressure and relative density (relative to halo center)