Pacific Northwest
NATIONAL LABORATORY

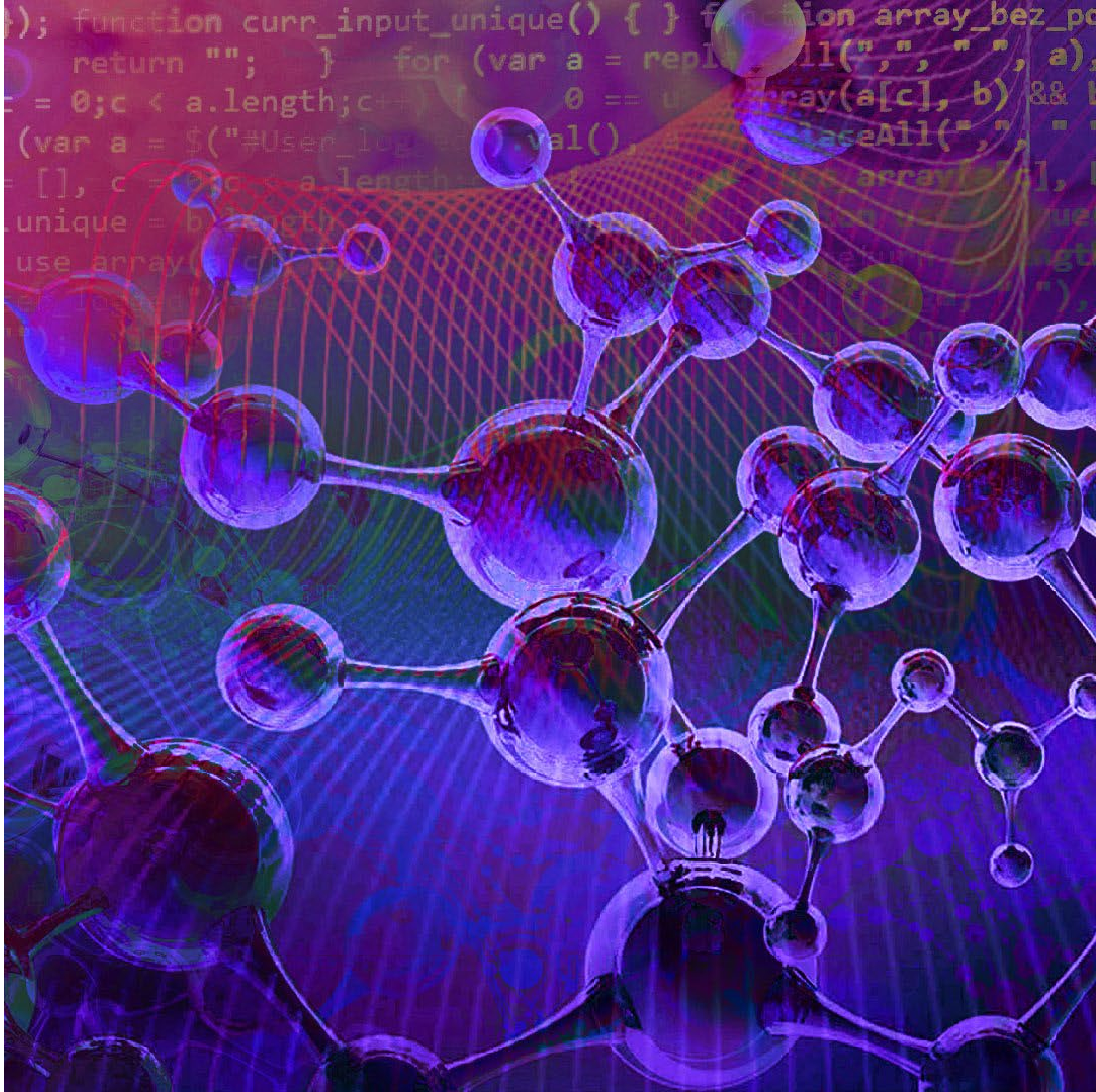
# A comparative study of dark matter flow & hydrodynamic turbulence and its applications

May 2022

**Zhijie (Jay) Xu**
Multiscale Modeling Team
Computational Mathematics Group
Physical & Computational Science Directorate
Zhijie.xu@pnnl.gov; zhijiexu@hotmail.com

U.S. DEPARTMENT OF ENERGY BATTELLE

PNNL is operated by Battelle for the U.S. Department of Energy



# Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!

# Data repository and relevant publications

Structural (halo-based) approach:

| 0. | Data https://dx.doi.org/10.5281/zenodo.6541230 |
|---|---|
| 1. | Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985 |
| 2. | Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244 |
| 3. | Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885 |
| 4. | The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665 |
| 5. | Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784 |
| 6. | Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054 |
| 7. | The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126 |
| 8. | Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676 |

Statistics (correlation-based) approach:

| 0. | Data https://dx.doi.org/10.5281/zenodo.6569898 |
|---|---|
| 1. | The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910 |
| 2. | The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991 |
| 3. | The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515 |
| 4. | Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240 |
| 5. | The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606 |
| 6. | The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899 |

# Structural (halo-based) approach for dark matter flow

# Inverse mass cascade in dark matter flow and effects on halo mass functions

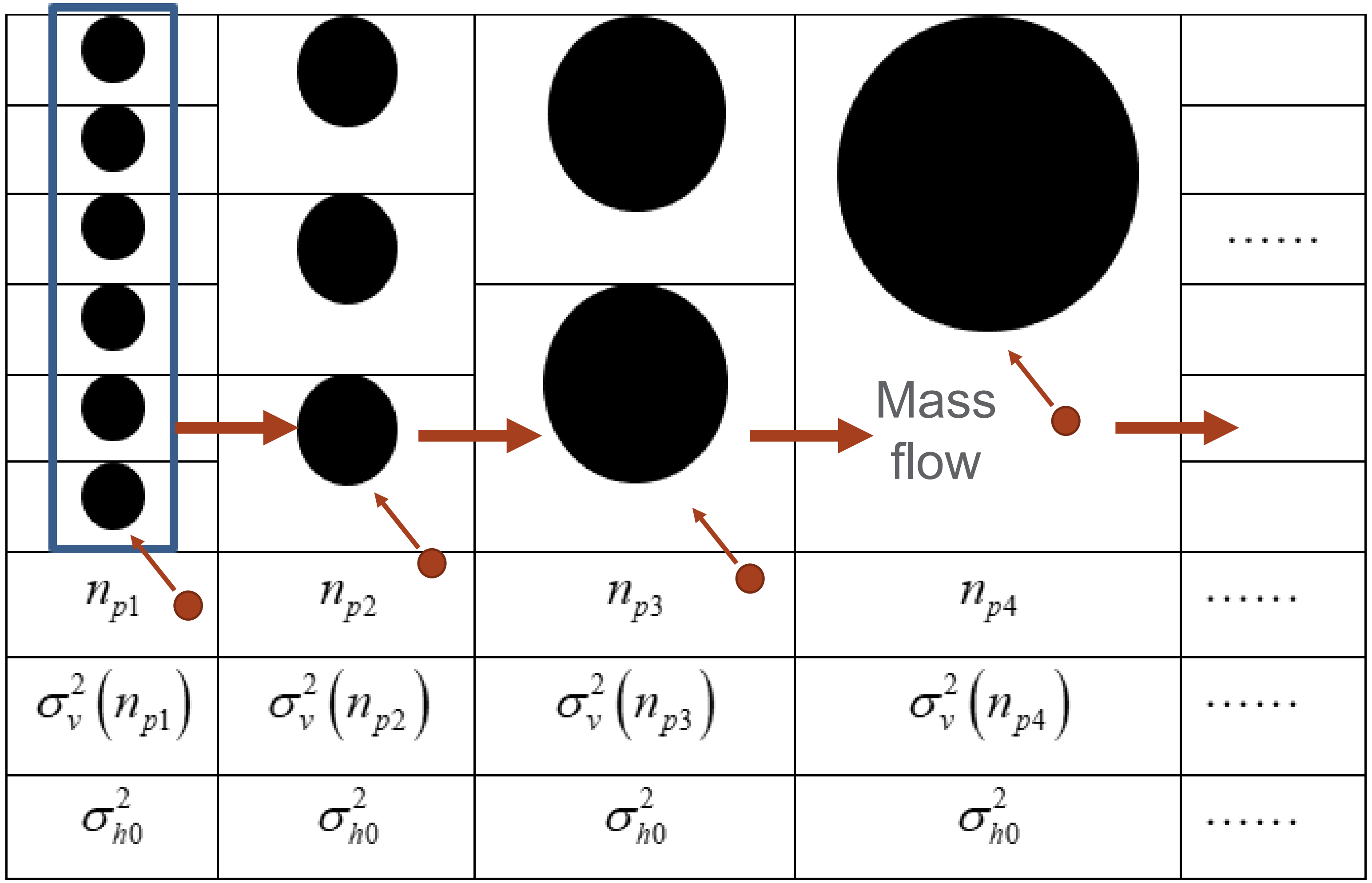
Xu Z., 2021, arXiv:2109.12244v1 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2109.12244

# Introduction

Review: In hydrodynamic turbulence, "energy cascade" involves the energy transfer from large eddies to small eddies with a scale-independent rate of energy cascade.

The dark matter flow, a self-gravitating collisionless flow, involves a continuous mass transfer from small to large mass scales with a scale-independent rate of mass cascade $\varepsilon_m$.

- Goal 1: Identify and formulate mass cascade
- Goal 2: Explore the random walk of halos in mass space
- Goal 3: Derive the halo mass function based on the theory of mass cascade

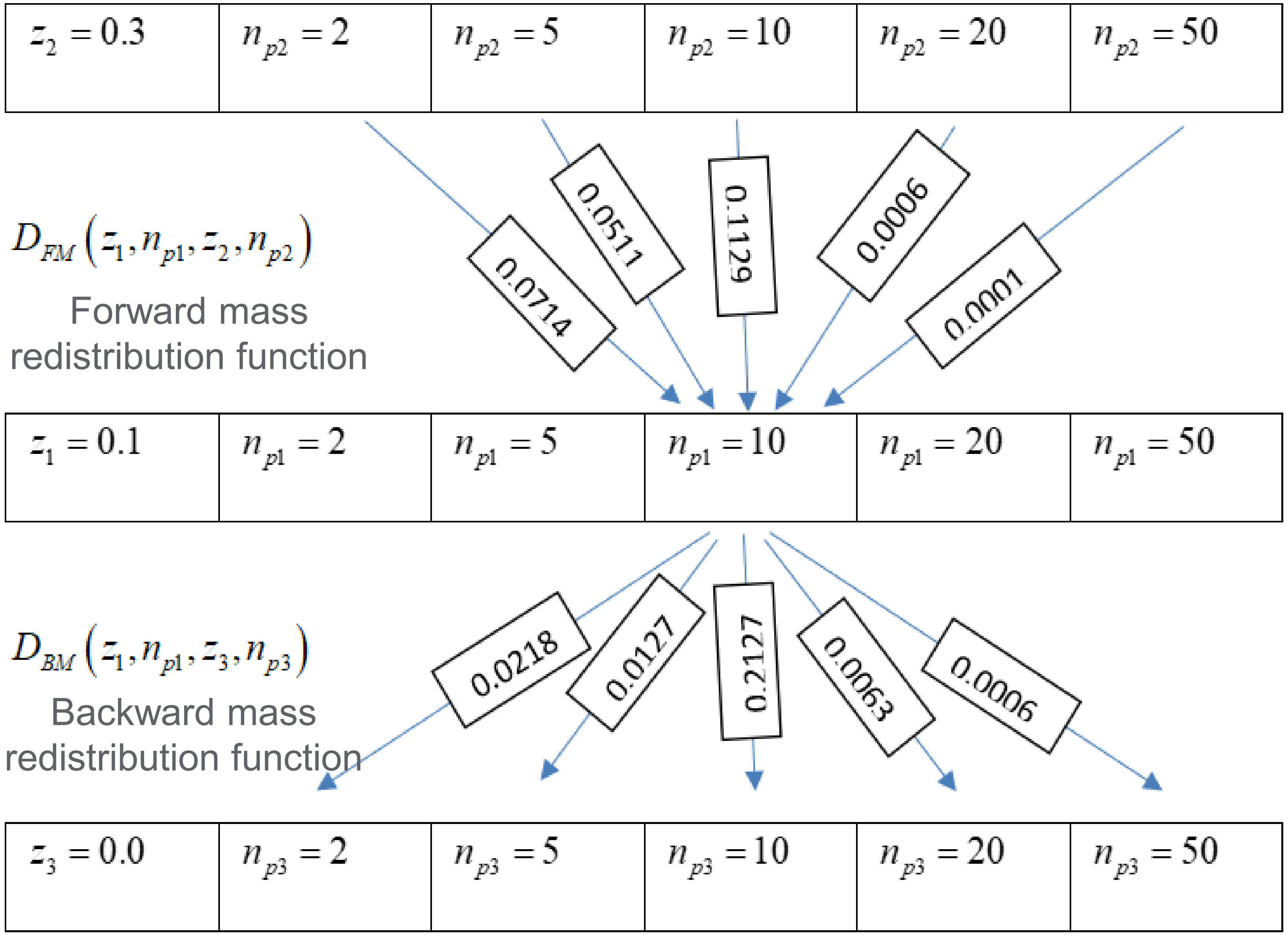



- Identify all halos of different sizes
- Group halos according to the halo size $n_p$
- Mass flow across halo groups from small to large mass scale (inverse) through the merging with "single merger"
- Cascade leads to random-walk of halos in mass space

# Mass redistribution among halo groups

Backward function: fraction of mass inherited from all other halo groups at an earlier time

Forward function: fraction of mass passed to all other halo groups at a later time



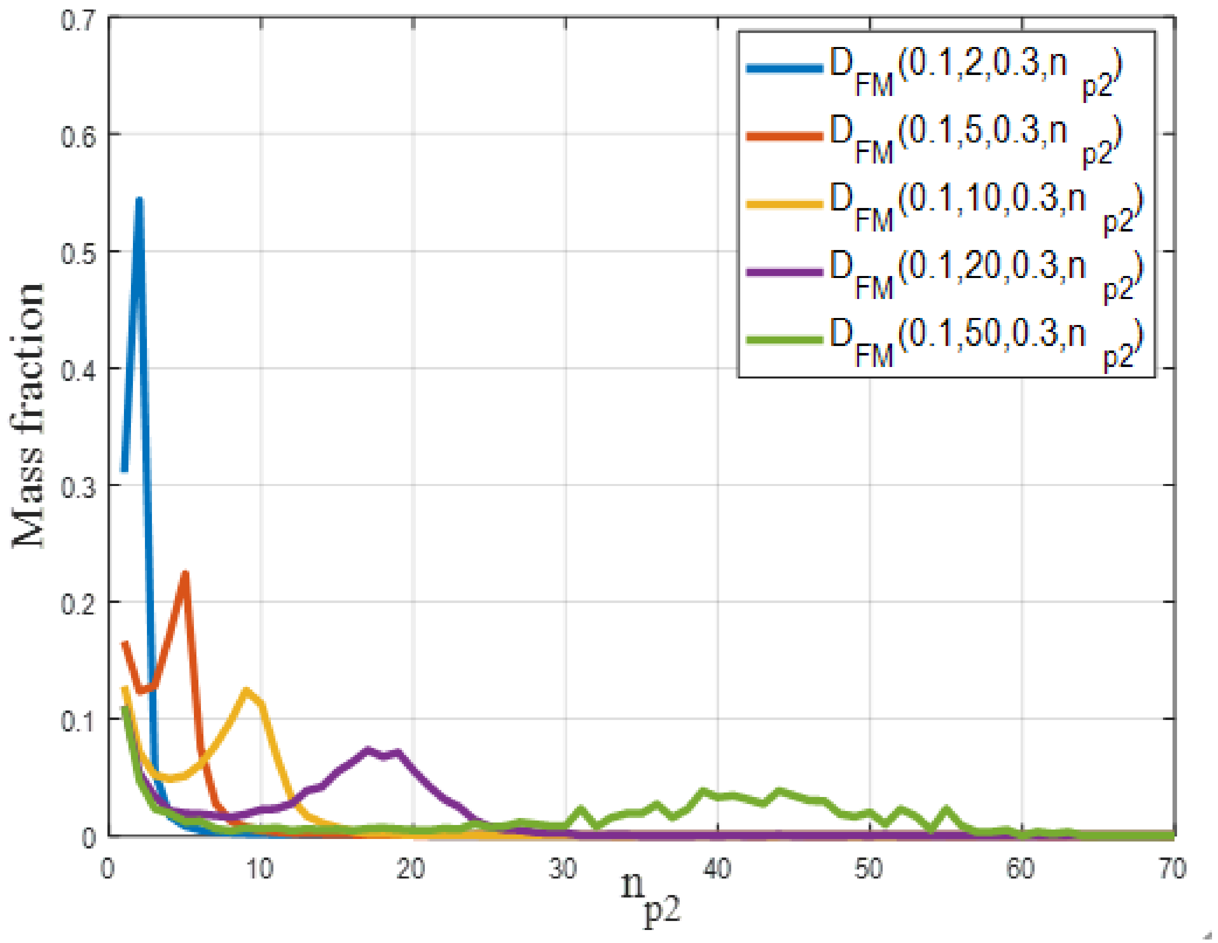

Minus sign

Backward mass redistribution function

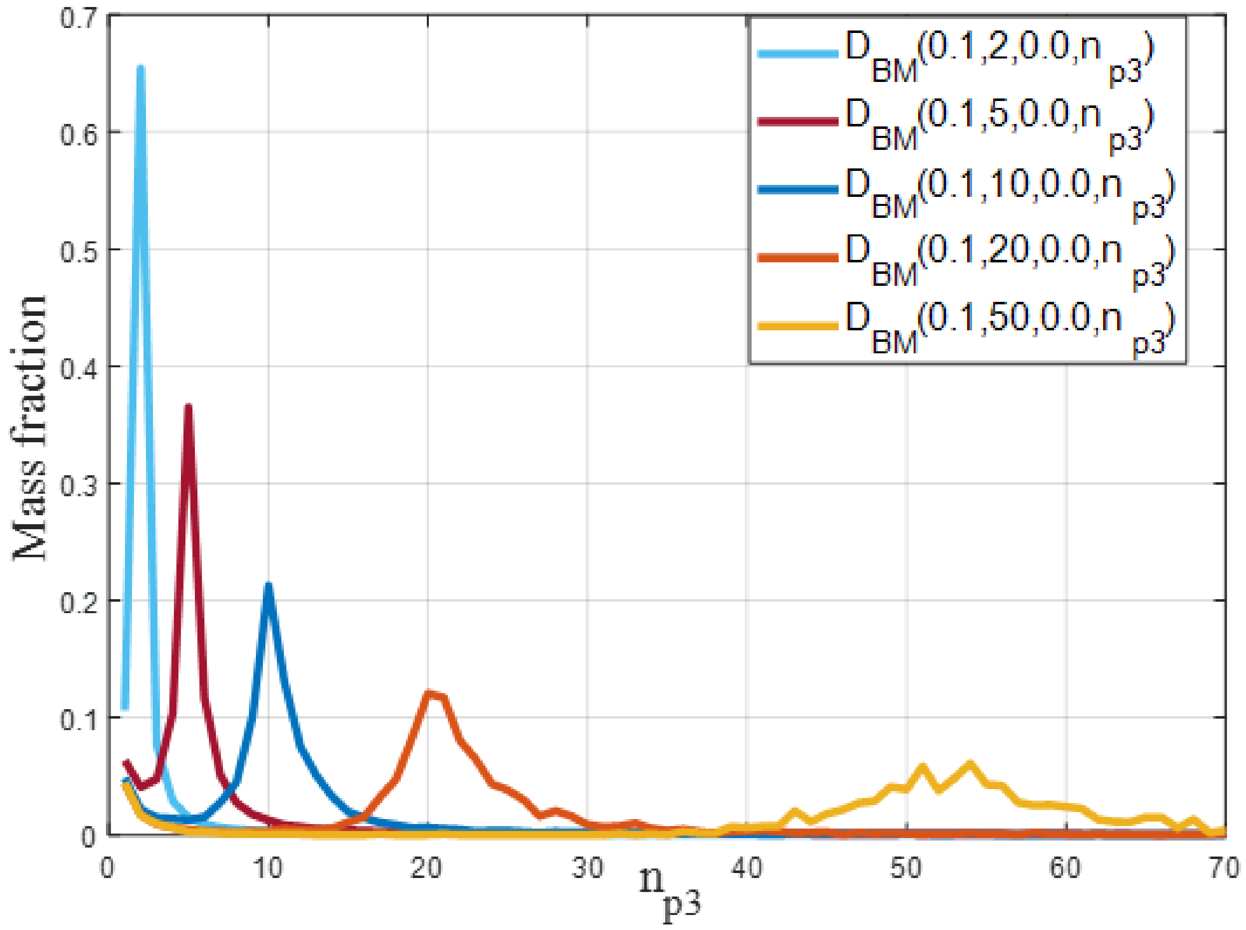
-- Forward mass redistribution function

= Net mass redistribution function

# Properties/features of mass cascade




Forward mass redistribution function



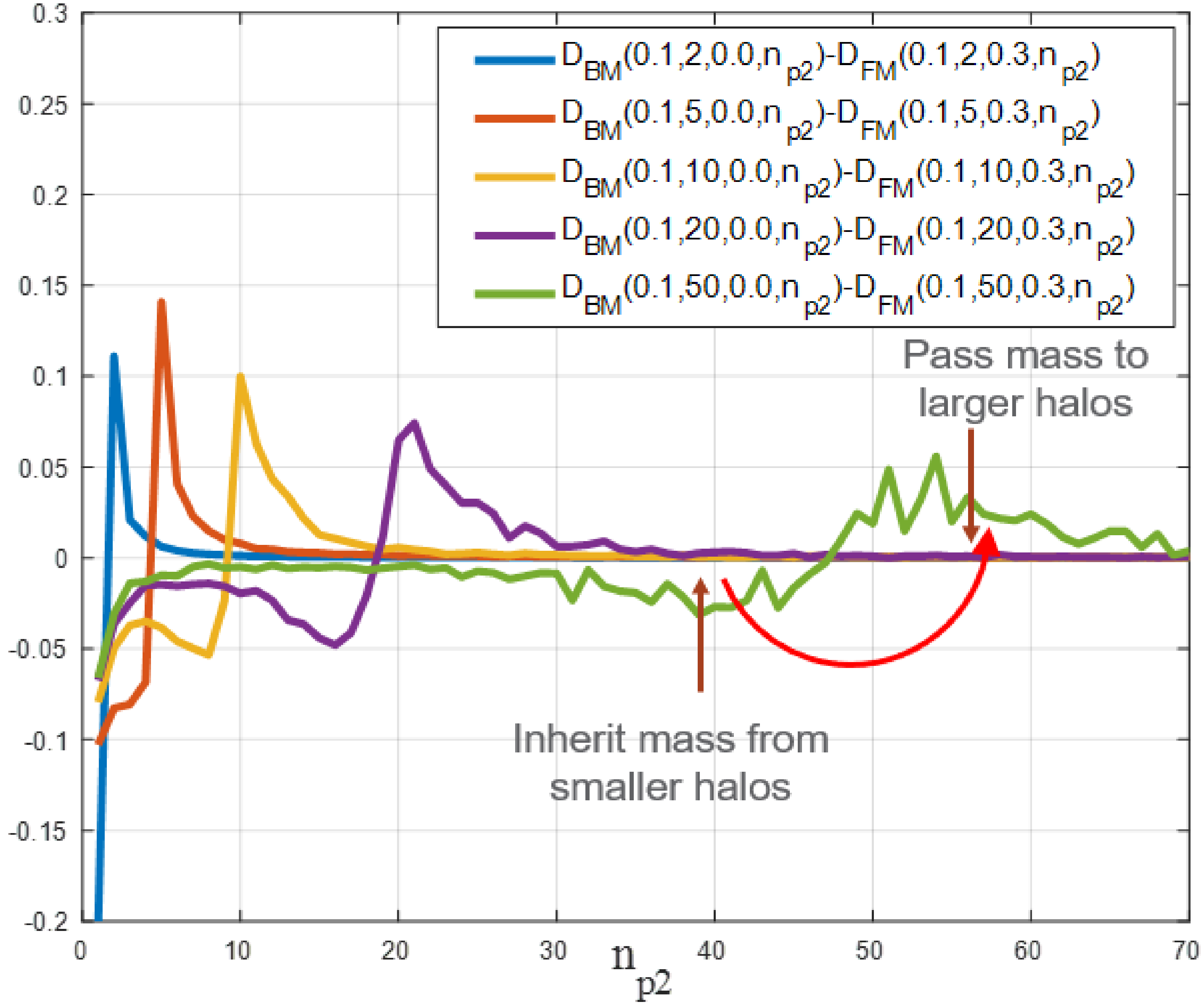

Backward mass redistribution function

- *Local: cascade is local in mass space*

*Halos inherit/pass their mass mostly from/to halos of the same or similar size.*

(energy cascade in turbulence is also local in wavenumber space)

# Properties of mass cascade




Net mass redistribution function

Net mass redistribution function $D_{NM}$:
<0: inherit more mass than pass mass
>0: pass more mass than inherit mass
Sum of $D_{NM}$ =0

Net effect: halos transfers mass from below to above.

- *Asymmetric: cascade is two-way in mass space but not symmetric*
- *Inverse: from small to large mass scales*

(energy cascade in turbulence is a direct cascade from large to small scales)

# Time and mass scales in inverse mass cascade

Average waiting time of a merging event with a single merger in a given halo group of halo mass $m_h$

$$\tau_h\left(m_h,a\right)$$

The rate at which mass is passed up from this group:

$$\varepsilon_m \sim -m_h/\tau_h$$

Average waiting time (halo lifespan) of a merging event for a given halo in halo group with $n_h$ halos of mass $m_h$

$$\tau_g\left(m_h,a\right)=n_h\tau_h=-\frac{m_h n_h}{\varepsilon_m}=-\frac{m_g}{\varepsilon_m}$$

Average time required to form halo of mass $m_h$ via a sequence of merging events ($n_p$ times):

$$\tau_f\left(m_h,a\right)=\tau_g n_p=\tau_g m_h/m_p$$

Time required to cascade entire mass $M_h$ in all halos:

$$\tau_M\left(a\right)=-M_h\left(a\right)/\varepsilon_m\left(a\right)\sim t$$

Time required to form halo of a characteristic mass $m_h^*$ should be on the order of the current physical time t:

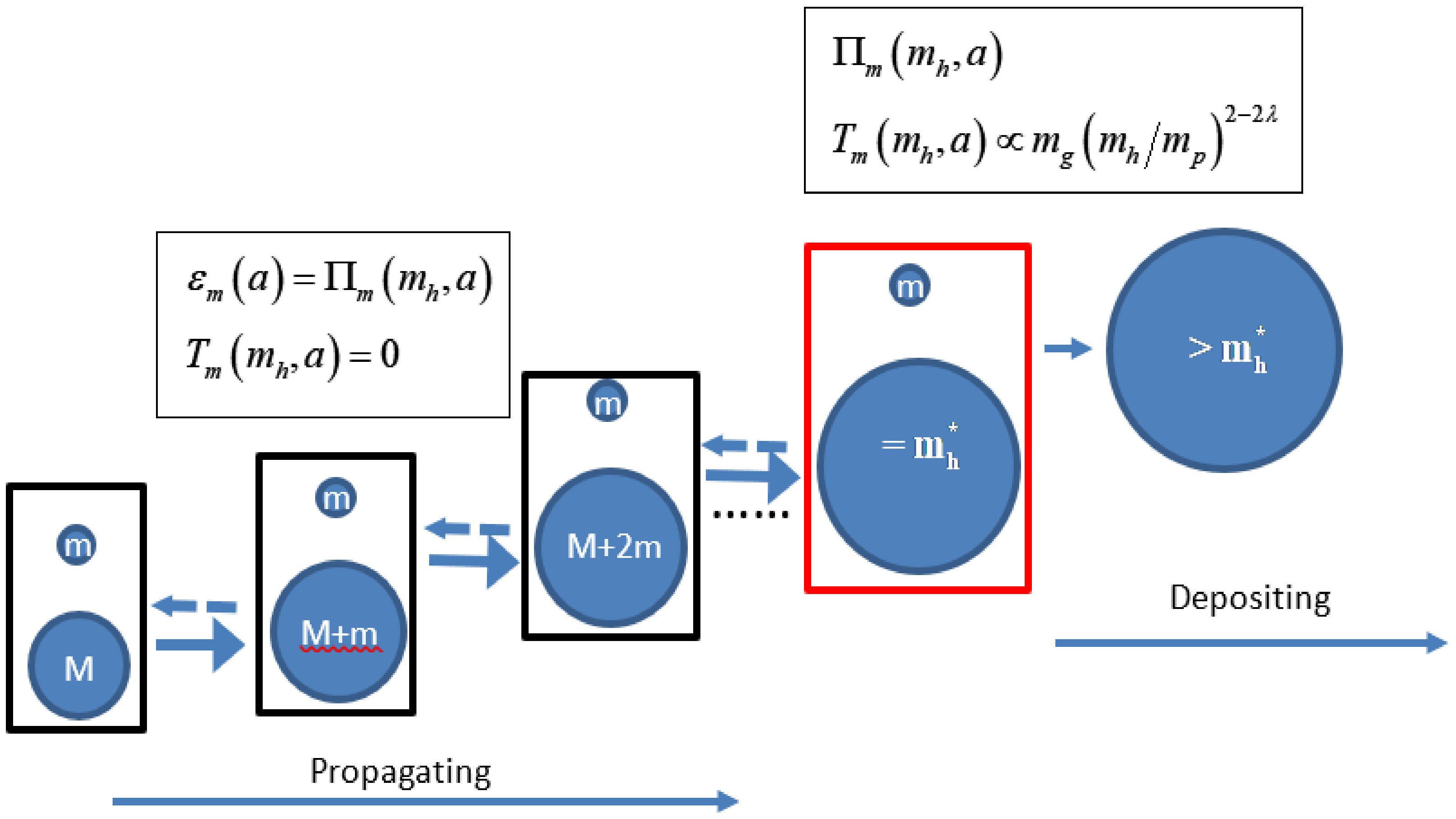
$$\tau_f\left(m_h^*,a\right)\sim t$$

$$\boxed{\tau_M\left(a\right)\geq\tau_f\left(m_h,a\right)\geq\tau_g\left(m_h,a\right)\geq\tau_h\left(m_h,a\right)}$$

$$m_h^* \sim \frac{M_h\left(a\right)m_p}{n_h^* m_h^*}\sim-\frac{\varepsilon_m\left(a\right)}{H n_h^* n_p^*}$$

# Chain reaction description of mass cascade

*"Little halos have big halos, That feed on their mass; And big halos have greater halos, And so on to growth"*

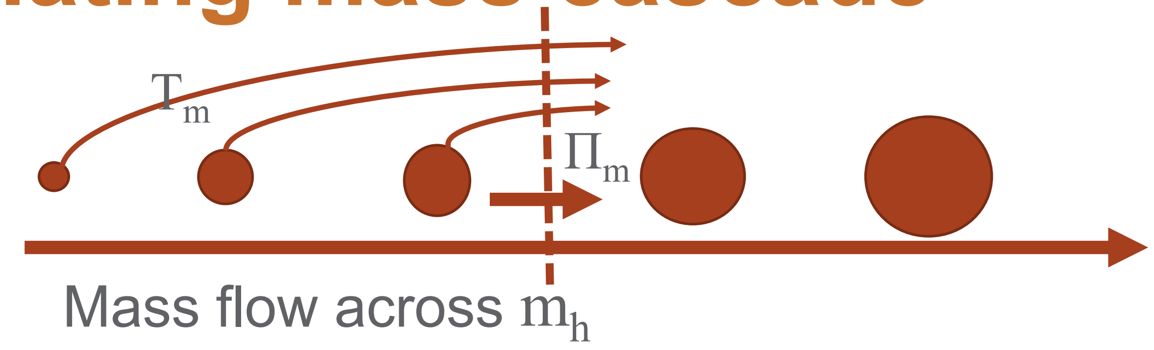



Chain reactions provide non-equilibrium systems a potential mechanism to continuously release energy and increase the system entropy.

- Mass cascade is Local, Asymmetric, Inverse;
- Justifies a chain reaction description of mass cascade;
- The initial stage: initiation/generation of the chain carriers (free radicals)
- The propagation stage: a sequence of accretion of single mergers to propagate the mass along the reaction chain
- The termination stage: the deposition of the mass cascaded from the scales below to grow halos

# Formulating mass cascade

Mass flux function (kg/s): total mass flux from all halos below $m_h$




$$\Pi_m(m_h,a) = -\frac{\partial}{\partial t}\left[M_h(a)\int_{m_h}^{\infty} f_M(m,m_h^*)\,dm\right]$$

Mass flux function: $\Pi_m(m_h,a)$

Total mass of all halos: $M_h(a)$

Halo mass function: $f_M(m_h,a)$

Halo group mass: $m_g(m_h,a)$

Mass transfer function (1/s): rate of mass transfer for halos of mass $m_h$

$$T_m(m_h,a) = \frac{\partial \Pi_m(m_h,a)}{\partial m_h} = \frac{\partial\left[M_h(a) f_M(m_h,m_h^*)\right]}{\partial t} = \frac{\partial m_g(m_h,a)}{m_p \partial t}$$

Halo mass: $m_h$

Particle mass: $m_p$

In mass propagation range: $m_h \ll m_h^*$

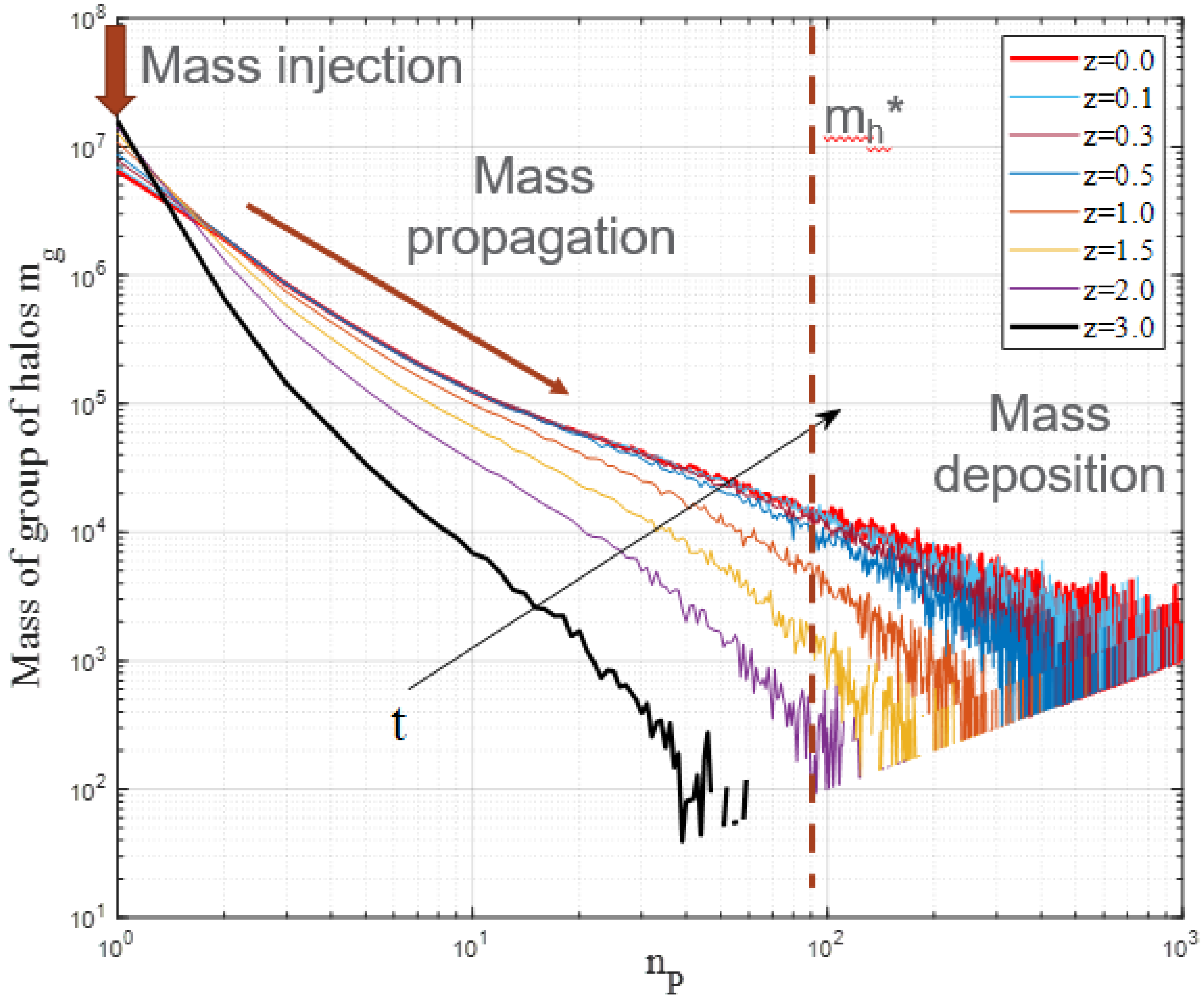
$$\varepsilon_m(a) = \Pi_m(m_h,a)$$

- Rate of mass cascade is Mass-scale independent;

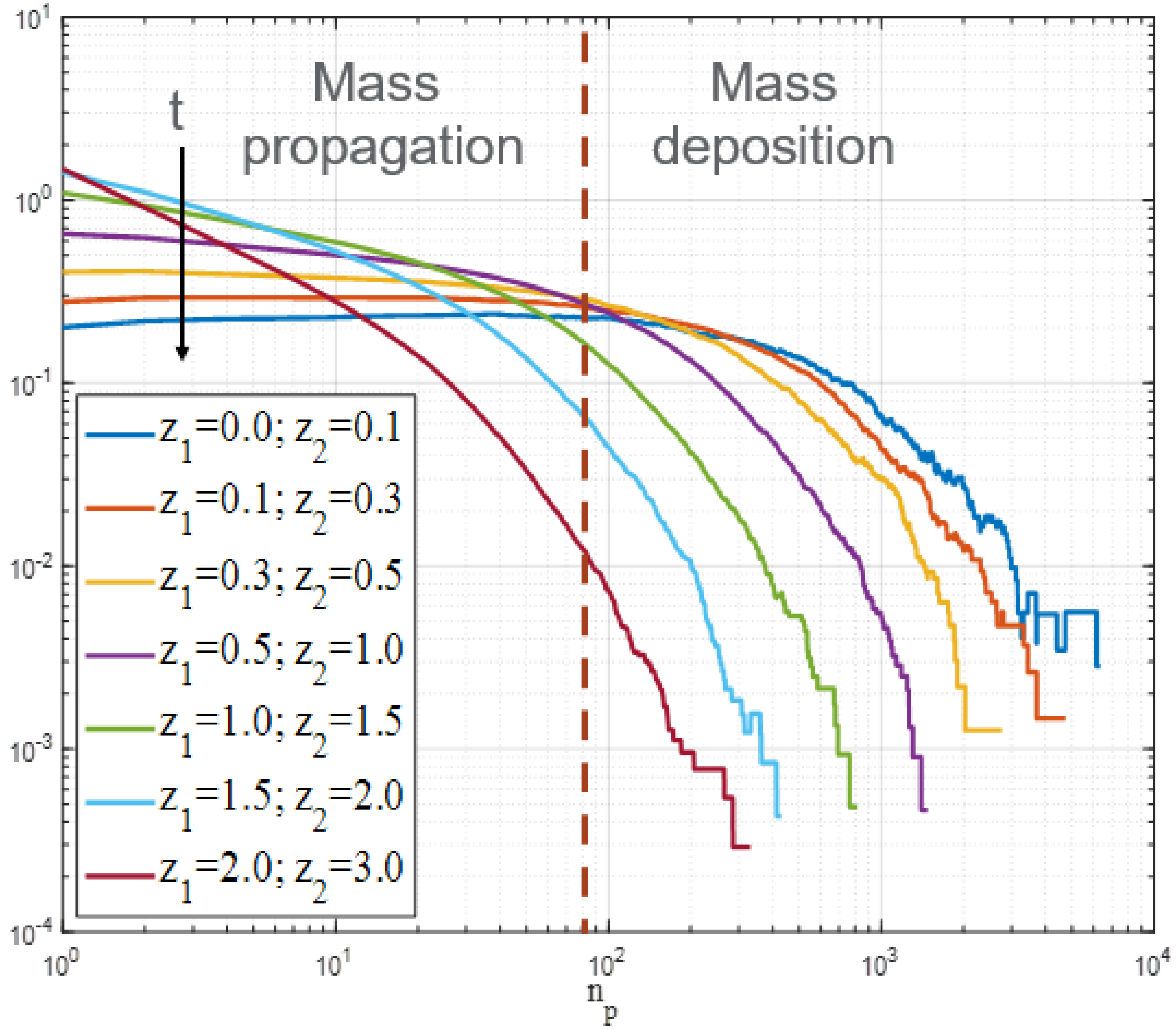
$$T_m(m_h,a) = \frac{\partial m_g(m_h,a)}{m_p \partial t} = 0 \qquad m_g(m_h) \equiv m_g(m_h,a)$$

- Halo group mass is time-independent (steady-state);

# Halo group mass and mass flux function




Halo group mass $m_g(m_h, a)$
(time-independent in mass propagation range)




Mass flux function $\Pi_m(m_h, a)$ (normalized by $Nm_p/t_0$) varying with halo size
(scale-independent in mass propagation range)

# Formulating mass cascade

In mass propagation range: $m_h \ll m_h^*$

$$\varepsilon_m(a) = \Pi_m(m_h = 0, a) = -\frac{\partial M_h(a)}{\partial t}$$

$$\varepsilon_m(a) = -m_h f_h(m_h, a)$$

$$f_h(m_h, a) = \underbrace{f_0(a) M_h(a) f_M(m_h, m_h^*(a)) \frac{m_p}{m_h}}_{1} \underbrace{\left(\frac{m_h}{m_p}\right)^{\lambda}}_{2}$$

Term 1: proportional to the number of halos in group;

Term 2: proportional surface area of halo in group;

$$\lambda \approx 2/3$$

Merging frequency for halo group: $f_h(m_h, a)$

Halo geometry parameter: $\lambda$

Fundamental frequency for merging of two single mergers: $f_0(a) \propto a^{-\tau_0}$

Independent variables: $m_h$ $a$

Free parameters: $m_p$ $\lambda$ $\tau_0$

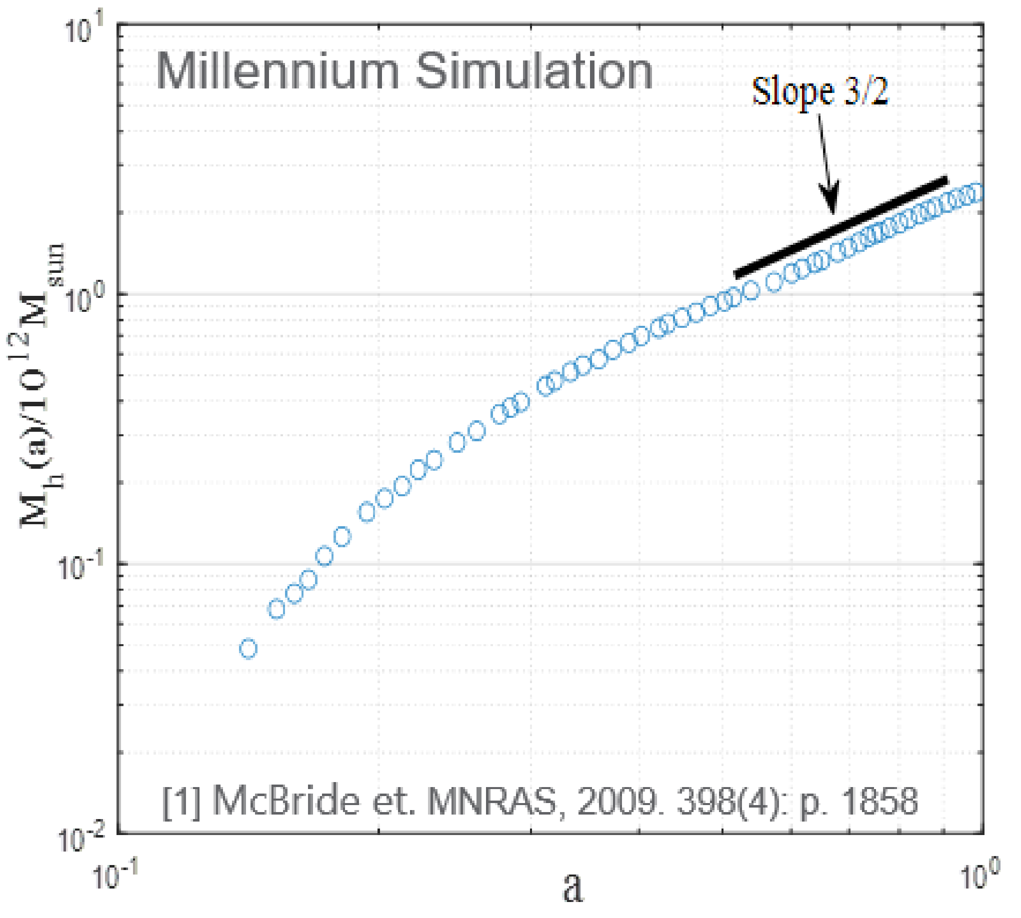
# Formulating mass cascade

In mass propagation range: $m_h \ll m_h^*$

Dimensional analysis requires mass function: $$f_M\left(m_h, m_h^*\right) = \beta_0 m_h^{-\lambda}\left(m_h^*\right)^{\lambda-1}$$

Table 2. List of dependence on the scale factor $a$ for different values of $\tau_0$ and $\lambda$

| $\lambda$ | $\tau_0$ | $f_0$ | $\varepsilon_m$ | $M_h$ | $f_M$ | $m_h^*$ | $\tau_h^*$ | $n_h^*$ | $m_g^*$ | $\tau_g^*$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\tau_0$ | $a^{-\tau_0}$ | $a^{-\tau_0}$ | $a^{3/2-\tau_0}$ | $a^{\tau_0-3/2}$ | $a^{\frac{(3/2-\tau_0)}{(1-\lambda)}}$ | $a^{\frac{(3/2-\lambda\tau_0)}{(1-\lambda)}}$ | $a^{-\left(\frac{3}{2}-\tau_0\right)\frac{(1+\lambda)}{(1-\lambda)}}$ | $a^{-\left(\frac{3}{2}-\tau_0\right)\frac{\lambda}{(1-\lambda)}}$ | $a^{\frac{(\tau_0-3\lambda/2)}{(1-\lambda)}}$ |
| $2/3$ | $1$ | $a^{-1}$ | $a^{-1}$ | $a^{1/2}$ | $a^{-1/2}$ | $a^{3/2}$ | $a^{5/2}$ | $a^{-5/2}$ | $a^{-1}$ | $a^{0}$ |
| $2/3$ | $1/2$ | $a^{-1/2}$ | $a^{-1/2}$ | $a^{1}$ | $a^{-1}$ | $a^{3}$ | $a^{7/2}$ | $a^{-5}$ | $a^{-2}$ | $a^{-3/2}$ |
| $3/4$ | $1$ | $a^{-1}$ | $a^{-1}$ | $a^{1/2}$ | $a^{-1/2}$ | $a^{2}$ | $a^{3}$ | $a^{-7/2}$ | $a^{-3/2}$ | $a^{-1/2}$ |




The halo mass for type II halos (the dominant type for large halos, Fig. 2 in ref. [1]) exhibits a power law scaling

# Dependence on halo mass $m_h$ and mass resolution $m_p$

Table 3. List of dependence on the halo mass $m_h$

| $f_M$ | $\varepsilon_m$ | $f_h$ | $\tau_h$ | $\tau_g$ | $\tau_f$ | $m_g$ | $n_h$ |
|---|---|---|---|---|---|---|---|
| $m_h^{-\lambda}$ | $m_h^0$ | $m_h^{-1}$ | $m_h^1$ | $m_h^{-\lambda}$ | $m_h^{1-\lambda}$ | $m_h^{-\lambda}$ | $m_h^{-1-\lambda}$ |

Table 4. List of dependence on the mass resolution $m_p$

| $f_M$ | $f_h$ | $\tau_f$ | $\varepsilon_m$ | $\tau_h^*$ | $\tau_h$ | $m_h^*$ | $M_h$ | $c_0$ | $\beta_0$ |
|---|---|---|---|---|---|---|---|---|---|
| $m_p^0$ | $m_p^0$ | $m_p^0$ | $m_p^0$ | $m_p^0$ | $m_p^0$ | $m_p^0$ | $m_p^0$ | $m_p^0$ | $m_p^0$ |
| $n_h$ | $\tau_g$ | $m_g$ | $n_p$ | $f_0$ | $b_0$ | $\lambda_0$ | $n_h^*$ | $m_g^*$ | $n_p^*$ |
| $m_p$ | $m_p$ | $m_p$ | $m_p^{-1}$ | $m_p^{\lambda-1}$ | $m_p^{\lambda-1}$ | $m_p^{1-\lambda}$ | $m_p$ | $m_p$ | $m_p^{-1}$ |

Fundamental frequency $f_0$ for merging between two single mergers depends on particle mass (same as cosmological redshift for photon frequency $f \sim a^{-1}$):

$$f_0 \propto a^{-1} m_p^{-1/3}$$

Can we detect $f_0$ from any experiment or observation?

# Random walk of halos and halo mass function

Merging frequency for halo group: $f_h(m_h, a)$

Characteristic merging time for halo group: $\tau_h(m_h, a) = 1/f_h$

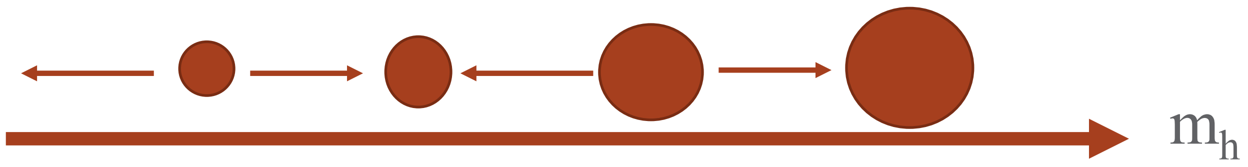
Characteristic merging time (lifetime) for a given halo: waiting time to merge

\# of halos in group

$$\tau_g(m_h, a) = n_h \tau_h$$

The exponential distribution of waiting time to merge:

$$P(\tau_{gr}) = \frac{1}{\tau_g}\exp\left(-\frac{\tau_{gr}}{\tau_g}\right)$$




1D Random walk equation in mass space:

$$\frac{\partial m_h(t)}{\partial t} = \frac{m_p \xi(t)}{\tau_g(m_h)} = \sqrt{2D_p(m_h)}\varsigma(t)$$

Fokker-Planck equation for distribution function:

$$\frac{\partial P_h}{\partial t} = \frac{\partial}{\partial m_h}\left[\sqrt{D_p}\frac{\partial}{\partial m_h}\left(\sqrt{D_p}P_h\right)\right] = D_{p0}\frac{\partial}{\partial m_h}\left[m_h^{\lambda}\frac{\partial}{\partial m_h}\left(m_h^{\lambda}P_h\right)\right]$$

Halo mass function:

$$f_M(m_h, a) = \frac{(1-\lambda)}{\sqrt{\pi\eta_0}}\left(\frac{m_h^*}{m_h}\right)^{\lambda}\frac{1}{m_h^*}\exp\left[-\frac{1}{4\eta_0}\left(\frac{m_h}{m_h^*}\right)^{2-2\lambda}\right]$$

Reduce to Press-Schechter (PS) mass function if λ=2/3 !

# Double-λ mass function from mass cascade

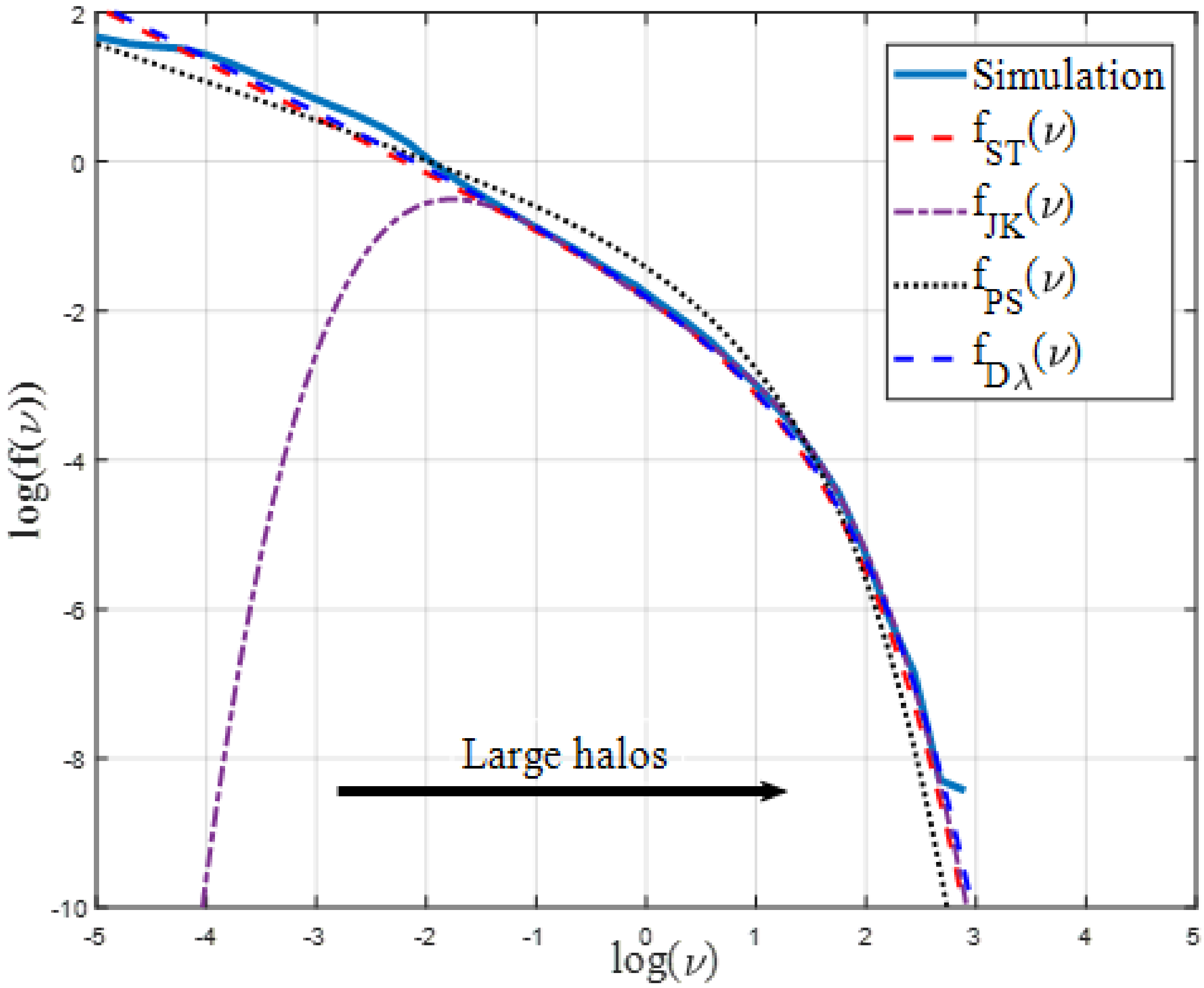
λ: halo geometry parameter; naturally, we can have different λ for different range.
$\lambda_1$ for mass propagation range (small halos);
$\lambda_2$ for mass deposition range (large halos);

$$f_M(m_h,a)=\frac{(1-\lambda)}{\sqrt{\pi\eta_0}}\left(\frac{m_h^*}{m_h}\right)^{\lambda_1}\frac{1}{m_h^*}\exp\left[-\frac{1}{4\eta_0}\left(\frac{m_h}{m_h^*}\right)^{2-2\lambda_2}\right]$$

Double-λ mass function:

$$f_{D\lambda}(\nu)=\frac{\left(2\sqrt{\eta_0}\right)^{-q}}{\Gamma(q/2)}\nu^{q/2-1}\exp\left(-\frac{\nu}{4\eta_0}\right)$$

- PS mass function
- ST model (modified PS) from ellipsoid collapse
- JK mass function by data fitting
- More generally, $\lambda_1$ can be a function of halo mass $m_h$




Comparison between different mass functions and simulation

# Summary and key words

| | | |
|---|---|---|
| Hydrodynamic turbulence | Dark matter flow | Mass redistribution |
| Direct energy cascade from large to small length scales | Inverse mass cascade from small to large mass scales | Random walk |
| “inertial range” & “dissipation range” | propagation range & deposition range | Heterogeneous diffusion |
| | | Waiting time |
| | | Chain-reaction |
| | | Halo mass function |

- Strong connections between dark matter flow and hydrodynamic turbulence
- The mass cascade is local, two-way, and asymmetric in mass space
- Scale-independent rate of mass cascade and time-independent halo group mass
- Chain reaction description for mass cascade to release energy and maximize entropy
- Random-walk of halos in mass space with an exponential distribution of waiting time
- Press-Schechter mass function is a special solution from halo random-walk
- New Double-λ halo mass function (based on the mass cascade)
- Extend double-λ halo mass function to consider λ as some function of halo size.