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A comparative study of dark matter flow & hydrodynamic turbulence and its applications

May 2022

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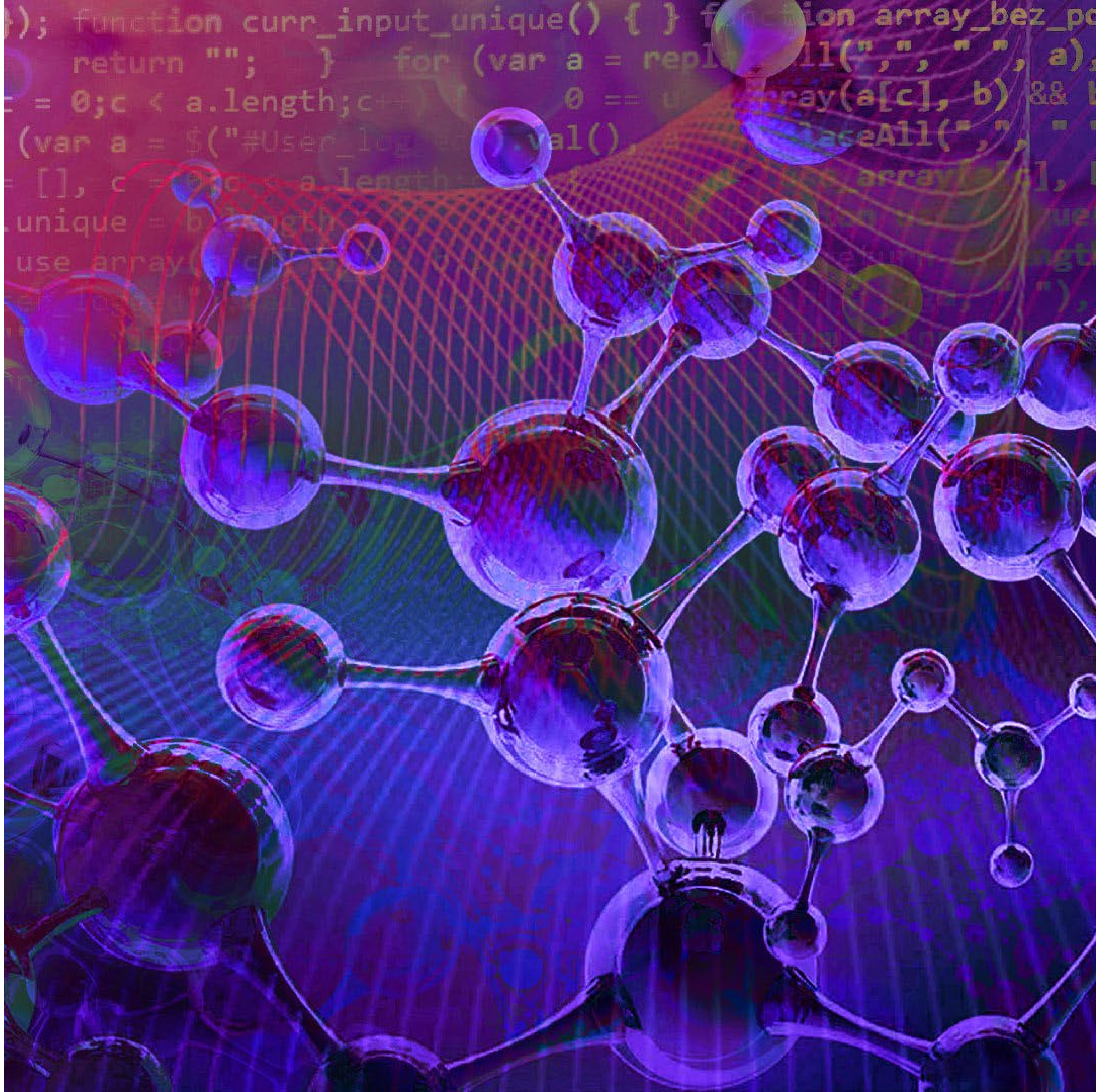

Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!

Data repository and relevant publications

Structural (halo-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6541230
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6569898
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

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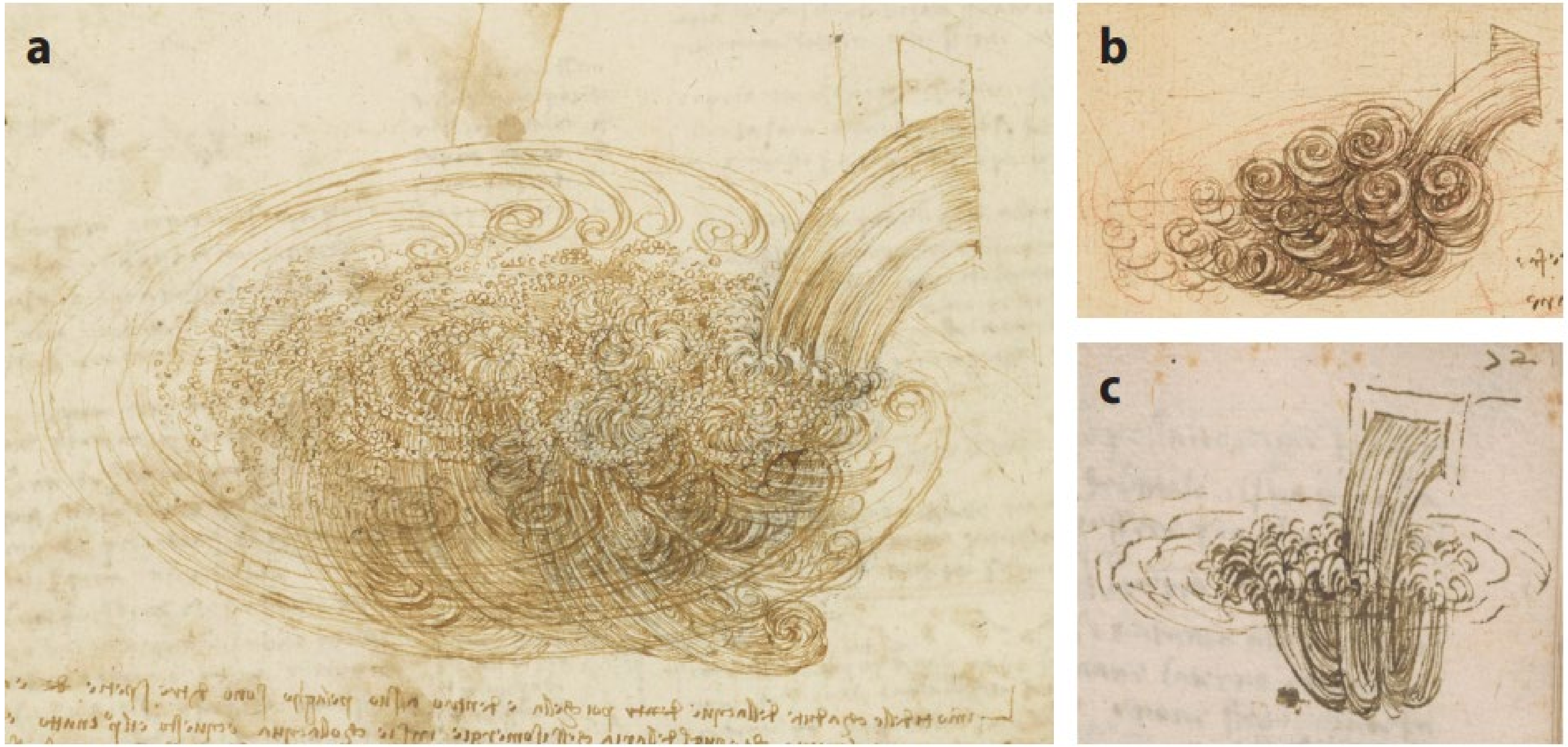
Comparison of two non-equilibrium systems:

Dark matter flow (DMF or SG-CFD)
vs.
Hydrodynamic turbulence

SG-CFD: Self-Gravitating Collisionless Fluid Dynamics

da Vinci's sketch of turbulence (~1500 AD)



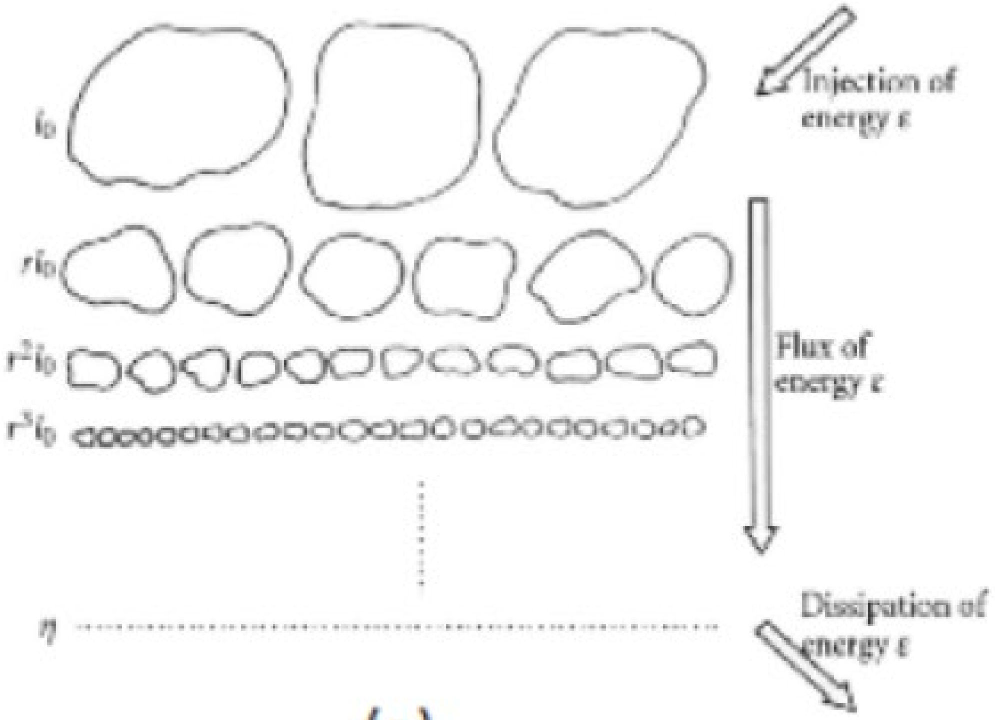
da Vinci sketch of turbulence: plunging water jet

- "turbolenza": the origin of modern word "turbulence"
- The pattern of flow with vortexes in fluid
- The random chaotic nature

". . . the smallest eddies are almost numberless, and large things are rotated only by large eddies and not by small ones, and small things are turned by small eddies and large."

Richardson’s direct cascade (1922)

“Big whorls have little whorls, That feed on their velocity;
And little whorls have lesser whorls, And so on to viscosity.”



(a)

(b)

(a) : Cascade of energy, (b) : Lewis Richardson

[1] "Weather Prediction by Numerical Process“, Richardson, L.F. 1922

Key attributes:

- Disorganized, chaotic, random;
- Nonrepeatability (sensitivity to initial conditions);

- Multiscale: large range of length and time scales;
- Dissipation mediated by viscosity;

- Three dimensionality;
- Time dependence;
- Rotationality (incompressible);
- Intermittency in space and time;

- <u>Cascade</u>: energy is injected at large scale, propagating, and dissipated at the smallest scale.

Existing approaches for turbulence

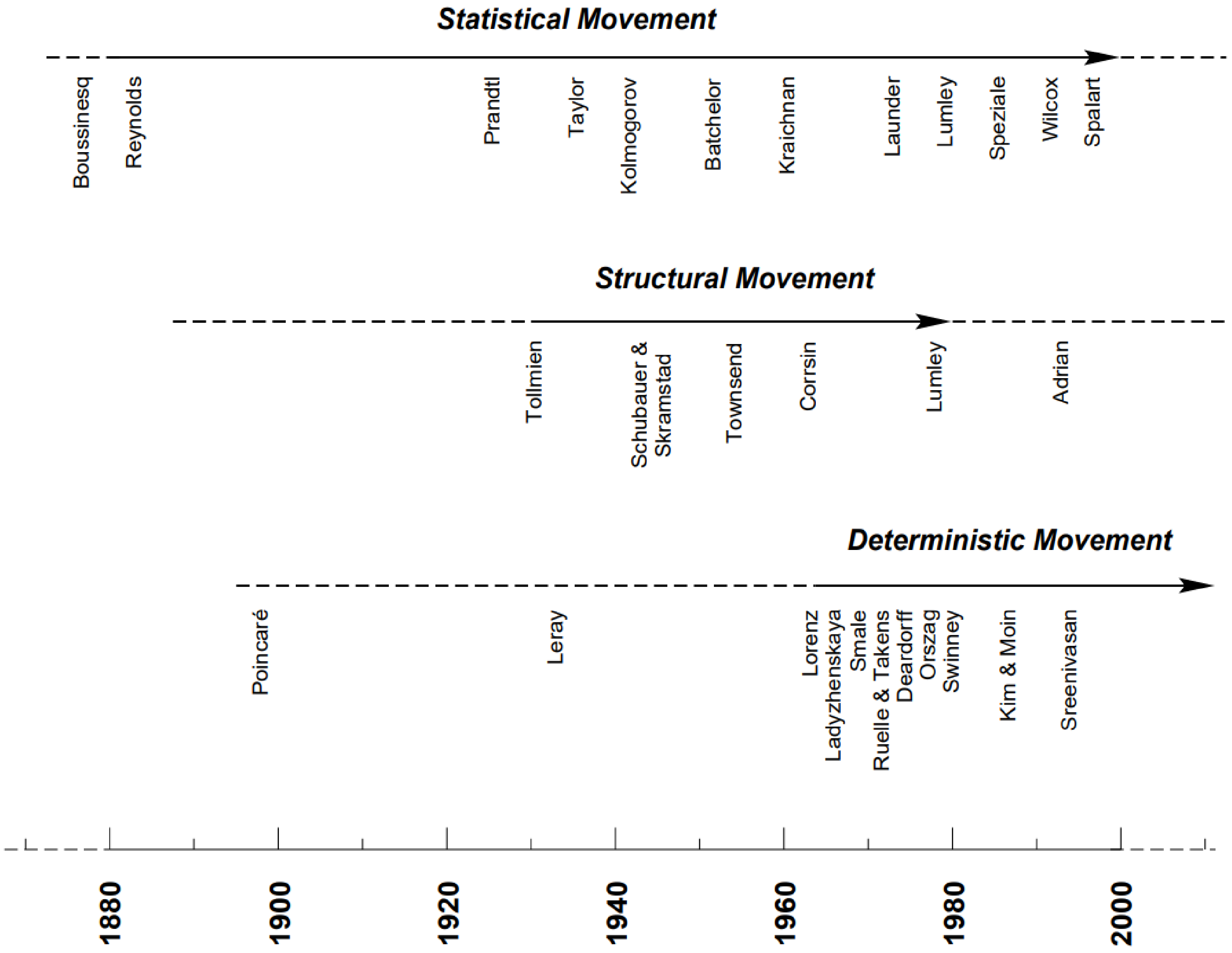


Figure 1.3: Movements in the study of turbulence, as described by Chapman and Tobak [1].

Statistic approach: (correlations etc.)

- Focusing on means and various averages
- Celebrated problem of closure
- Structureless without power of conceptualization

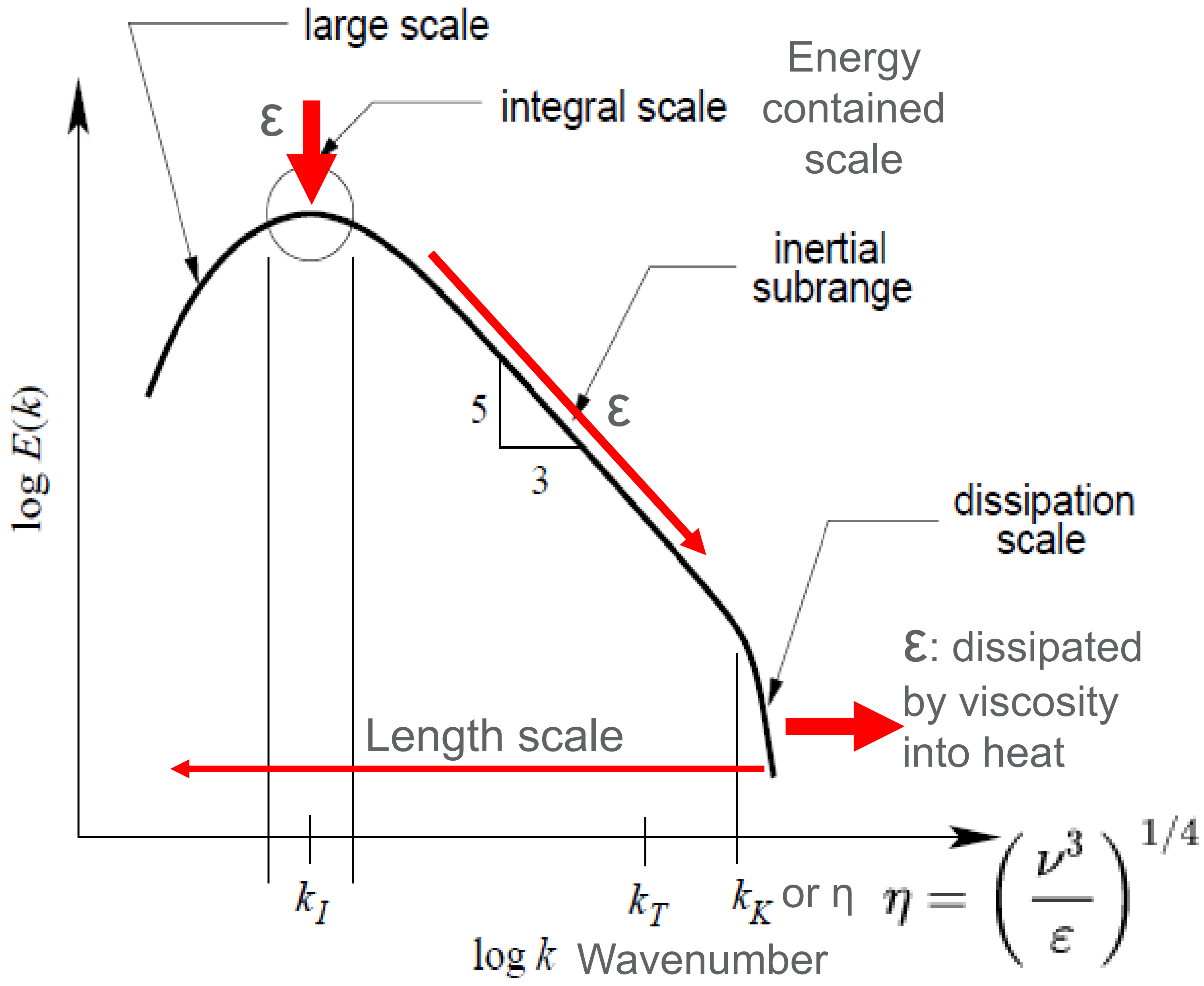
Structural approach: (vortex ect.)

- Existence of coherent structures
- Detecting and analyzing coherent structures in turbulent flows

Deterministic: (should be explored in DMF?)

- Chaotic behavior in simple deterministic systems
- Deterministic chaotic behavior can occur after just a few bifurcations
- Bifurcation theory, strange attractors, fractals, and renormalization group

Direct energy cascade in turbulence



- Freely decaying vs. forced stationary
- Integral scale: energy injection
- Inertial range: inertial >> viscous force
- Dissipation range: viscous dominant
- Dissipation scale: determined by viscosity (m^2/s) and rate of cascade (m^2/s^3)
- Is there energy cascade in dark matter flow? If yes, how it initiates, propagates, and dies ??

Inertial range, scaling laws, and intermittence

- There exist an inertial range with a scale-independent rate of energy cascade (ε does not depend on eddy size *l*) for eddy size η< *l* <L. *L is the integral length scale where energy is injected.*

Dissipation (Kolmogorov) scale:

$$\eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

- In this range, inertial force is dominant over viscous force. For eddies with a characteristic velocity *u* and size *l* , the lifetime (turnaround time) of eddy is *l*/*u*. The rate ε can be computed as the kinetic energy passed per lifetime.

$$\varepsilon \approx \frac{u^2}{(l/u)} \approx \frac{u^3}{l} \Rightarrow u^3 \propto l$$

turnaround time

- In this range, a general scaling for velocity structure functions for pairwise velocity can be identified (the most important results in turbulence)

$$S_m(r,a) = \left\langle (\Delta u_L)^m \right\rangle = \left\langle (u_L' - u_L)^m \right\rangle \Rightarrow S_m(r) \propto (\varepsilon_u)^{m/3} r^{m/3}$$

m=2

$$S_2 \propto (-\varepsilon_u)^{2/3} r^{2/3}$$

two-thirds law in hydrodynamic turbulence

Velocity correlation $R_2(r,a) = \langle \mathbf{u} \cdot \mathbf{u}' \rangle$

- Intermittence of cascade in space and time can be identified from the deviation from ideal scaling law

- What is the dissipation scale η in DMF?
- Is there any simple expression for ε?
- What are the scaling laws in DMF?
- What about the intermittence in DMF?
  - Touched here but need to be further studied.

Large scale dynamics of freely decaying turbulence

- Freely decaying turbulence is free from any external force to maintain the turbulence (Coffee example).
- There is no energy injection on large scale and total energy is continuously decaying with time.
- Both integral scale l (energy-contained scale) and energy dissipation rate ε vary with time.
- What is the large-scale dynamics of freely decaying turbulence? How does energy evolve with time?

$$\varepsilon \equiv A\frac{u^2}{(l/u)} = A\frac{u^3}{l}$$

Loitsyansky integral invariant (integral of velocity correlation):

$$\int \langle \mathbf{u}\cdot\mathbf{u}' \rangle \mathbf{r}^2 d\mathbf{r} \approx u^2 l^5 = const$$

$$u^2 \sim t^{-10/7}$$

$$l \sim t^{2/7}$$

$$\varepsilon \sim t^{-17/7}$$

Due to the formation and virilization of halos, the kinetic energy in dark matter flow continuously increases with time. In this regard, dark matter flow is a freely growing turbulence.

- What is the large-scale dynamics in DMF?
- How energy and momentum (both radial and angular) evolve on large scale?
- Loitsyansky integral invariant is related to the conservation of angular momentum
- Do we have similar integral "constants" of motion in dark matter flow? Are they still constant or varying with time?

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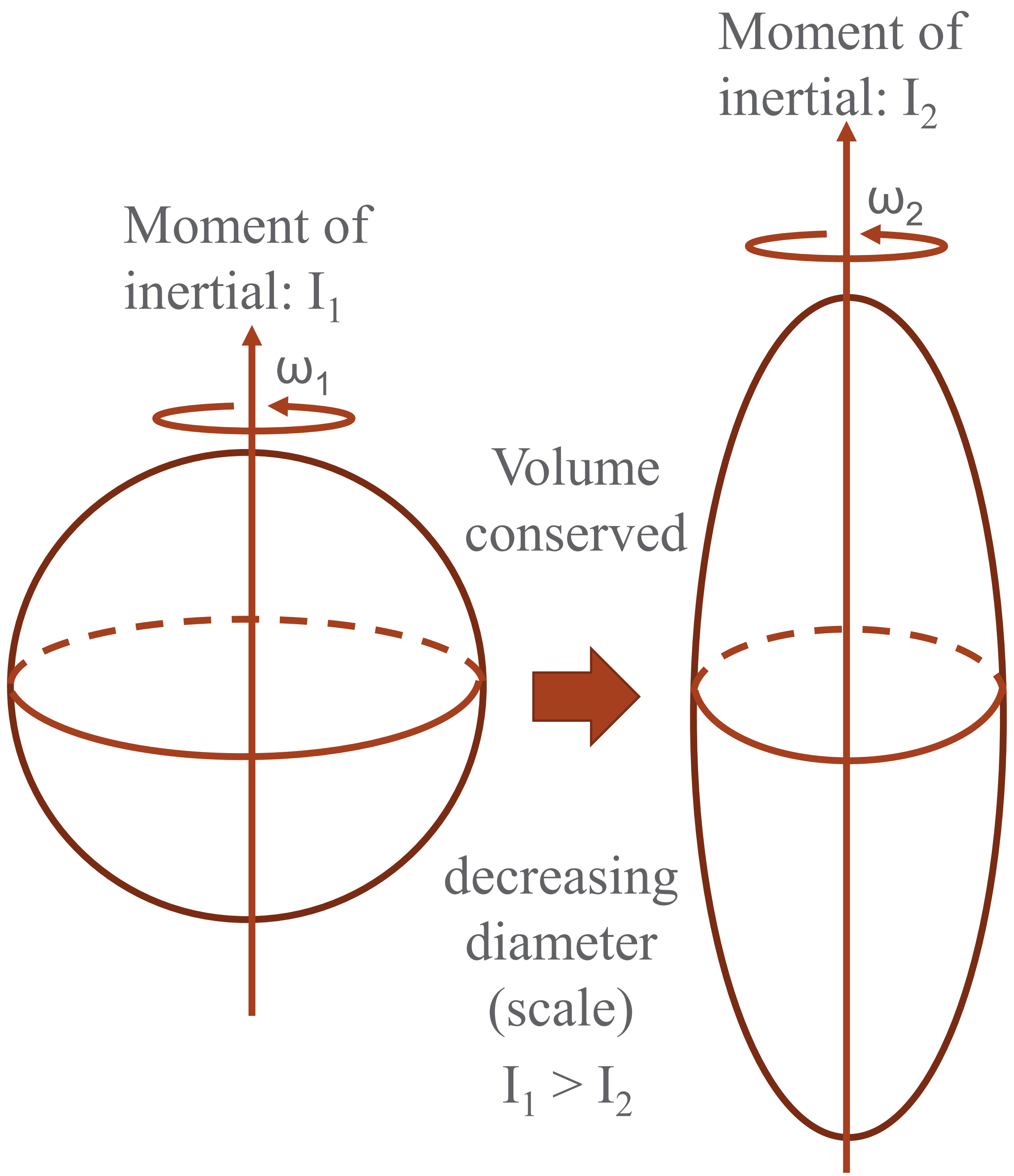
Vortex Stretching mechanism for energy cascade



Conservation of angular momentum:

$$I_1\omega_1 = I_2\omega_2 \Rightarrow \omega_2 > \omega_1$$

Ratio of rotational kinetic energy:

$$\frac{I_2\omega_2^2}{I_1\omega_1^2} = \frac{I_1}{I_2} \Rightarrow I_2\omega_2^2 > I_1\omega_1^2$$

Rotational kinetic energy is passing down the scales (direct energy cascade) !

- Does similar mechanism hold for halos in dark matter flow?
- What is the major mechanism for energy cascade in dark matter flow? (facilitated by mass cascade)

Reynolds stress for energy transfer between mean flow and random fluctuation

Reynolds decomposition: $u_i = \overline{u_i} + u_i'$,

Navier–Stokes equation (self-closed):

$$\nabla \cdot \mathbf{u} = 0$$

$$\rho\left[\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right] = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right)$$

Reynolds Averaged Navier–Stokes (RANS, not closed):

Reynolds stress

$$\rho\left[\frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j}\right] = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \overline{u_i}}{\partial x_j} - \rho \overline{u_i' u_j'}\right)$$

- Reynolds stress facilitates the **one-way** energy exchange from coherent (mean) flow to random fluctuation and enhances system entropy.
- Eddy viscosity models the Reynolds stress using the rate of strain of mean flow $\tau_{ij}^{ev} = -2\nu_{sgs}\overline{S}_{ij}$.

Jeans' equation (not self-closed):

Mean flow — Pressure from Fluctuation — Potential

$$\rho\left[\frac{\partial \langle u_i \rangle}{\partial t} + \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j}\right] = -\frac{\partial \langle \rho \sigma_{ij}^2 \rangle}{\partial x_j} - \rho \frac{\partial \Phi}{\partial x_i}$$

$$\sigma_{ij}^2 = \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle = \langle u_i' u_j' \rangle$$

- Is it possible to obtain a self-closed equation for dark matter flow? (closure problem)
  - Any similar concept as eddy viscosity in dark matter flow?
- How energy/momentum exchanges between mean flow and random fluctuation in dark matter flow?

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Brief timeline for turbulence research (~500 years)

- **1500s**: Earliest recognition of turbulence as a physical phenomenon
- **1877**: Boussinesq's concept of "eddy viscosity" replacing molecular viscosity
- **1894**: Reynolds number for transition from laminar to turbulence; Reynolds's decomposition (RANS) $v = \langle v \rangle + v'$ Reynolds stress;
- **1925**: Prandtl "mixing-length theory" predicts the eddy viscosity
- **1930s**: Taylor's statistical methods involving correlations power spectra; Karman and Howarth
- **1940s**: Kolmogorov: the "K41 theory" Two-thirds law -5/3 law
- **1963**: Lorenz propose possible links between "deterministic chaos" and turbulence
- **1967**: Kraichnan's inverse cascade in 2-dimensional turbulence

RANS: Reynolds-averaged Navier-Stokes Equation;

For all concepts listed here, we can identify their counterparts in dark matter flow!

Hydrodynamic turbulence vs. dark matter flow

Key attributes of hydrodynamic turbulence:

- Disorganized, chaotic, random;
- Nonrepeatability (sensitivity to initial conditions);
- Multiscale in length and time scales;
- Intermittency in space and time;
- Dissipative and collisional
- Short-range interaction
- Velocity fluctuation
- Vortex as fundamental building block
- Maximum entropy distribution (Gaussian)
- Incompressible on all scales $\nabla \cdot \mathbf{v} = 0$
  - Divergence-free
  - Constant density
- Energy cascade from large to small length scales
- Vortex stretching responsible for energy cascade
  - Volume conserving
  - Shape changing
  - Uniform density
- Reynolds decomposition
- Reynolds stress for energy transfer between mean flow and random motion (turbulence)
- Closure problem, eddy viscosity, etc…
- Statistical theory: correlation/structure functions scaling laws in inertial range

Key attributes of dark matter flow:

- Disorganized, chaotic, random;
- Nonrepeatability;
- Multiscale in mass/length/time scales;
- Intermittency in space and time;
- Dissipationless and collisionless
- Long-range gravity
- Velocity & acceleration fluctuation → Critical MOND acceleration a_0?
- Halos as fundamental building block
- Maximum entropy distribution?? (X dist.) → Deep MOND?
- Flow behavior is scale-dependent (peculiar velocity)
  - Small scale: constant divergence $\nabla \cdot \mathbf{v} = \theta$
  - Large scale: irrotational (curl-free) $\nabla \times \mathbf{v} = 0$
- Mass/energy cascade from small to large mass scales
- Role of halos for energy cascade??
  - Halos are growing, rotating, with nonuniform density
  - Is halo shape changing important?
  - Mass cascade facilitates energy cascade?
- Velocity/acceleration decomposition?
- What facilitates the energy transfer between mean flow and random motion??
- Self-closed model (analogue of NS) ?? Closure problem?
- Statistical theory: Kinematic and dynamic relations? Scaling laws?

← Common features (first four attributes of each)

Theory and applications of dark matter flow

Theory of dark matter flow

- Structural (halo-based) approach
  - Inverse mass cascade in dark matter flow
    - Impact on halo mass functions
    - Impact on halo energy and density profiles
  - Energy cascade in dark matter flow
  - Properties of spherical, axisymmetric, rotating, and growing halos (from mass accretion)
  - Maximum entropy distributions in dark matter flow
  - Halo mass function from maximum entropy distribution
  - Two-body collapse model (TBCM): an elementary step of mass cascade
  - Energy and momentum evolution and integral constants

- Statistical (correlation-based) approach
  - One-point statistics: velocity, density, acceleration distributions in dark matter flow
  - Two-point statistics:
    - Kinematic relations for second order statistics (correlation, structure, spectrum functions)
    - Kinematic and dynamic relations for high order statistics

Applications

- Predicting dark matter mass and properties
- Origin of MOND acceleration
- Baryonic-to-halo mass relation and total baryons in halos