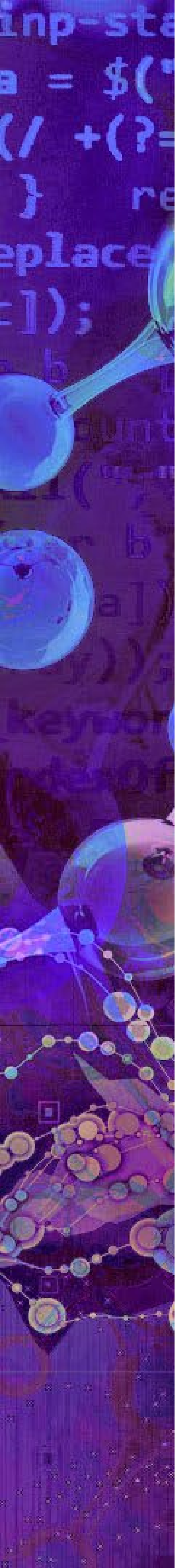

Pacific Northwest
NATIONAL LABORATORY

A comparative study of dark matter flow & hydrodynamic turbulence and its applications

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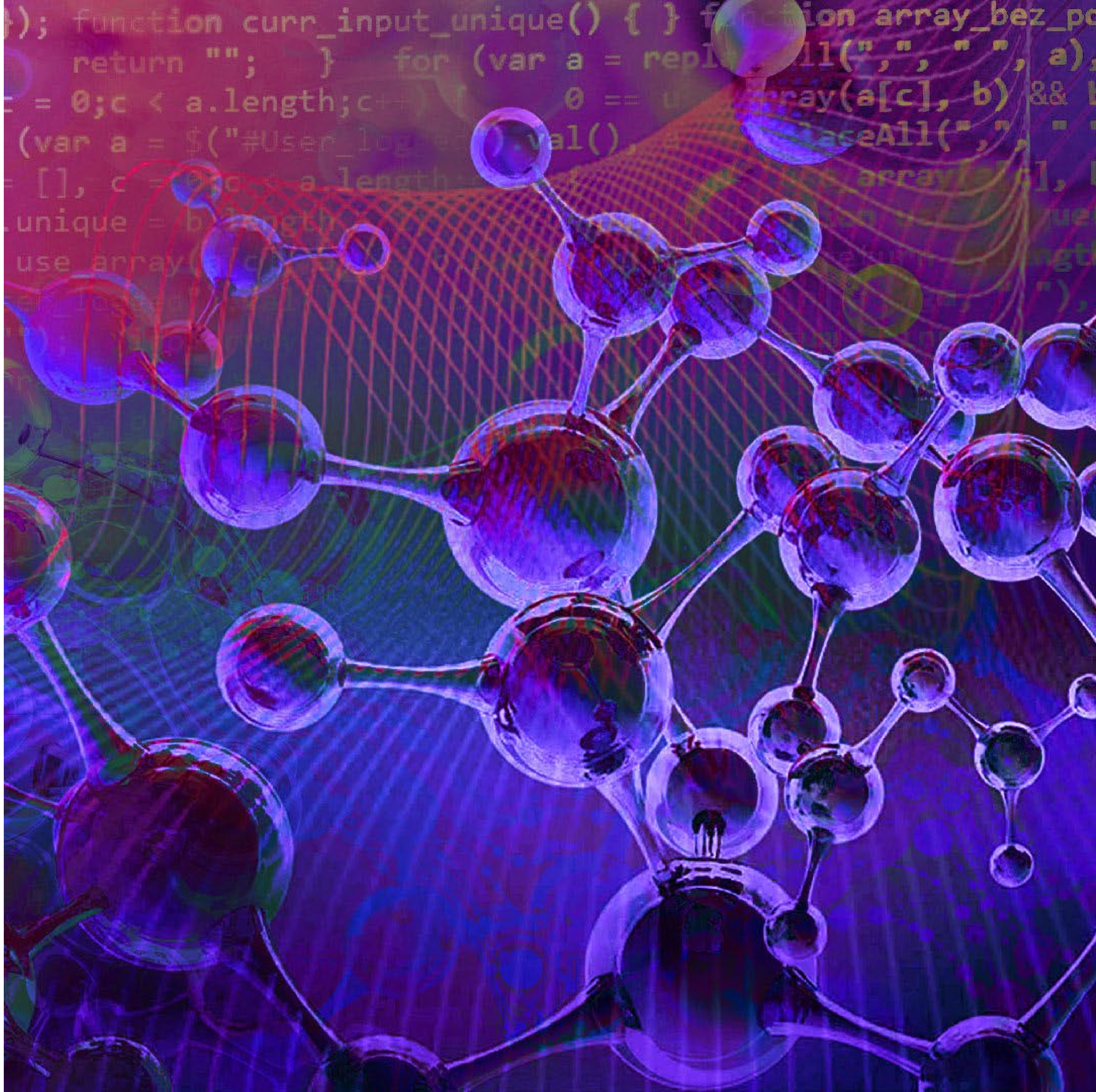
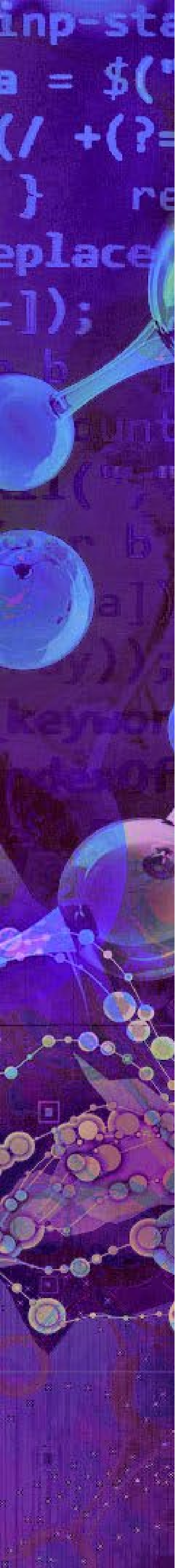


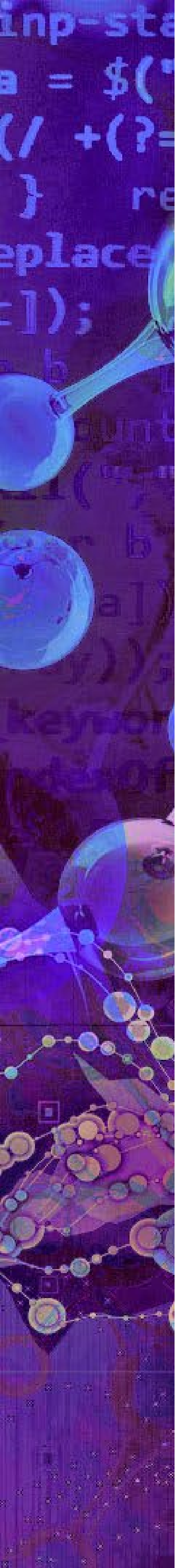


Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!



Data repository and relevant publications

Structural (halo-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6541230
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6569898
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

Statistical (correlation-based) approach for dark matter flow

The statistical theory of dark matter flow (high order)

Xu Z., 2022, arXiv:2202.02991 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2202.02991

Introduction

Review:
Statistical theory in hydrodynamic turbulence

- Kinematic relations between statistical measures (2nd and 3rd order)
- Dynamic relations between statistical measures of different order (from NS equations of velocity)
- Reynolds decomposition
- Closure problem, eddy viscosity, etc…

Current statistical theory of dark matter flow is not satisfactory:

- Dark matter flow is intrinsically complex with different nature of flow on different scales, i.e. a constant divergence flow on small scale and an irrotational flow on large scale.
- The kinematic and dynamic relations need to be developed separately for both types of flow on different scales.
- Dynamic equations of velocity (Jeans' equation) are not self-closed. No dynamic relations can be derived without a self-closed dynamics for velocity evolution.

- Existing work mostly focus on the 1st and 2nd order velocity statistics, while the peculiar velocity field contains much richer information beyond the second order.
- Finally, very challenging to explore high order statistics, as that inherently involves tensor and vector calculus of great complexity.

❖ Most kinematic relations between statistical measures (2nd)
Need to extend to high and arbitrary order

❖ Develop self-consistent dynamic equation for velocity field

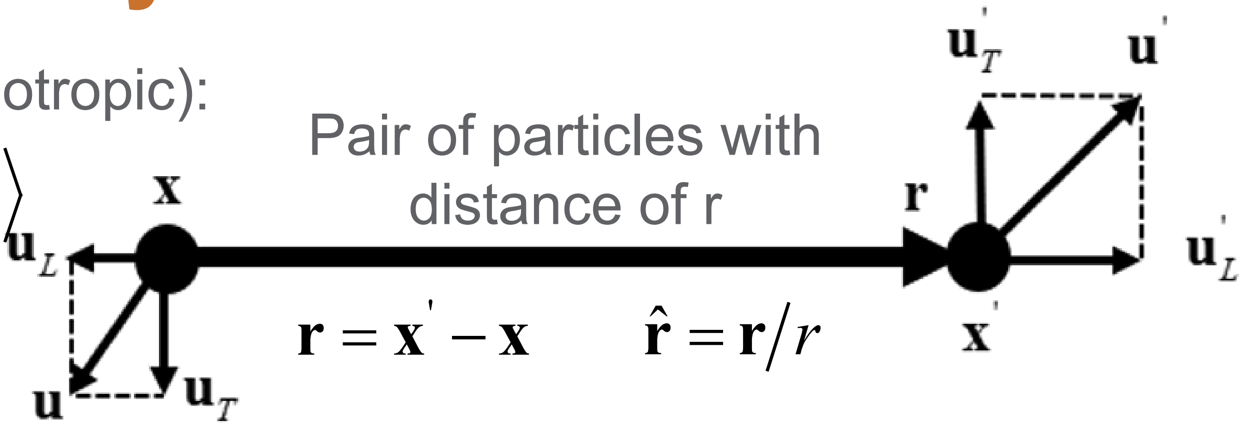
❖ Develop dynamic relations between statistical measures of different order

❖ Derive the "eddy" (artificial) viscosity from velocity fluctuation

Two-point third order velocity correlation tensors

Third order velocity correlation tensor (homogeneous and isotropic):

$$Q_{ijk}\left(\mathbf{x},\mathbf{r}\right)=Q_{ijk}\left(\mathbf{r}\right)=Q_{ijk}\left(r\right)=\left\langle u_i\left(\mathbf{x}\right)u_j\left(\mathbf{x}\right)u_k\left(\mathbf{x}^{'}\right)\right\rangle=\left\langle u_i u_j u_k^{'}\right\rangle$$



Pair of particles with distance of r

$\mathbf{r}=\mathbf{x}^{'}-\mathbf{x}$ $\quad\hat{\mathbf{r}}=\mathbf{r}/r$

General form of isotropic third order tensor:

$$Q_{ijk}\left(r\right)=A_3\left(r\right)r_i r_j r_k+B_3\left(r\right)\left(r_i\delta_{jk}+r_j\delta_{ki}\right)+D_3\left(r\right)r_k\delta_{ij}$$

Divergence of second order tensor:

$$Q_{ijk,k}=\frac{\partial\left\langle u_i u_j u_k^{'}\right\rangle}{\partial r_k}=\left(5A_3+\frac{\partial A_3}{\partial r}r+\frac{2}{r}\frac{\partial B_3}{\partial r}\right)r_i r_j+\left(2B_3+\frac{\partial D_3}{\partial r}r+3D_3\right)\delta_{ij}$$

$$Q_{ijk,k}=\left\langle\left(u_i\left(\mathbf{x}\right)u_j\left(\mathbf{x}\right)\right)\left(\nabla^{'}\cdot u_j\left(\mathbf{x}^{'}\right)\right)\right\rangle=0$$ ← Incompressible flow

$$Q_{ijk,k}=\left\langle\left(u_i\left(\mathbf{x}\right)u_j\left(\mathbf{x}\right)\right)\left(\nabla^{'}\cdot u_j\left(\mathbf{x}^{'}\right)\right)\right\rangle=\theta\left\langle\left(u_i\left(\mathbf{x}\right)u_j\left(\mathbf{x}\right)\right)\right\rangle\neq 0$$ ← Constant divergence flow

Use this to derive Kinematic relations

Longitudinal velocity:

$$u_L=\mathbf{u}\cdot\hat{\mathbf{r}}=u_i\hat{r}_i$$

$$u_L^{'}=\mathbf{u}^{'}\cdot\hat{\mathbf{r}}=u_i^{'}\hat{r}_i$$

Transverse velocity:

$$\mathbf{u}_T=-\left(\mathbf{u}\times\hat{\mathbf{r}}\times\hat{\mathbf{r}}\right)$$

$$\mathbf{u}_T^{'}=-\left(\mathbf{u}^{'}\times\hat{\mathbf{r}}\times\hat{\mathbf{r}}\right)$$

Velocity difference or Pairwise velocity: $\Delta u_L=u_L^{'}-u_L$

Velocity sum: $\sum u_L=u_L^{'}+u_L$

Curl of second order tensor:

$$\nabla\times Q_{mni}\left(r\right)=\varepsilon_{ijk}Q_{mnk,j}=\left(A_3-\frac{1}{r}\frac{\partial B_3}{\partial r}\right)\left(\varepsilon_{imk}r_n r_k+\varepsilon_{ink}r_m r_k\right)=0$$ ← Irrotational flow

Different odd order kinematic relations for incompressible flow and constant divergence flow

Two-point third order velocity correlation functions

Using index contraction of third order tensor to define four scalar correlation functions

Two total correlation functions:

$$R_3(r) = \frac{1}{2} Q_{ijk}\left(\delta_{ik}\hat{r}_j + \delta_{jk}\hat{r}_i\right) = \left\langle u_L \mathbf{u}\cdot\mathbf{u}' \right\rangle = A_3 r^3 + \left(4B_3 + D_3\right) r$$

$$R_{31}(r) = Q_{ijk}\delta_{ij}\hat{r}_k = \left\langle \mathbf{u}\cdot\mathbf{u}u_L' \right\rangle = A_3 r^3 + \left(2B_3 + 3D_3\right) r$$

Longitudinal triple correlation function:

$$L_3(r) = Q_{ijk}\hat{r}_i\hat{r}_j\hat{r}_k = \left\langle u_L^2 u_L' \right\rangle = A_3 r^3 + \left(2B_3 + D_3\right) r$$

Transverse third-order correlation function:

$$T_3(r) = \left\langle u_L \mathbf{u}_T\cdot\mathbf{u}_T' \right\rangle / 2 = \left(R_3 - L_3\right)/2 = B_3 r$$

$$R_3(r) = L_3(r) + 2T_3(r)$$

Relation to third correlation tensor:

$$Q_{iki,k} = Q_{ijk,i}\delta_{jk} = Q_{ikk,i} = \frac{1}{r^2}\left(r^2 R_3\right)_{,r}$$

$$Q_{iik,k} = \frac{1}{r^2}\left(r^2 R_{31}\right)_{,r}$$

Correlation functions of any order (pth order):

$$L_{(p,q)} = \left\langle u^q u_L^{p-q-1} u_L' \right\rangle$$

$$R_{(p,q+1)} = \left\langle u^q u_L^{p-q-2} u_i u_i' \right\rangle = \left\langle u^q u_L^{p-q-2} \mathbf{u}\cdot\mathbf{u}' \right\rangle$$

$$R_{(p,q+1)} = L_{(p,q)} + 2T_{(p,q)}$$

Goal is to identify kinematics relations between correlations functions of same order

Kinematic relations for third order correlation functions

For incompressible flow: $\nabla \cdot \mathbf{u} = 0$

$$R_3 = \frac{1}{2r^3}\left(r^4 L_3\right)_{,r} \qquad T_3 = \frac{1}{4r}\left(r^2 L_3\right)_{,r} \qquad r^2\left(r^2 R_3\right)_{,r} = 2\left(r^4 T_3\right)_{,r} \qquad R_{31}(r) = \left\langle \mathbf{u} \cdot \mathbf{u} u_L^{'} \right\rangle = 0$$

Relations between correlation functions

$$Q_{ijk}(r) = \frac{L_3 - rL_3^{'}}{2}\hat{r}_i\hat{r}_j\hat{r}_k + \frac{2L_3 + rL_3^{'}}{4}\left(\hat{r}_i\delta_{jk} + \hat{r}_j\delta_{ki}\right) - \frac{L_3}{2}\hat{r}_k\delta_{ij}$$

Correlation tensor in terms of correlations

For constant divergence flow: $\nabla \cdot \mathbf{u} = \theta$ Reduced to incompressible flow with Θ=0

$$R_3 + \frac{1}{2}\left\langle u_L^2 \right\rangle \theta r = \frac{1}{2r^3}\left(r^4 L_3\right)_{,r}$$

$$\left\langle u^2 \right\rangle \theta = \frac{1}{r^2}\left(r^2 R_{31}\right)_{,r}$$

$$\left\langle u^2 \right\rangle \approx 3\left\langle u_L^2 \right\rangle$$

$$R_3 + \frac{1}{6r}\left(r^2 R_{31}\right)_{,r} = \frac{1}{2r^3}\left(r^4 L_3\right)_{,r}$$

For irrotational flow: $\nabla \times \mathbf{u} = 0$

$$\left(rR_3\right)_{,r} + R_{31} = \frac{1}{r^3}\left(r^4 L_3\right)_{,r} \qquad 3L_3 - R_{31} = 2\left(rT_3\right)_{,r} \qquad 3R_3 - R_{31} = \frac{2}{r^3}\left(r^4 T_3\right)_{,r}$$

Scaling laws for two-point third order velocity structure function (review)

Structure functions as moments of pairwise velocity:

$$S_3^{lp}(r)=\left\langle(\Delta u_L)^3\right\rangle=\left\langle(u_L^{'}-u_L)^3\right\rangle=6L_3(r)-2\left\langle u_L^3\right\rangle \qquad S_m^{lp}=\left\langle(\Delta u_L)^m\right\rangle=\left\langle(u_L^{'}-u_L)^m\right\rangle$$

Two-thirds law for even order (reduced) structure function:

$$S_{2n}^{lp}(r)-S_{2n}^{lp}(0)\propto(-\varepsilon_u)^{2/3}\,r^{2/3}$$

ε_u : rate of energy cascade.

Generalized stable clustering hypothesis (GSCH)

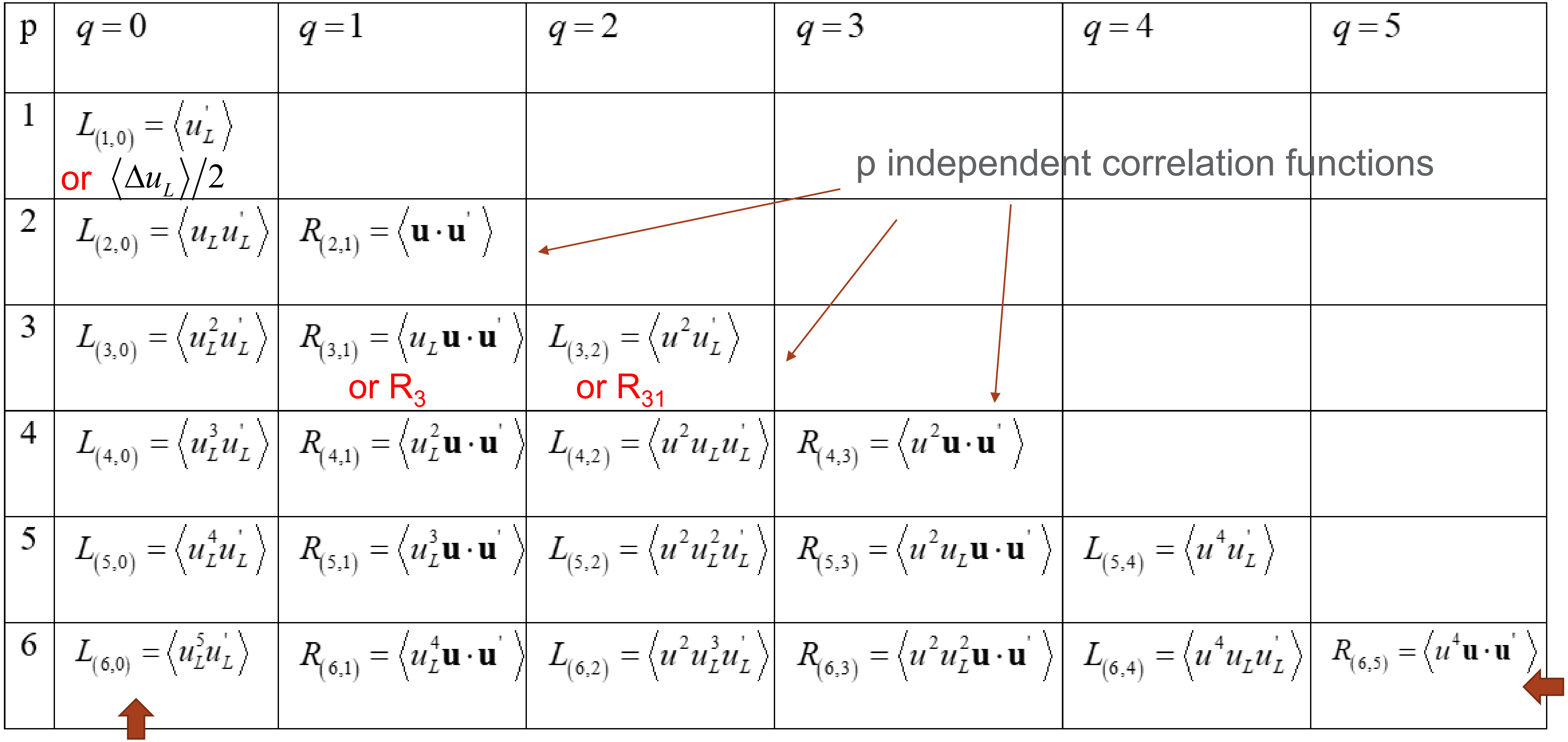
$$S_{2n+1}^{lp}(r)=(2n+1)S_1^{lp}(r)S_{2n}^{lp}(r)$$

$$S_{2n+1}^{lp}(r)=-(2n+1)HarS_{2n}^{lp}(0)=-2^n(2n+1)K_{2n}(\Delta u_L,0)Haru^{2n}\propto r$$

$K_{2n}(\Delta u_L,0)$: Generalized kurtosis of the distribution of pairwise velocity

Velocity correlation functions of any (pth) order $L_{(p,q)}$ and $R_{(p,q)}$

Table 2. The velocity correlation functions of different order



p	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$
1	$L_{(1,0)}=\langle u_L^{'}\rangle$ or $\langle \Delta u_L\rangle/2$					
2	$L_{(2,0)}=\langle u_L u_L^{'}\rangle$	$R_{(2,1)}=\langle \mathbf{u}\cdot\mathbf{u}^{'}\rangle$				
3	$L_{(3,0)}=\langle u_L^2 u_L^{'}\rangle$	$R_{(3,1)}=\langle u_L \mathbf{u}\cdot\mathbf{u}^{'}\rangle$ or R_3	$L_{(3,2)}=\langle u^2 u_L^{'}\rangle$ or R_{31}			
4	$L_{(4,0)}=\langle u_L^3 u_L^{'}\rangle$	$R_{(4,1)}=\langle u_L^2 \mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(4,2)}=\langle u^2 u_L u_L^{'}\rangle$	$R_{(4,3)}=\langle u^2 \mathbf{u}\cdot\mathbf{u}^{'}\rangle$		
5	$L_{(5,0)}=\langle u_L^4 u_L^{'}\rangle$	$R_{(5,1)}=\langle u_L^3 \mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(5,2)}=\langle u^2 u_L^2 u_L^{'}\rangle$	$R_{(5,3)}=\langle u^2 u_L \mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(5,4)}=\langle u^4 u_L^{'}\rangle$	
6	$L_{(6,0)}=\langle u_L^5 u_L^{'}\rangle$	$R_{(6,1)}=\langle u_L^4 \mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(6,2)}=\langle u^2 u_L^3 u_L^{'}\rangle$	$R_{(6,3)}=\langle u^2 u_L^2 \mathbf{u}\cdot\mathbf{u}^{'}\rangle$	$L_{(6,4)}=\langle u^4 u_L u_L^{'}\rangle$	$R_{(6,5)}=\langle u^4 \mathbf{u}\cdot\mathbf{u}^{'}\rangle$

p independent correlation functions

Kinematic relations (for same order p)

Dynamic relations (for different order p)

$$L_{(p,q)}=\left\langle u^q u_L^{p-q-1} u_L^{'}\right\rangle \quad R_{(p,q+1)}=\left\langle u^q u_L^{p-q-2} u_i u_i^{'}\right\rangle=\left\langle u^q u_L^{p-q-2}\mathbf{u}\cdot\mathbf{u}^{'}\right\rangle \quad R_{(p,q+1)}=L_{(p,q)}+2T_{(p,q)}$$

Correlation functions in the limit of small and large scale

For odd order p

$$\lim_{r\to 0}\frac{\left\langle u^{q}u_L^{p-q-1}\right\rangle}{\left\langle u_L^{p-1}\right\rangle}=\frac{p}{p-q} \qquad \lim_{r\to \infty}\frac{\left\langle u^{q}u_L^{p-q-1}\right\rangle}{\left\langle u_L^{p-1}\right\rangle}=\frac{p}{p-q}$$

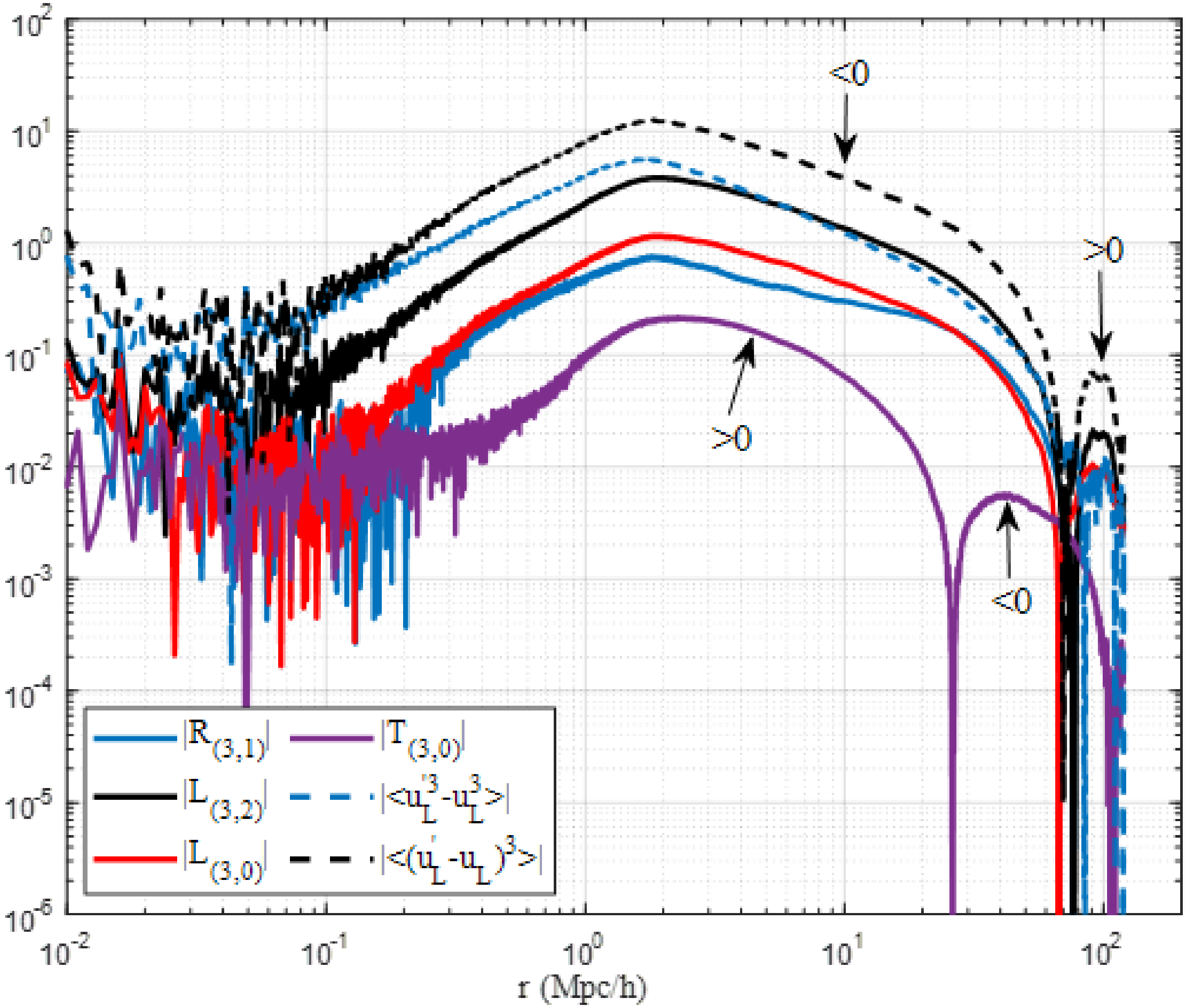
$$\lim_{r\to 0,\infty}\frac{L_{(p,q)}}{L_{(p,0)}}=\lim_{r\to 0,\infty}\frac{\left\langle u^{q}u_L^{p-q-1}u_L^{'}\right\rangle}{\left\langle u_L^{p-1}u_L^{'}\right\rangle}=\lim_{r\to 0,\infty}\frac{\left\langle u^{q}u_L^{p-q-1}\right\rangle}{\left\langle u_L^{p-1}\right\rangle}=\frac{p}{p-q}$$

For even order p

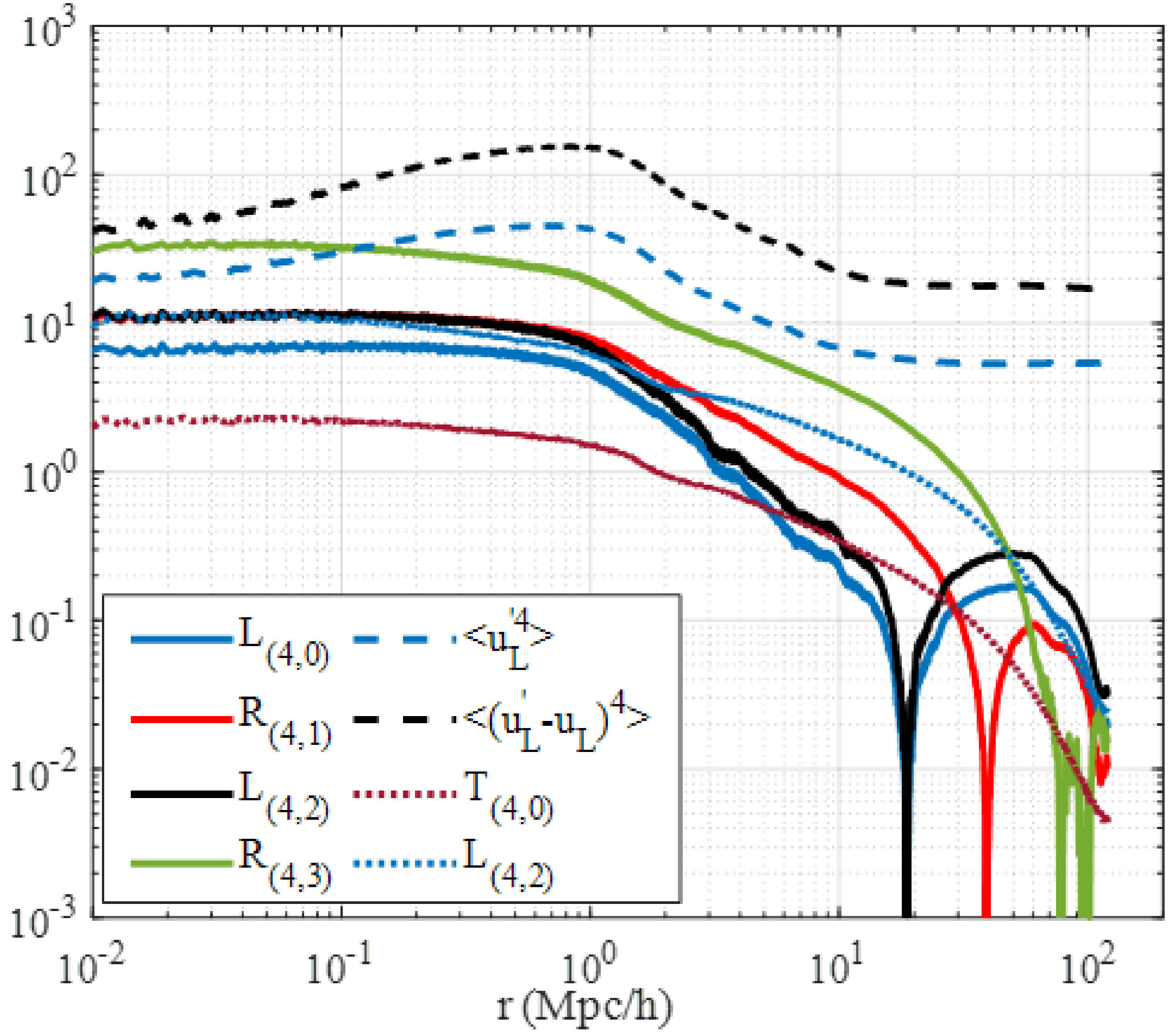
$$\lim_{r\to 0}\frac{R_{(p,q+1)}}{L_{(p,0)}}=\lim_{r\to 0}\frac{\left\langle u^{q}u_L^{p-q-2}\mathbf{u}\cdot\mathbf{u}^{'}\right\rangle}{\left\langle u_L^{p-1}u_L^{'}\right\rangle}=\frac{p+1}{p-q-1} \qquad \lim_{r\to 0,\infty}\frac{L_{(p,q)}}{L_{(p,0)}}=\lim_{r\to 0,\infty}\frac{\left\langle u^{q}u_L^{p-q-1}u_L^{'}\right\rangle}{\left\langle u_L^{p-1}u_L^{'}\right\rangle}=\frac{p+1}{p+1-q}$$

- The collisionless nature has effects on the limits of correlations functions oat both small and large scales.
- These results can be confirmed by N-body simulation data

Correlation and structure functions from N-body simulation



Two-point third order velocity correlation and structure functions (normalized by u^3) at z=0



Two-point fourth order velocity correlation and structure functions (normalized by u^4) at z=0

Kinematic relations for correlation functions $L_{(p,q)}$ and $R_{(p,q)}$ of any (pth) order (derivation skipped)

For incompressible flow: $\nabla\cdot\mathbf{u}=0$

$$\left(p-q-1\right)R_{(p,q+1)}=\frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$$

$$2\left(p-q-1\right)T_{(p,q)}=\frac{1}{r}\left(r^{2}L_{(p,q)}\right)_{,r}$$

$$\left(r^{2}R_{(p,q+1)}\right)_{,r}=\frac{2}{r^{p-q-1}}\left(r^{p-q+1}T_{(p,q)}\right)_{,r}$$

For irrotational flow: $\nabla\times\mathbf{u}=0$

$$\left(R_{(p,q+1)}r\right)_{,r}+\left(p-q-2\right)L_{(p,q+2)}=\frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$$

$$\left(p-q\right)R_{(p,q+1)}-\left(p-q-2\right)L_{(p,q+2)}=\frac{2}{r^{p-q}}\left(r^{p-q+1}T_{(p,q)}\right)_{,r}$$

$$\left(p-q\right)L_{(p,q)}-\left(p-q-2\right)L_{(p,q+2)}=2\left(rT_{(p,q)}\right)_{,r}$$

If $\Theta=0$

For constant divergence flow: $\nabla\cdot\mathbf{u}=\theta$

$$\left(p-q-1\right)R_{(p,q+1)}+\left\langle u^{q}u_{L}^{p-q-1}\right\rangle\theta r=\frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$$

If $\Theta\neq0$ and p is even: $\lim\limits_{r\to0}\left\langle u^{q}u_{L}^{p-q-1}\right\rangle=0$

$$\left(p-q-1\right)R_{(p,q+1)}=\frac{1}{r^{p-q}}\left(r^{p-q+1}L_{(p,q)}\right)_{,r}$$

If $\Theta\neq0$ and p is odd:

$$\left\langle u^{p-1}\right\rangle\theta r=\frac{1}{r}\left(r^{2}L_{(p,p-1)}\right)_{,r}$$

$$\left(p-1\right)R_{(p,1)}+\left\langle u_{L}^{p-1}\right\rangle\theta r=\frac{1}{r^{p}}\left(r^{p+1}L_{(p,0)}\right)_{,r}$$

If $\Theta\neq0$ and p=1:

$$\theta=\frac{1}{r^{2}}\left(r^{2}L_{(1,0)}\right)_{,r}=\frac{1}{2r^{2}}\left(r^{2}\left\langle\Delta u_{L}\right\rangle\right)_{,r}$$

Kinematic relations for even order correlations of constant divergence flow should be the same as that of incompressible flow

Kinematic relations validated by N-body simulations

Original Kinematic relations

On small scale, kinematic relations for even order (even p) correlations are the same as those for incompressible flow:

$$H_{(p,q)}^{S}\left(r\right)=\frac{\left(p-q-1\right)}{r^{p-q+1}L_{(p,q)}}\int_0^r R_{(p,q+1)}r^{p-q}dr=1$$

On small scale, kinematic relations for odd order (odd p) correlations are the same as those for incompressible flow:

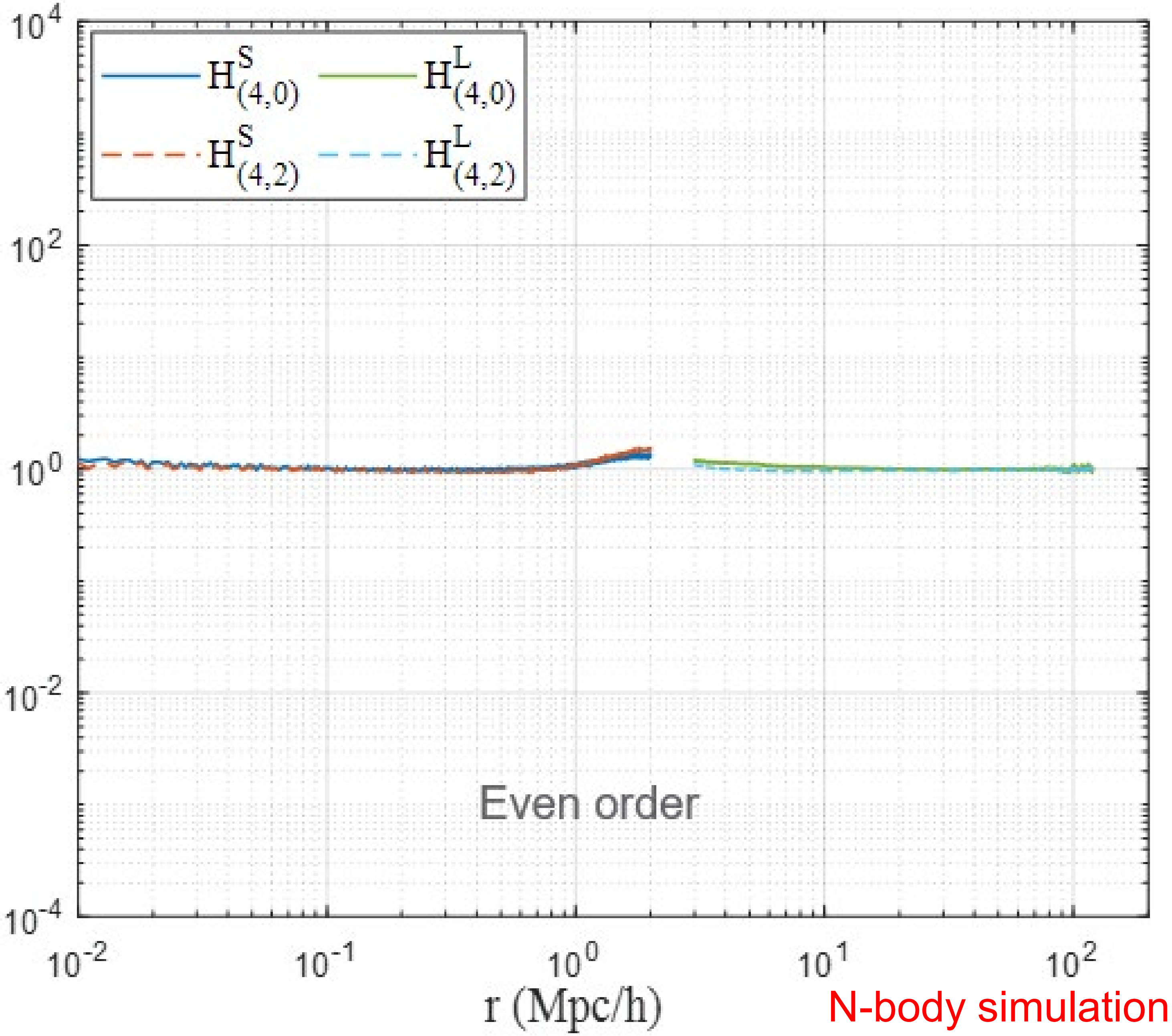
$$H_{(p,q)}^{S}\left(r\right)=\frac{\left(p-q-1\right)}{r^{p-q+1}L_{(p,q)}}\int_0^r\left(R_{(p,q+1)}-\frac{L_{(p,p-1)}}{p-q}\right)r^{p-q}dr+\frac{1}{\left(p-q\right)}\cdot\frac{L_{(p,p-1)}}{L_{(p,q)}}=1$$

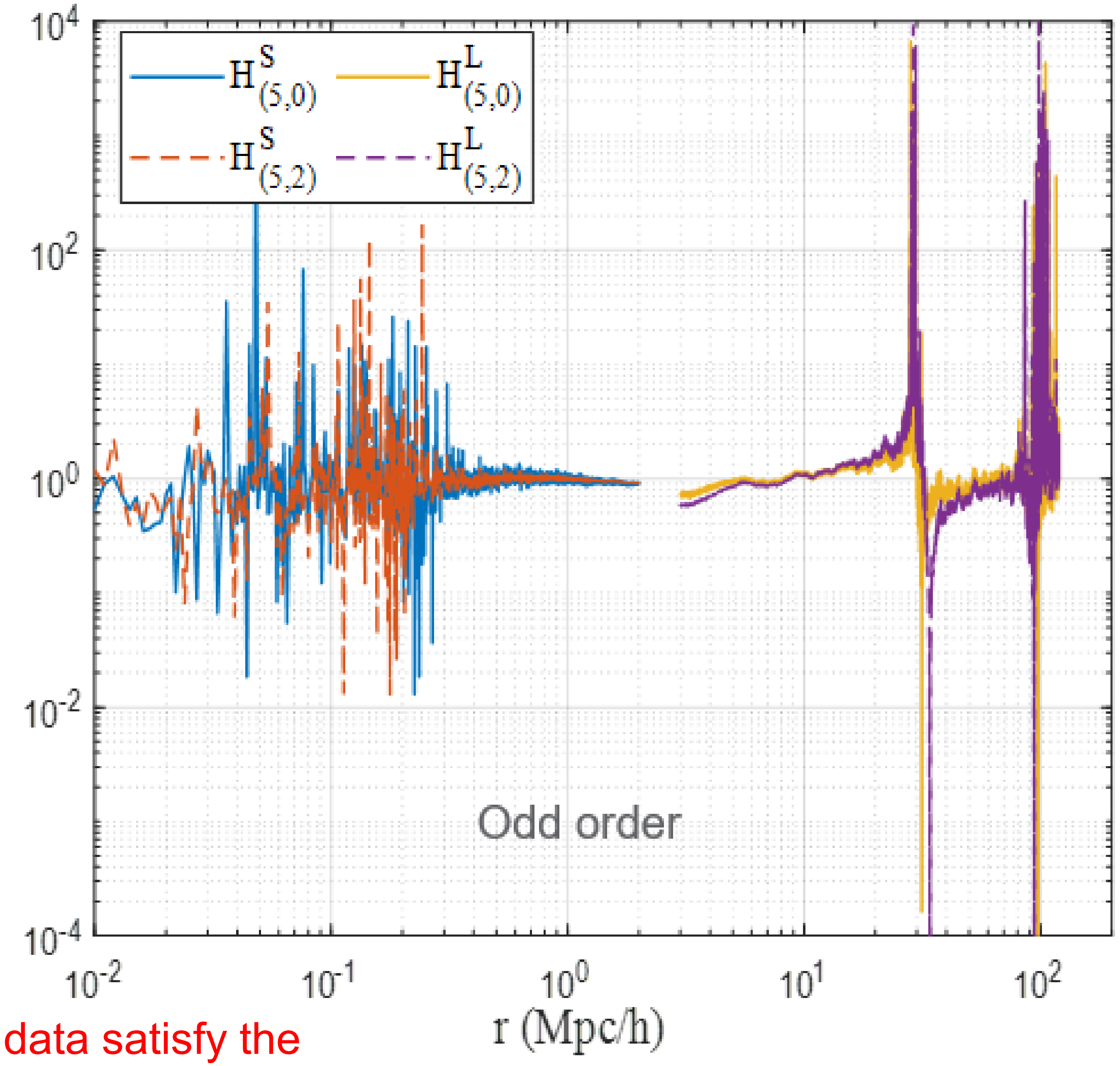
On large scale, kinematic relations for irrotational flow:

$$H_{(p,q)}^{L}\left(r\right)=\frac{1}{2r^{p-q+1}T_{(p,q)}}\int_0^r\left[\left(p-q\right)R_{(p,q+1)}-\left(p-q-2\right)L_{(p,q+2)}\right]r^{p-q}dr=1$$

- To validate kinematic relations with N-body data, we need to construct equivalent relations.
- Extract high order correlation functions from N-body simulation data
- Dark matter flow is of constant divergence on small scale and irrotational on large scale
- Check the equivalent kinematic relations against simulation data

Kinematic relations validated by N-body simulations





N-body simulation data satisfy the kinematic relations.

Dynamic relations from dynamics on large scale

- Kinematic relations are relations between correlation and structure functions of the same order;
- Dynamic relations are relations between correlation functions of different orders and can only be obtained from the self-closed dynamic evolution of velocity.
- However, closure problem is well known for Jeans' equations which are not self-closed.
- Self-closed dynamic equations of velocity must be introduced on small and large scale.
- Dynamic equations are subsequently converted into dynamic relations.

Self-closed adhesion approximation on large scale : $\nabla \times \mathbf{v} = 0$

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}\mathbf{v} \cdot \nabla \mathbf{v} = c(a)\mathbf{v} + \nu(a)\nabla^2 \mathbf{v} \xrightarrow{\text{Neglect second order}} \frac{\partial \mathbf{v}}{\partial t} = c(a)\mathbf{v}$$

Damping

"Artificial " viscosity

Neglect second order

Zeldovich approximation

Using identity:

$$\mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{2}\nabla(\mathbf{u} \cdot \mathbf{u}) + (\nabla \times \mathbf{u}) \times \mathbf{u}$$

$$c(a) = \left(\frac{4\pi G \rho_0}{Hf(\Omega_m)} - H\right) = \frac{1}{2}H$$

Matter dominant

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2a}\nabla(\mathbf{v} \cdot \mathbf{v}) = c(a)\mathbf{v} + \nu(a)\nabla^2 \mathbf{v}$$

$$\frac{\partial v_j}{\partial t} + \frac{1}{2a}\frac{\partial(v_i v_i)}{\partial x_j} = cv_j + \nu\nabla^2 v_j \quad \times v_i'$$

Index Eq. at location x

$$+ \quad \frac{\partial v_i'}{\partial t} + \frac{1}{2a}\frac{\partial(v_j' v_j')}{\partial x_i'} = cv_i' + \nu\nabla'^2 v_i' \quad \times v_j$$

Index Eq. at location x'

$$= \frac{\partial\langle v_j v_i'\rangle}{\partial t} + \frac{1}{2a}\left\langle v_i' \frac{\partial(v_k v_k)}{\partial x_j} + v_j \frac{\partial(v_k' v_k')}{\partial x_i'}\right\rangle = c\langle v_j v_i' + v_i' v_j\rangle + \nu\nabla^2\langle v_j v_i' + v_i' v_j\rangle$$

Dynamic relations from dynamics on large scale

Time evolution of the second order correlation tensor Q_{ij}:

$$\frac{\partial Q_{ij}}{\partial t} = \frac{1}{2a}\left(\frac{\partial Q_{kki}}{\partial r_j} + \frac{\partial Q_{kkj}}{\partial r_i}\right) + 2cQ_{ij} + 2\nu\nabla^2 Q_{ij} \quad \times\delta_{ij}$$

Time evolution of the second order correlation function R_2:

$$\frac{\partial R_2}{\partial t} = \boxed{2\Gamma(r)} + 2cR_2 + 2\nu\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_2}{\partial r}\right)\right)$$

Fourier transform: ↓ E_u: Energy spectrum

$$\frac{\partial E_u}{\partial t} = \boxed{T(k,t)} + 2cE_u(k,t) - 2\nu k^2 E_u(k,t)$$

$$\Gamma(r) = \frac{1}{2a}\frac{\partial Q_{kki}}{\partial r_i} = \frac{1}{2ar^2}\left(r^2 R_{31}\right)_{,r}$$

← Real-space energy transfer function

$$T(k) = \frac{2}{\pi}\int_0^\infty \Gamma(r)\, kr\sin(kr)\, dr$$

← Spectral energy transfer function

$$L_{(3,2)}(r) = R_{31}(r) = -2a\nu\frac{\partial R_2}{\partial r}$$

Dynamic relation between 2nd and 3rd correlation functions

Density correlation:

$$\xi(r) = -\frac{1}{\left(aHf(\Omega_m)\right)^2}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R_2}{\partial r}\right)\right]$$

$$\Gamma(r) = \nu\left(aHf(\Omega_m)\right)^2\xi(r) = \frac{\nu a_0 u^2}{rr_2}\exp\left(-\frac{r}{r_2}\right)\left[\left(\frac{r}{r_2}\right)^2 - 7\left(\frac{r}{r_2}\right) + 8\right]$$

Third order correlation:

$$R_{31} = \left\langle u^2 u_L^{'}\right\rangle = -\nu Ha^2 f(\Omega_m)^2\left\langle \Delta u_L\right\rangle = -\frac{2a_0u^2a\nu}{r_2}\exp\left(-\frac{r}{r_2}\right)\left(\frac{r}{r_2} - 4\right)$$

$$T(k) = a_0u^2\frac{16\nu}{\pi r_2}\frac{1}{\left(1 + 1/(kr_2)^2\right)^3}$$

Modeling high order correlation functions on large scale

The same model can be generalized to high order correlation functions:

$$L_{(3,2)} = R_{31} = \left\langle u^2 u_L^{'} \right\rangle = a_3 u^3 \exp\left(-\frac{r}{r_2}\right)\left(\frac{r}{r_2} - b_3\right)$$

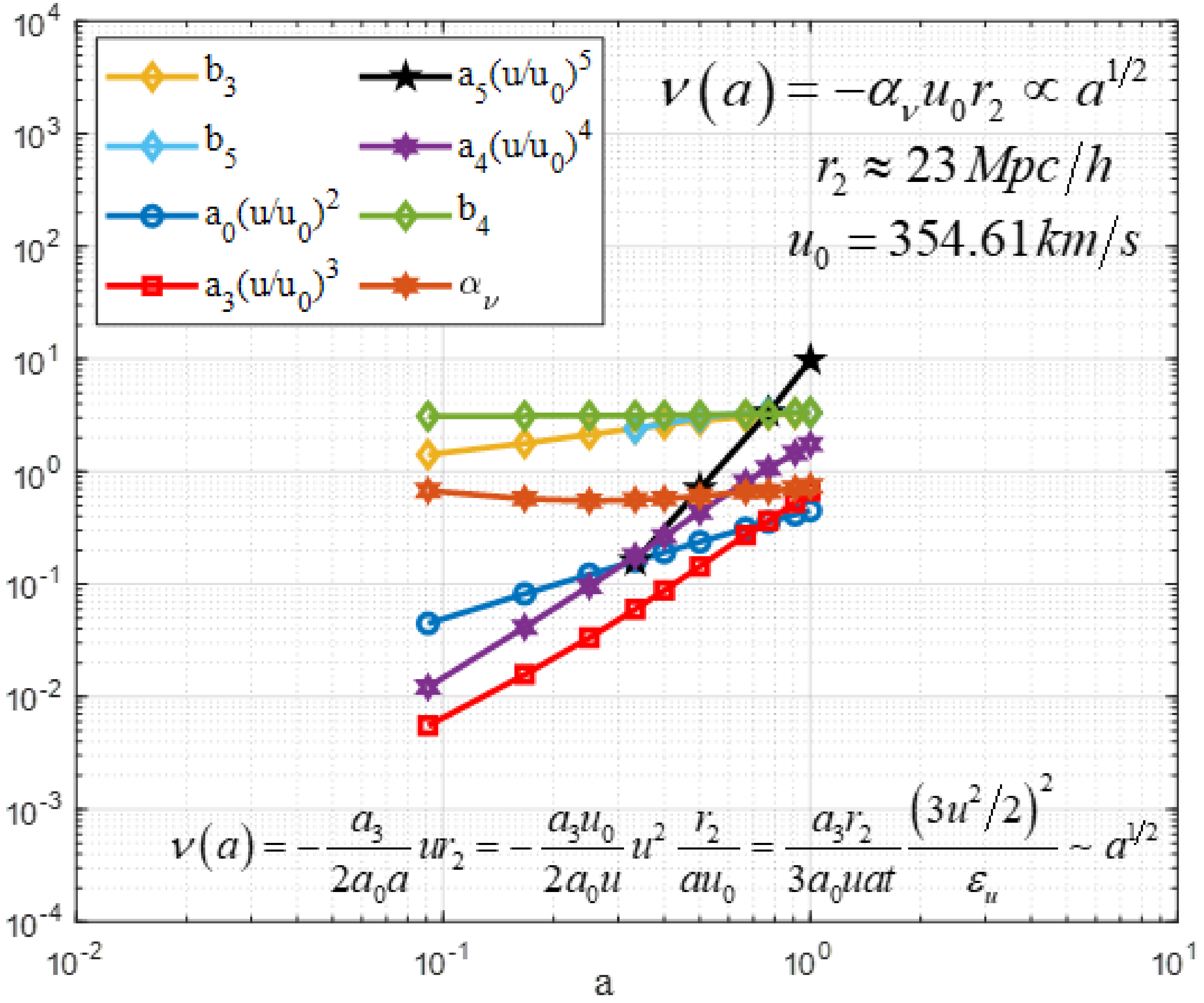
$$R_{(4,3)} = \left\langle u^2 \mathbf{u} \cdot \mathbf{u}^{'} \right\rangle = a_4 u^4 \exp\left(-\frac{r}{r_2}\right)\left(b_4 - \frac{r}{r_2}\right)$$

$$L_{(5,4)} = \left\langle u^4 u_L^{'} \right\rangle = a_5 u^5 \exp\left(-\frac{r}{r_2}\right)\left(\frac{r}{r_2} - b_5\right)$$

Generalize to any order correlation functions:

$$L_{(q+1,q)} = \left\langle u^q u_L^{'} \right\rangle \propto u^q \left\langle u_L^{'} \right\rangle \propto \left(\nu H a^2\right)^{q/2} L_{(1,0)} \propto a^{(q+3)/2}$$

$$R_{(q,q-1)} = \left\langle u^{q-2} \mathbf{u} \cdot \mathbf{u}^{'} \right\rangle \propto u^{q-2} \left\langle \mathbf{u} \cdot \mathbf{u}^{'} \right\rangle \propto \left(\nu H a^2\right)^{(q-2)/2} R_{(2,1)} \propto a^{q/2}$$



$$\nu(a) = -\alpha_\nu u_0 r_2 \propto a^{1/2}$$

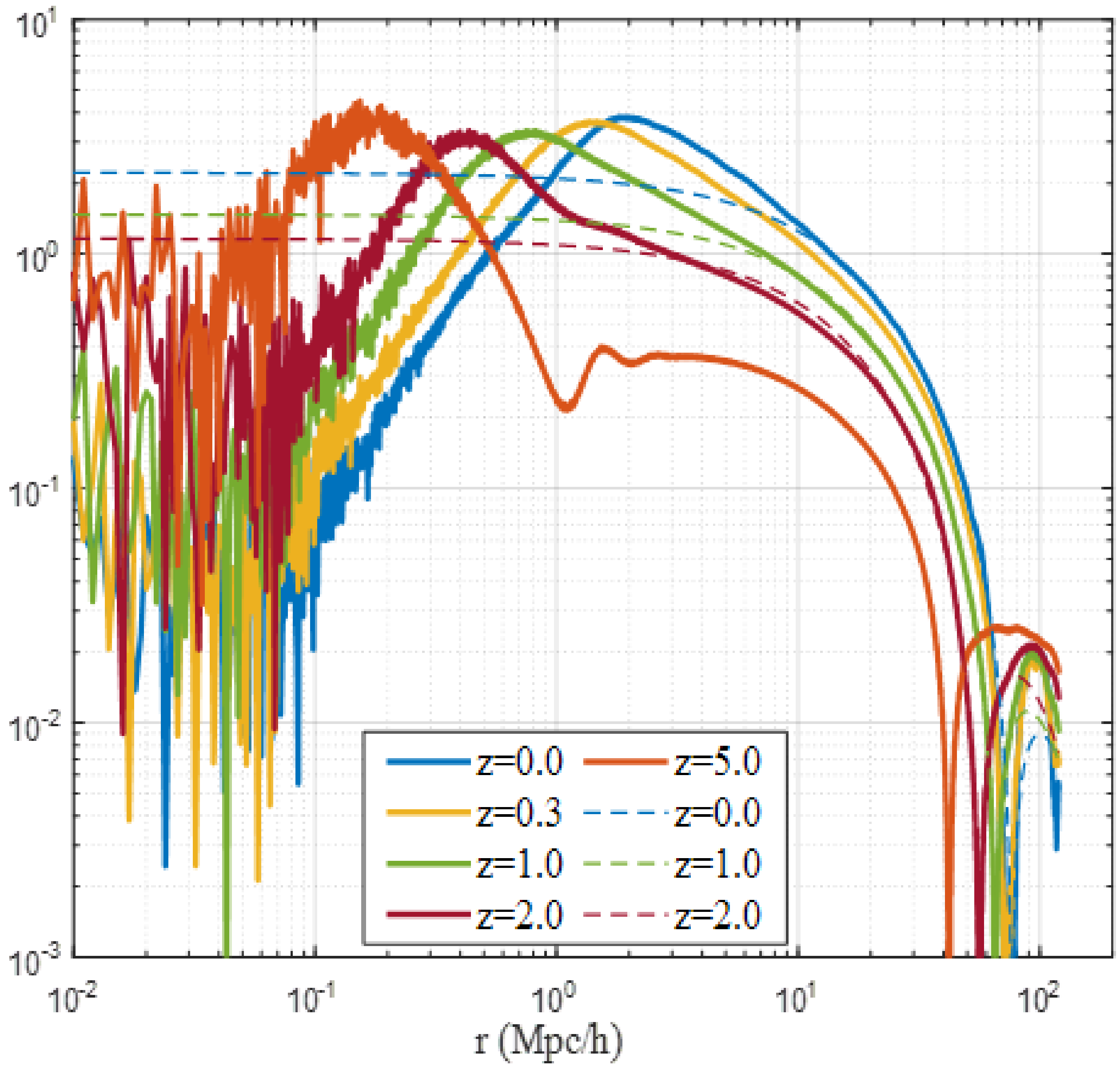
$$r_2 \approx 23\,Mpc/h$$

$$u_0 = 354.61\,km/s$$

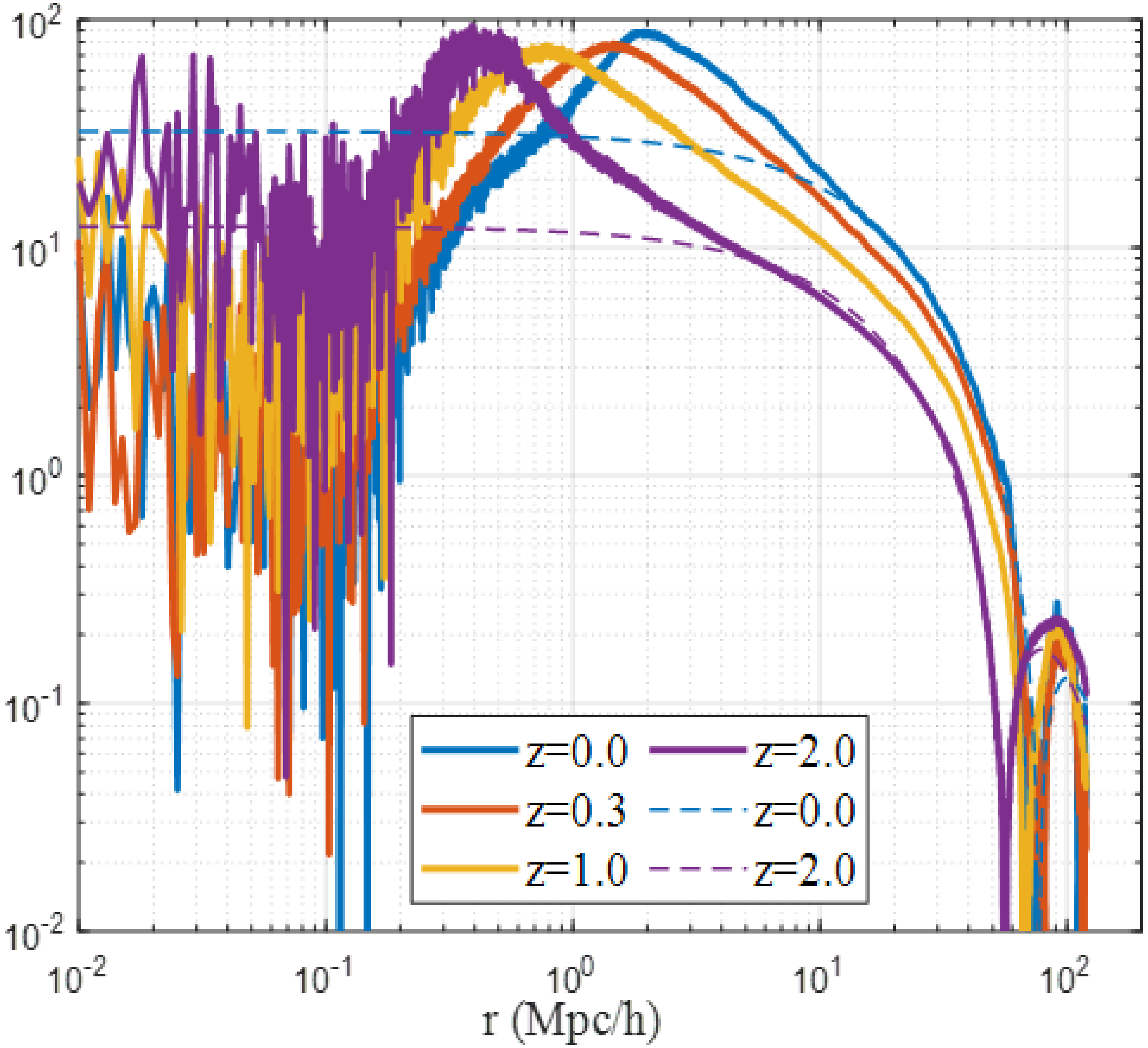
$$\nu(a) = -\frac{a_3}{2a_0 a} u r_2 = -\frac{a_3 u_0}{2a_0 u} u^2 \frac{r_2}{a u_0} = \frac{a_3 r_2}{3a_0 u a t} \frac{\left(3u^2/2\right)^2}{\varepsilon_u} \sim a^{1/2}$$

V(a) is artificial viscosity

Modeling high order correlation functions on large scale



Two-point third order velocity correlation $L_{(3,2)}$



Two-point fifth order velocity correlation $L_{(5,4)}$

Dynamic relations from dynamics on large scale

$$\langle\theta\rangle=\langle\nabla\cdot\mathbf{u}\rangle=\frac{1}{2r^2}\left(r^2\langle\Delta u_L\rangle\right)_{,r}$$ ⬅ Kinematic relation

From pair conservation equation:

$$\langle\Delta u_L\rangle\approx-\frac{2}{3}Har\bar{\xi}(r,a)=-\frac{2Ha}{r^2}\int_0^r\xi(y)y^2dy$$

⬇

$$\boxed{\begin{aligned}&\langle\theta\rangle=\langle\nabla\cdot\mathbf{u}\rangle=-Ha\xi(r)\\&\langle\theta\rangle=\langle\nabla\cdot\mathbf{u}\rangle=-aHf(\Omega_m)\langle\delta\rangle\end{aligned}}$$

Dynamic equation on large scale

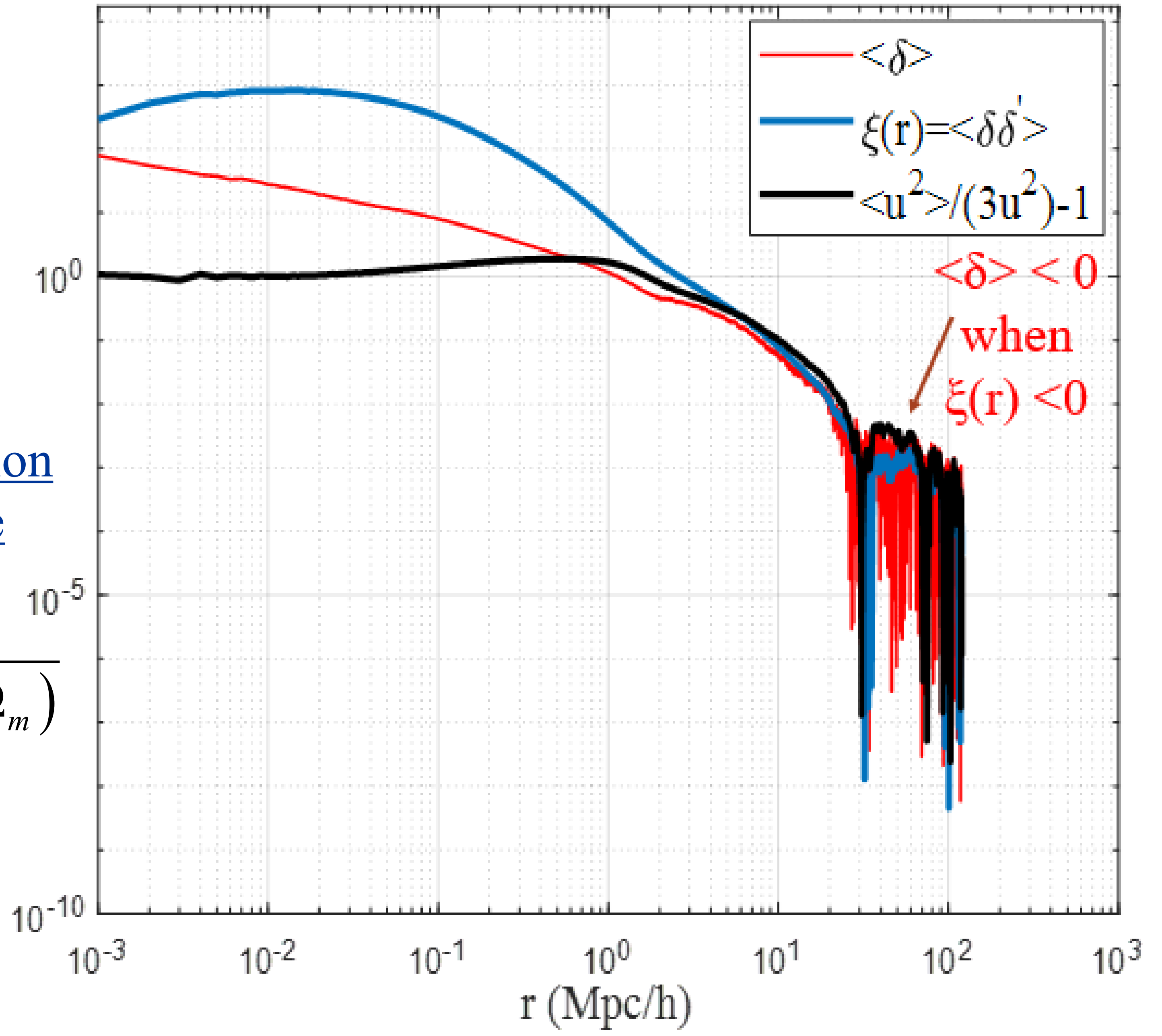
⬇

⬅ $$\delta\approx\eta=-\frac{\nabla\cdot\mathbf{u}}{aHf(\Omega_m)}$$

⬇

$$f(\Omega_m)\langle\delta\rangle=f(\Omega_m)\langle\delta+\delta'\rangle/2=\xi(r)=\langle\delta\delta'\rangle$$

On large scale, mean density at two locations is proportional to density correlation on the same scale



Dynamic relations from dynamics on large scale

Use dynamic equations at locations x and x':

$$\frac{\partial v_j}{\partial t}+\frac{1}{2a}\frac{\partial\left(v_i v_i\right)}{\partial x_j}=cv_j+\nu\nabla^2 v_j \quad \times\hat{r}_j$$

$$-\quad \frac{\partial v_i^{'}}{\partial t}+\frac{1}{2a}\frac{\partial\left(v_j^{'} v_j^{'}\right)}{\partial x_i^{'}}=cv_i^{'}+\nu\nabla^{'2} v_i^{'} \quad \times\hat{r}_i$$

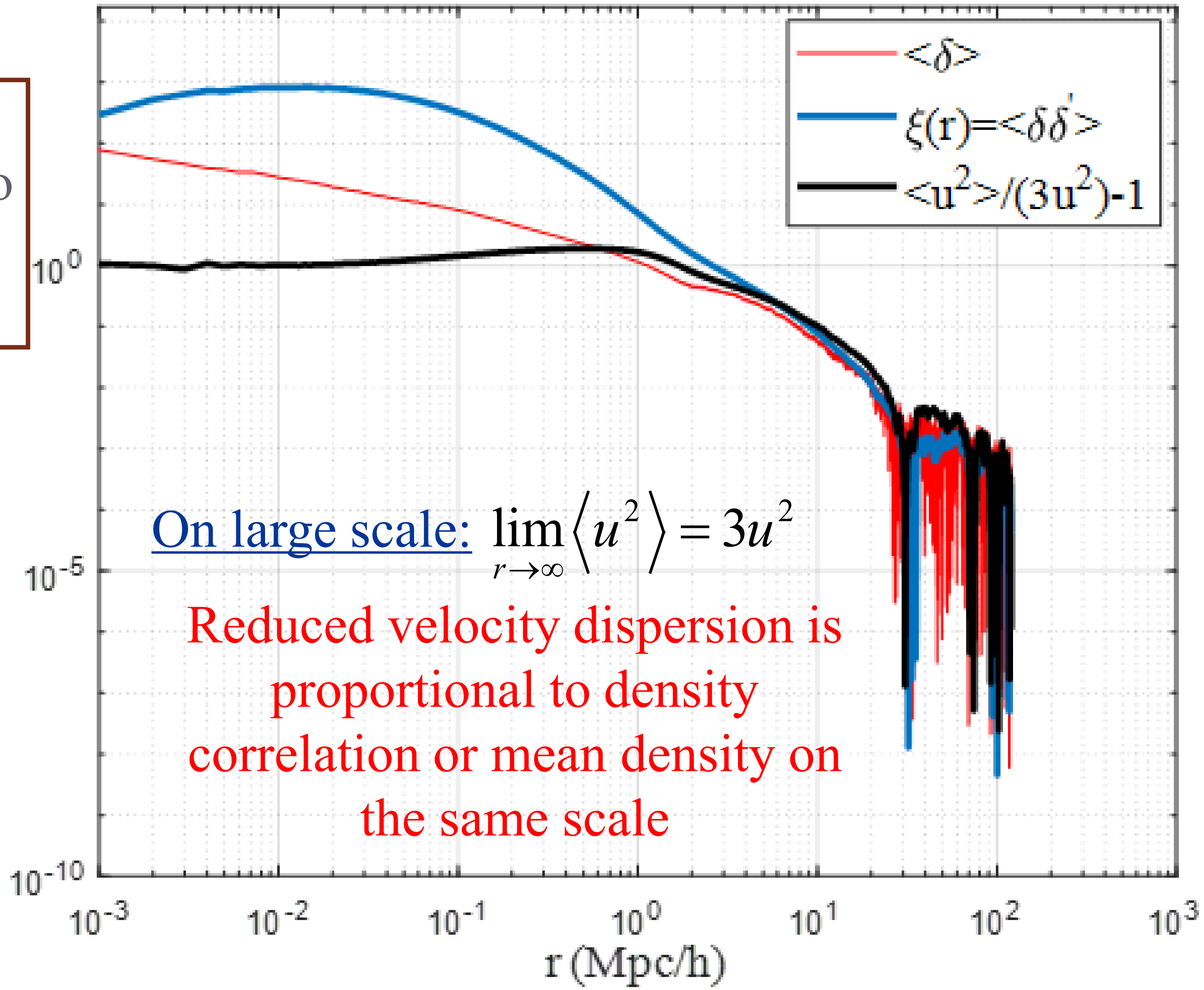
Unit vector between two particles $\hat{\mathbf{r}}=\mathbf{r}/r$

$$=\left\langle \hat{r}_i\frac{\partial v_i^{'}}{\partial t}-\hat{r}_j\frac{\partial v_j}{\partial t}\right\rangle+\frac{1}{a}\frac{\partial\left\langle u^2\right\rangle}{\partial r}=c\left\langle \Delta u_L\right\rangle+2\nu\frac{\partial\left\langle\theta\right\rangle}{\partial r}$$

$$\frac{\partial\mathbf{v}}{\partial t}=c\left(a\right)\mathbf{v} \Rightarrow \begin{array}{l}\hat{r}_i\dfrac{\partial v_i}{\partial t}=c\left(a\right)\hat{r}_i v_i=c\left(a\right)u_L\\ \hat{r}_i\dfrac{\partial v_i^{'}}{\partial t}=c\left(a\right)\hat{r}_i v_i^{'}=c\left(a\right)u_L^{'}\end{array}$$

Use $\left\langle\theta\right\rangle=\left\langle\nabla\cdot\mathbf{u}\right\rangle=-Ha\xi\left(r\right)$ and $f\left(\Omega_m\right)\left\langle\delta\right\rangle=\xi\left(r\right)$

$$\frac{\partial\left\langle u^2\right\rangle}{\partial r}=2\nu a\frac{\partial\left\langle\theta\right\rangle}{\partial r}=-2\nu a^2 Hf\left(\Omega_m\right)\frac{\partial\left\langle\delta\right\rangle}{\partial r}=-2\nu Ha^2\frac{\partial\xi\left(r\right)}{\partial r} \Rightarrow \frac{\left\langle u^2\right\rangle}{3u^2}-1=-\frac{2\nu Ha^2\xi\left(r\right)}{3u^2}=-\frac{2\nu Ha^2}{3u^2}f\left(\Omega_m\right)\left\langle\delta\right\rangle$$



On large scale: $\lim_{r\to\infty}\left\langle u^2\right\rangle=3u^2$

Reduced velocity dispersion is proportional to density correlation or mean density on the same scale

Divergence of velocity on all scales

Kinematic relation (good for all scales):

$$\langle\theta\rangle = \langle\nabla\cdot\mathbf{u}\rangle = \frac{1}{2r^2}\left(r^2\langle\Delta u_L\rangle\right)_{,r}$$

From pair conservation equation:
(for large scale)

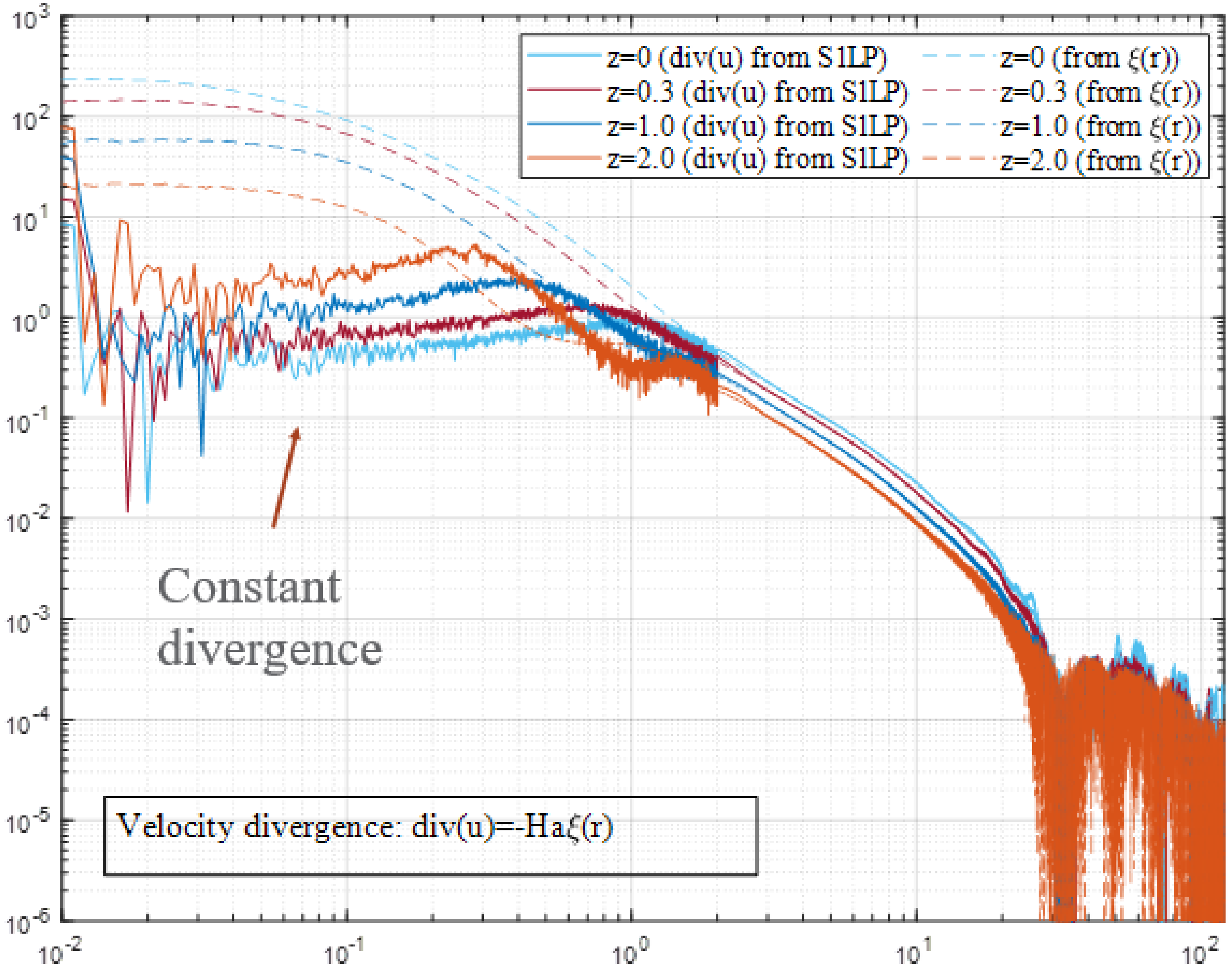
$$\langle\Delta u_L\rangle = -\frac{2Ha}{r^2}\int_0^r \xi(y)\,y^2dy$$

On large scale:

$$\langle\theta\rangle = \langle\nabla\cdot\mathbf{u}\rangle = -Ha\xi(r)$$

Dynamic equation on large scale

$$\delta = -\frac{\nabla\cdot\mathbf{u}}{aHf(\Omega_m)} = -\frac{\theta}{aHf(\Omega_m)}$$



Velocity divergence on different scales
(normalized by Ha)

Deriving exponential velocity correlation functions on large scale

- The exponential function was proposed for second order transverse velocity correlation T_2 on large scale.
- This is not a coincidence and must be deeply rooted in the dynamics and kinematics on large scale.

Velocity dispersion function for kinetic energy contained in all scales above r:

$$\sigma_u^2(r) = \frac{1}{3}\int_{-\infty}^{\infty} E_u(k) W(kr)^2 \, dk$$

$$W(x) = \frac{3}{x^3}\left[\sin(x) - x\cos(x)\right] = 3\frac{j_1(x)}{x}$$

Window function

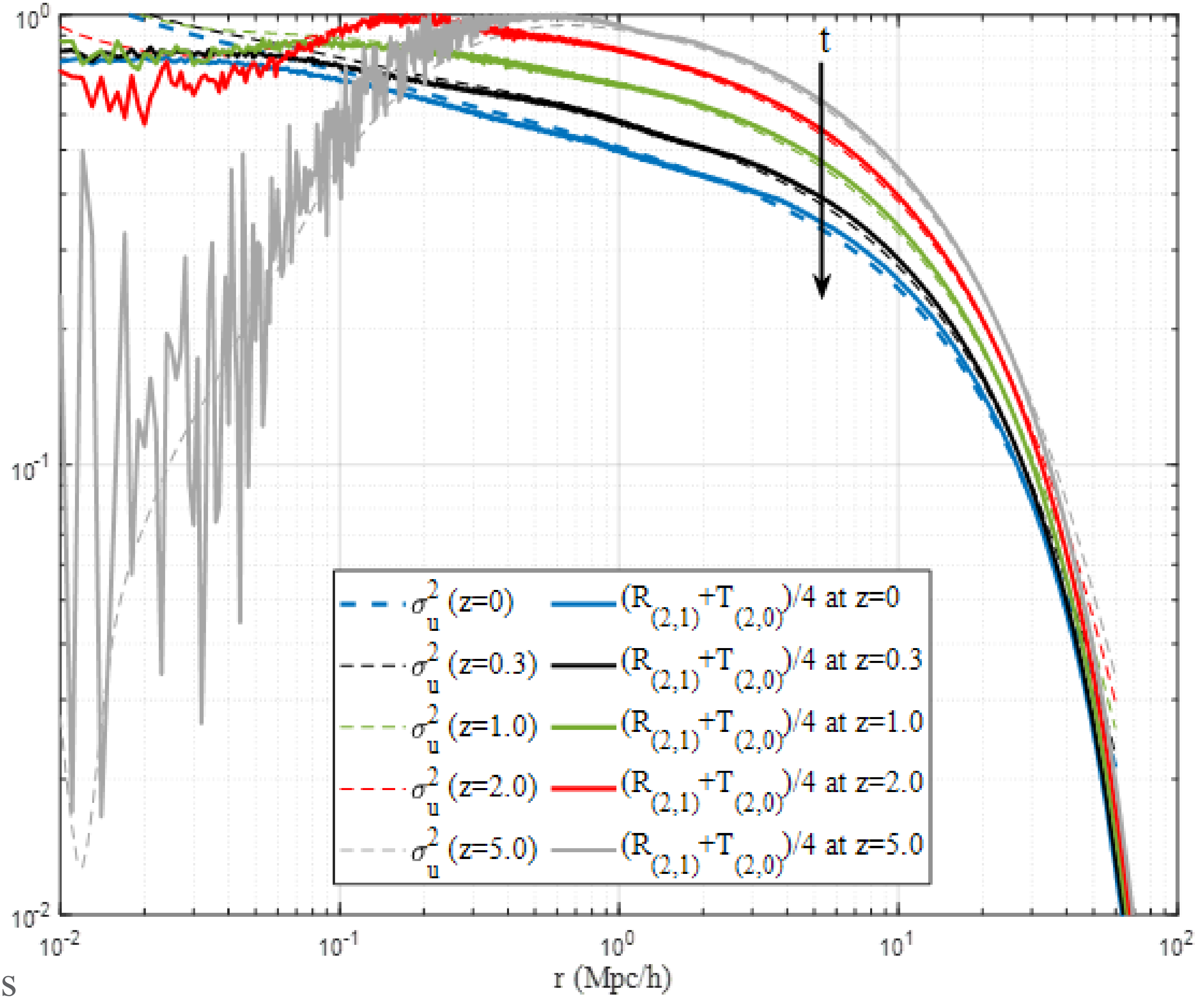
On large scale, velocity dispersion function can be approximated by:

$$\sigma_u^2(r) \approx \frac{1}{4}\left[R_{(2,1)}(r) + T_{(2,0)}(r)\right]$$

Relate to velocity correlation functions (Equipartition)

3 translational

1 rotational



Deriving exponential velocity correlation functions on large scale

On large scale velocity dispersion function can be approximated as,

$$\sigma_u^2(r) \approx \frac{1}{4}\left[R_{(2,1)}(r) + T_{(2,0)}(r)\right]$$

Relate to velocity correlation functions (Equipartition)

On large scale, the rate of energy cascade (m^2/s^3):

$$\Pi_u \propto \frac{\sigma_u^2(r)}{(ar)/u}$$

Kinetic energy in scales above r

Turnaround time for energy cascade

$$\Pi_u \propto \frac{\langle u^3 \rangle}{ar} \propto \frac{L_{(3,2)}(r)}{ar}$$

⬇

$$L_{(3,2)}(r) \propto u\sigma_u^2(r)$$

From dynamic relation on large scale:

$$L_{(3,2)}(r) = -2a\nu \frac{\partial R_{(2,1)}}{\partial r}$$

⬇

$$\frac{8\nu a}{\alpha_r u}\frac{\partial R_{(2,1)}}{\partial r} = \left[R_{(2,1)}(r) + T_{(2,0)}(r)\right]$$

From kinematic relation on large scale for irrotational flow:

$$R_{(2,1)} = \frac{1}{r^2}\left(r^3 T_{(2,0)}\right)_{,r}$$

⬇

Exponential second order transverse correlation function:

$$T_{(2,0)} = Const \cdot \exp\left(-\frac{r}{r_2}\right) \quad \text{with} \quad r_2 = -\frac{8\nu a}{\alpha_r u}$$

Deriving power-law velocity correlation functions on small scale

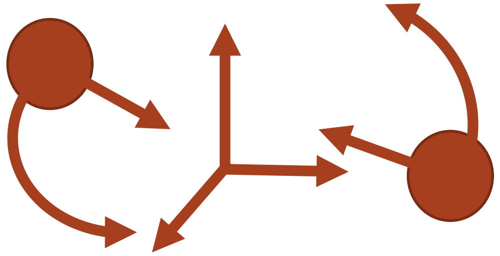
- Similar idea can be applied to determine the power-law exponent of correlation functions on small scale
- On small scale, velocity dispersion function can be approximated as

$$\sigma_u^2(r) \approx \frac{1}{5}\left[R_{(2,1)}(r)+T_{(2,0)}(r)+L_{(2,0)}(r)\right]$$

3 translational

1 internal rotational (two-body is planar)

1 internal longitudinal relative motion



$$S_2^l = 2u^2\left(r/r_1\right)^n$$

Power-law that can be related to virial theorem

From kinematic relations on small scale:

$$L_2(r) = u^2 - \frac{S_2^l}{2} = u^2\left[1-\left(\frac{r}{r_1}\right)^n\right]$$

$$T_2 = \frac{1}{2r}\left(r^2 L_2\right)_{,r} = u^2\left[1-\frac{2+n}{2}\left(\frac{r}{r_1}\right)^n\right]$$

$$R_2 = \frac{1}{r^2}\left(r^3 L_2\right)_{,r} = u^2\left[3-(3+n)\left(\frac{r}{r_1}\right)^n\right]$$

See slides

$$\sigma_d^2(r) = u^2 - \sigma_u^2(r) \Rightarrow \boxed{\sigma_d^2(r) = \left(1+\frac{3}{10}n\right)u^2\left(\frac{r}{r_1}\right)^n \qquad \sigma_d^2(r) = \frac{24\cdot 2^n}{(4+n)(6+n)}u^2\left(\frac{r}{r_1}\right)^n}$$

See slides

n =0.27 ≈ ¼, the one-forth law on small scale

Dynamic relations from dynamics on small scale

- Self-closed equations for velocity evolution on small scale seems not exist.
- we will first formulate the self-close equations for velocity on small scale.
- These equations are subsequently applied to derive the dynamic relations on small scale.

Jeans equation (not self-closed):

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v}+H\mathbf{v}=-\frac{1}{a}\frac{\nabla\cdot\mathbf{p}}{\rho}-\frac{1}{a}\nabla\phi$$

$$\mathbf{p}=\rho\boldsymbol{\sigma}^2$$

Stress tensor

Velocity dispersion tensor

- γ=1/2 for small scale dynamic equation.
- γ=1 for large scale dynamic equation.

Decompose total velocity into halo velocity and velocity in halos

$$\mathbf{v}(\mathbf{x},t)=\mathbf{v}_h(\mathbf{x}_h,t)+\mathbf{v}_v(\mathbf{r},t)$$

Decompose velocity in halos into radial and azimuthal flow

$$\mathbf{v}_v=\mathbf{v}_r+\mathbf{v}_\varphi$$

Polar flow is neglected

Self-closed description of mean flow (derivation skipped):

$$\nabla\cdot\mathbf{v}=\theta(t)$$

Four equations and four unknowns

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v}+H\mathbf{v}=-\frac{1}{a}\nabla\phi^*+\gamma\frac{1}{a}\underbrace{(\nabla\times\mathbf{v})\times\mathbf{v}}_{1}$$

Centripetal acceleration, significant on small scale

$$\Updownarrow$$

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}(1-\gamma)\mathbf{v}\cdot\nabla\mathbf{v}+\frac{\gamma}{2a}\nabla(\mathbf{v}\cdot\mathbf{v})+H\mathbf{v}=-\frac{1}{a}\nabla\phi^*$$

$$\Updownarrow$$

$$\frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}(1-\gamma)\nabla\cdot(\mathbf{v}\otimes\mathbf{v})+\frac{\gamma}{2a}\nabla(\mathbf{v}\cdot\mathbf{v})+\left[H-\frac{1}{a}(1-\gamma)\theta\right]\mathbf{v}=-\frac{1}{a}\nabla\phi^*$$

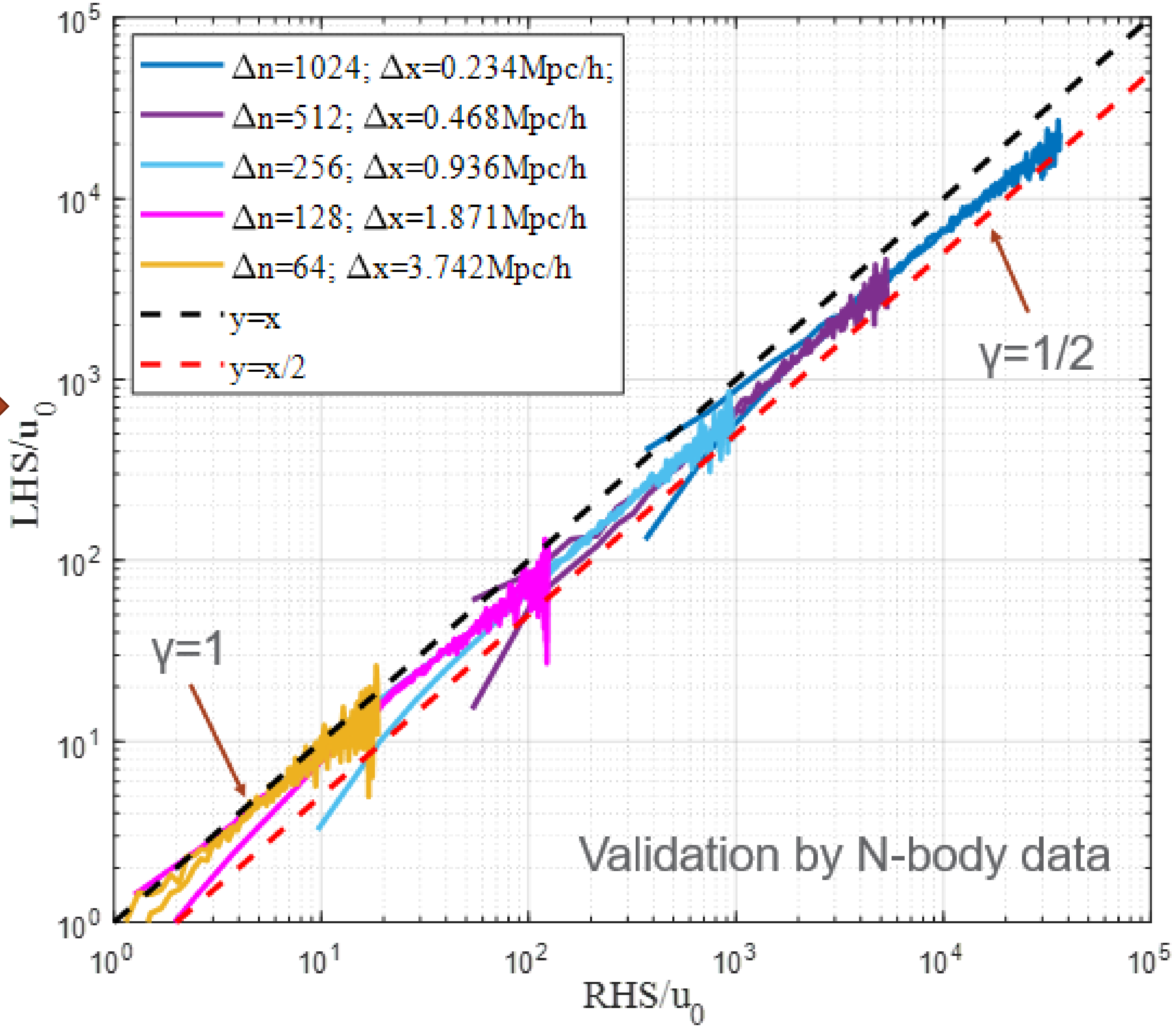
Self-closed description of dynamics

Taking curl on both sides:

$$\nabla\times\quad \frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v}+H\mathbf{v}=-\frac{1}{a}\nabla\phi^{*}+\gamma\frac{1}{a}\underbrace{(\nabla\times\mathbf{v})\times\mathbf{v}}_{1}$$

Equation for vorticity: $\boldsymbol{\omega}=\nabla\times\mathbf{v}$

$$\underbrace{\frac{\partial \boldsymbol{\omega}}{\partial t}+\frac{1}{a}\nabla\times(\mathbf{v}\cdot\nabla\mathbf{v})+H\boldsymbol{\omega}}_{LHS}=\gamma\underbrace{\frac{1}{a}\nabla\times[\boldsymbol{\omega}\times\mathbf{v}]}_{RHS}$$



Validation by N-body data

- On large scale (large grid size Δx), γ≈1
- On small scale (small grid size Δx), γ≈1/2.
- There is a transition between the two regimes.

Averaged dynamic equations for velocity and the origin of effective viscosity

With the self-closed description of velocity, we can derive the effective equations for mean flow

Similar to Reynolds decomposition, decompose velocity and potential into mean and fluctuation in time,

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}' \quad \phi^* = \overline{\phi^*} + \phi^{*'}$$

Averaging is essentially a filtering process with a cutoff resolution to separate variables into resolved and unresolved parts

Substitute into the self-closed description:

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{a}(1-\gamma)\mathbf{v}\cdot\nabla\mathbf{v} + \frac{\gamma}{2a}\nabla(\mathbf{v}\cdot\mathbf{v}) + H\mathbf{v} = -\frac{1}{a}\nabla\phi^*$$

$$\frac{\partial \overline{\mathbf{v}}}{\partial t} + \frac{1}{a}(1-\gamma)\overline{\mathbf{v}}\cdot\nabla\overline{\mathbf{v}} + \frac{\gamma}{2a}\nabla(\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}) + H\overline{\mathbf{v}} = -\frac{1}{a}\nabla\overline{\phi^*} - \left(\frac{1-\gamma}{a}\underbrace{\overline{\mathbf{v}'\cdot\nabla\mathbf{v}'}}_{1} + \frac{\gamma}{2a}\underbrace{\overline{\nabla(\mathbf{v}'\cdot\mathbf{v}')}}_{2}\right)$$

$$\nabla\overline{\phi^*} = -3Ha\overline{\mathbf{v}}/2 \quad \text{and} \quad \gamma = 1$$

$$\frac{\partial \overline{\mathbf{v}}}{\partial t} + \frac{1}{2a}\nabla(\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}) = \frac{1}{2}Ha\overline{\mathbf{v}} - \frac{1}{2a}\nabla\left(\overline{\mathbf{v}'\cdot\mathbf{v}'}\right)$$

Compare to dynamic equation on large sale:

$$\frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2a}\nabla(\mathbf{v}\cdot\mathbf{v}) = c(a)\mathbf{v} + \nu(a)\nabla^2\mathbf{v}$$

$$-\frac{1}{2a}\nabla\left(\overline{\mathbf{v}'\cdot\mathbf{v}'}\right) = \nu\nabla^2\overline{\mathbf{v}} = \nu\nabla(\nabla\cdot\overline{\mathbf{v}})$$ Subgrid model

Force as the gradient of kinetic energy in unresolved fluctuation

Force from Newtonian law of viscosity for mean flow

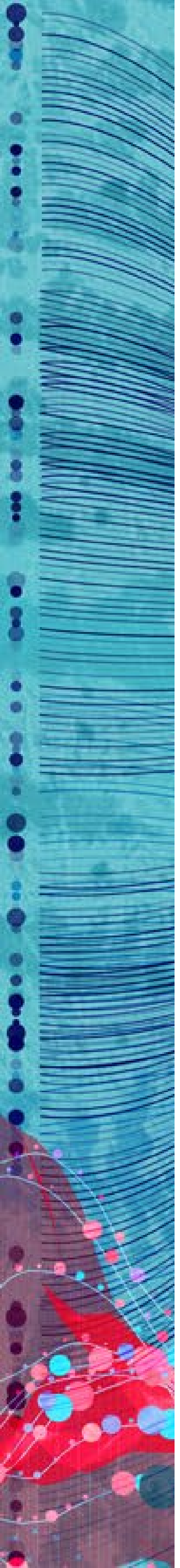
Divergence proportional to overdensity δ

The artificial viscosity on large scale origins from the unresolved velocity fluctuations

Use $\overline{\delta} = -\frac{\nabla\cdot\overline{\mathbf{v}}}{aHf(\Omega_m)}$ and integrate both sides of subgrid model

$$\overline{\mathbf{v}'^2} = F(t) + 2\nu a^2 Hf(\Omega_m)\overline{\delta}$$

The larger mean density (higher resolution), the smaller unresolved velocity fluctuations



Dynamic evolution of vorticity, enstrophy, and energy

Taking curl on both sides of self-closed description:

$$\nabla\times\quad \frac{\partial \mathbf{v}}{\partial t}+\frac{1}{a}(1-\gamma)\mathbf{v}\cdot\nabla\mathbf{v}+\frac{\gamma}{2a}\nabla(\mathbf{v}\cdot\mathbf{v})+H\mathbf{v}=-\frac{1}{a}\nabla\phi^*$$

Equation for vorticity: $\boldsymbol{\omega}=\nabla\times\mathbf{v}$

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+H\boldsymbol{\omega}=\frac{1}{a}(\gamma-1)\nabla\times(\mathbf{v}\cdot\nabla\mathbf{v})$$

Dynamic evolution of vorticity:

$$\frac{\partial\boldsymbol{\omega}}{\partial t}+\frac{1-\gamma}{a}\underbrace{\mathbf{v}\cdot\nabla\boldsymbol{\omega}}_{1}+\left[1+(1-\gamma)\frac{\theta}{Ha}\right]\underbrace{H\boldsymbol{\omega}}_{2}=\frac{1-\gamma}{a}\underbrace{\boldsymbol{\omega}\cdot\nabla\mathbf{v}}_{3}$$

1: Transport of vorticity

2: Destroy of vorticity on large scale

3: Generation of vorticity on small scale

Dynamic evolution of enstrophy:

$$\frac{\partial\boldsymbol{\omega}^2/2}{\partial t}+\frac{1-\gamma}{a}\underbrace{\mathbf{v}\cdot\nabla\frac{\boldsymbol{\omega}^2}{2}}_{1}+\left[1+(1-\gamma)\frac{\theta}{Ha}\right]\underbrace{H\boldsymbol{\omega}^2}_{2}=\frac{1-\gamma}{a}\underbrace{\boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla\mathbf{v})}_{3}$$

Taking scalar product on both sides:

$$\mathbf{v}\cdot\quad \frac{\partial\mathbf{v}}{\partial t}+\frac{1}{a}\mathbf{v}\cdot\nabla\mathbf{v}+H\mathbf{v}=-\frac{1}{a}\nabla\phi^*+\gamma\frac{1}{a}\underbrace{(\nabla\times\mathbf{v})\times\mathbf{v}}_{1}$$

$$\frac{\partial\mathbf{v}^2/2}{\partial t}=-\frac{1}{a}\nabla\cdot\left[\left(\frac{1}{2}\mathbf{v}^2+\phi^*\right)\mathbf{v}\right]-H\mathbf{v}^2+\frac{1}{a}\left(\frac{1}{2}\mathbf{v}^2+\phi^*\right)\nabla\cdot\mathbf{v}$$

Specific kinetic energy:

$$K=\int_V\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\,dV$$

Total energy:

$$E=\frac{1}{2}\mathbf{v}^2+\phi^*$$

Virial relation:

$$\int_V\left(2\mathbf{v}^2+\beta\phi^*\right)dV=0$$

Dynamic evolution of energy E at different location:

$$\nabla^2E+Ha\theta\left(1+\frac{\partial\ln\theta}{\partial\ln a}\right)=(1-\gamma)\left(\left[\mathbf{v}\cdot\left(\nabla^2\mathbf{v}-\nabla\theta\right)+\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right]\right)$$

Decay on large scale

Velocity gradient

Rotational contribution

Dynamic relations from dynamics on small scale

Self-closed dynamic equations at two locations x and x’:

$$\frac{\partial v_i}{\partial t}+\frac{1-\gamma}{a}\frac{\partial\left(v_i v_k\right)}{\partial x_k}+\frac{\gamma}{2a}\frac{\partial\left(v_k v_k\right)}{\partial x_i}+\left[1-\frac{\left(1-\gamma\right)}{aH}\theta\right]Hv_i=-\frac{1}{a}\frac{\partial\phi^*}{\partial x_i}\qquad\times v_j^{'}$$

$$+\ \frac{\partial v_j^{'}}{\partial t}+\frac{1-\gamma}{a}\frac{\partial\left(v_j^{'}v_k^{'}\right)}{\partial x_k^{'}}+\frac{\gamma}{2a}\frac{\partial\left(v_k^{'}v_k^{'}\right)}{\partial x_j^{'}}+\left[1-\frac{\left(1-\gamma\right)}{aH}\theta\right]Hv_j^{'}=-\frac{1}{a}\frac{\partial\phi^{*'}}{\partial x_j^{'}}\qquad\times v_i$$

With self-closed dynamic equations on small scale, we are ready to covert it into dynamic relations. Same approach was applied for irrotational flow on large scale.

$$\frac{\partial Q_{ij}}{\partial t}+2\left[1-\frac{\left(1-\gamma\right)}{aH}\theta\right]HQ_{ij}=\frac{2-2\gamma}{a}\frac{\partial Q_{ikj}}{\partial r_k}+\frac{\gamma}{a}\frac{\partial Q_{kkj}}{\partial r_i}-\frac{1}{a}\left[\frac{\partial\left\langle\phi^* v_j^{'}\right\rangle}{\partial x_i}+\frac{\partial\left\langle\phi^{*'}v_i\right\rangle}{\partial x_j^{'}}\right]\qquad\times\delta_{ij}$$

$$\frac{\partial R_{(2,1)}}{\partial t}+2\left[1-\frac{\left(1-\gamma\right)}{aH}\theta\right]HR_{(2,1)}=\frac{1}{ar^2}\left[\frac{\partial}{\partial r}\left(r^2\left[\left(2-2\gamma\right)R_{(3,1)}+\gamma L_{(3,2)}\right]\right)\right]+\frac{2}{a}\theta\left\langle\phi^*\right\rangle$$

Dynamic relations between second and third order correlations on small scale

$$\frac{-\left\langle\phi^*\right\rangle}{u^2}=\frac{\left\langle u^2\right\rangle}{\beta^* u^2}=\underbrace{\frac{5}{u^2 r^3}\int_0^r R_{(2,1)}\left(y\right)y^2 dy}_{1}\underbrace{-\frac{1}{Haru^2}\left(R_{(3,1)}+\frac{1}{2}L_{(3,2)}\right)}_{2}$$

$$\left(R_{(3,1)}+\frac{1}{2}L_{(3,2)}\right)=-Hau^2r=\left\langle\Delta u_L\right\rangle u^2=\frac{4}{9}\varepsilon_u ar$$

Dynamic relations from dynamics on small scale

Dynamic relations:

$$\left(R_{(3,1)} + \frac{1}{2}L_{(3,2)}\right) = -Hau^2 r = \left\langle \Delta u_L \right\rangle u^2 = \frac{4}{9}\varepsilon_u a r$$

$\Downarrow$

GSCH:

$$\left\langle \left(\Delta u_L\right)^3 \right\rangle = 3\left\langle \left(\Delta u_L\right)^2 \right\rangle \left\langle \Delta u_L \right\rangle \approx 6u^2 \left\langle \Delta u_L \right\rangle$$

$\Downarrow$

$$\left(R_{(3,1)} + \frac{1}{2}L_{(3,2)}\right) = \frac{1}{6}\left\langle \left(\Delta u_L\right)^3 \right\rangle$$

$\Downarrow$

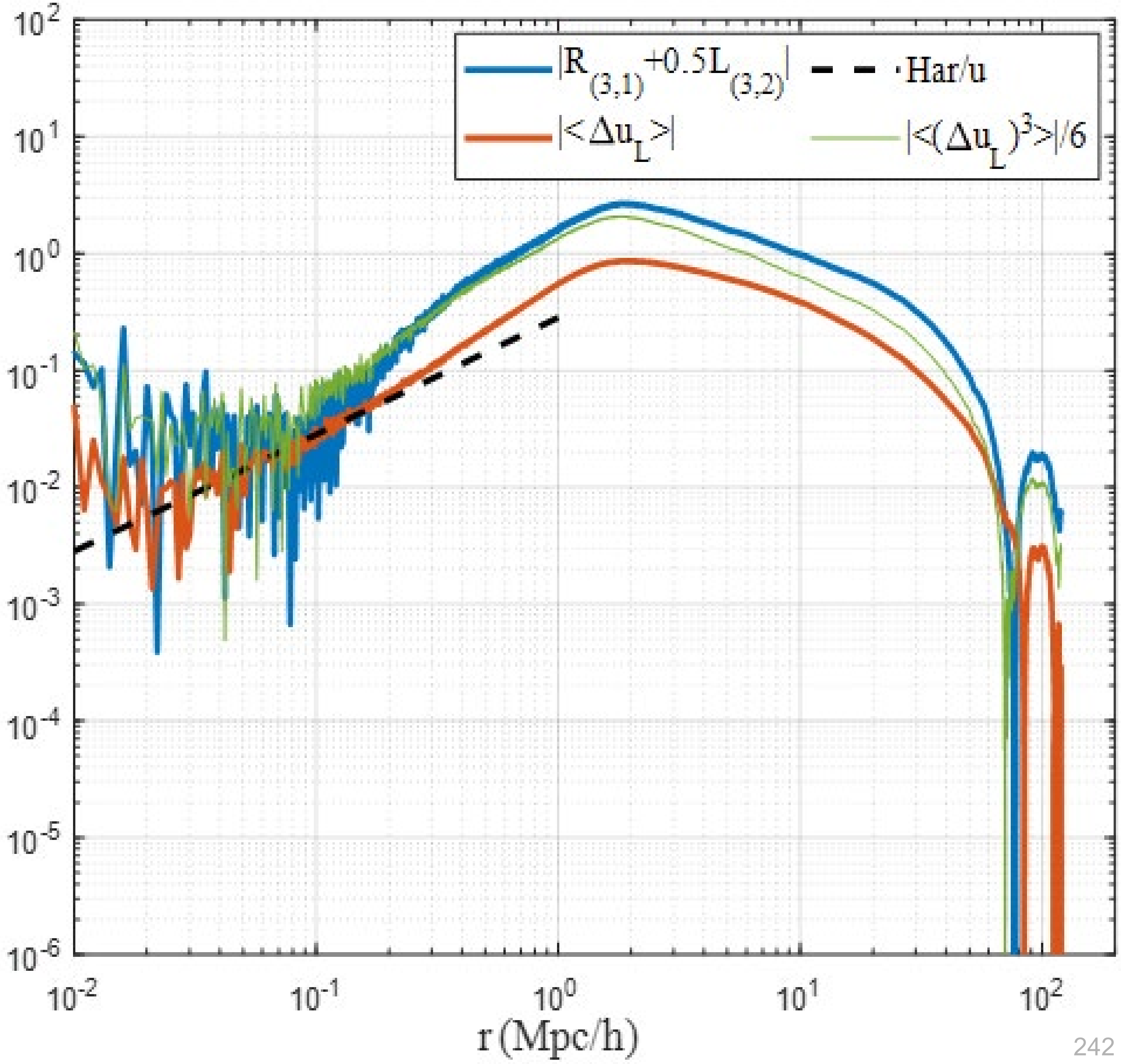
$$\left\langle \left(\Delta u_L\right)^3 \right\rangle = \frac{8}{3}\varepsilon_u a r \quad \Longleftrightarrow$$

$$\left\langle \left(\Delta u_L\right)^3 \right\rangle = -\frac{4}{5}\varepsilon_u r$$

For comparison, the four-fifths law for incompressible flow

$\Downarrow$

$$\varepsilon_u = \frac{3}{8}\frac{\left\langle \left(\Delta u_L\right)^3 \right\rangle}{ar}$$



Summary and keywords

Third order velocity correlation tensor	Vorticity, Energy and Enstrophy	Self-closed velocity equation
Effective viscosity	Kinematic relations	Dynamic relations

- Analogy between dark matter flow and homogeneous isotropic turbulence is established for development of statistical theory in terms of correlation, structure, dispersion, and spectrum functions;

- General kinematic relations for two-point velocity statistics are developed on small and large scales respectively;

- On large scale, the redshift dependence of qth order velocity correlations follows $\sim a^{(q+2)/2}$ for odd q and $\sim a^{q/2}$ for even q; The overdensity is proportional to density correlation on the same scale, i.e. $\langle\delta\rangle=\langle\delta\delta'\rangle$; (Negative) Effective viscosity in adhesion model originates from velocity fluctuations.

- On small scale, self-closed description for velocity is developed such that the dynamic relation can be obtained, which can be validated by N-body simulation.