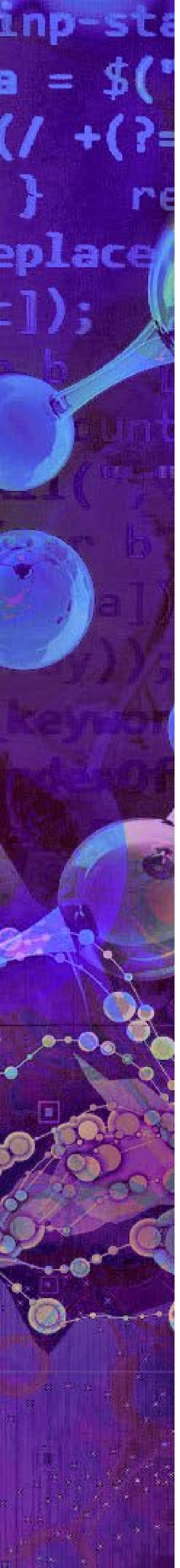

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A comparative study of dark matter flow & hydrodynamic turbulence and its applications

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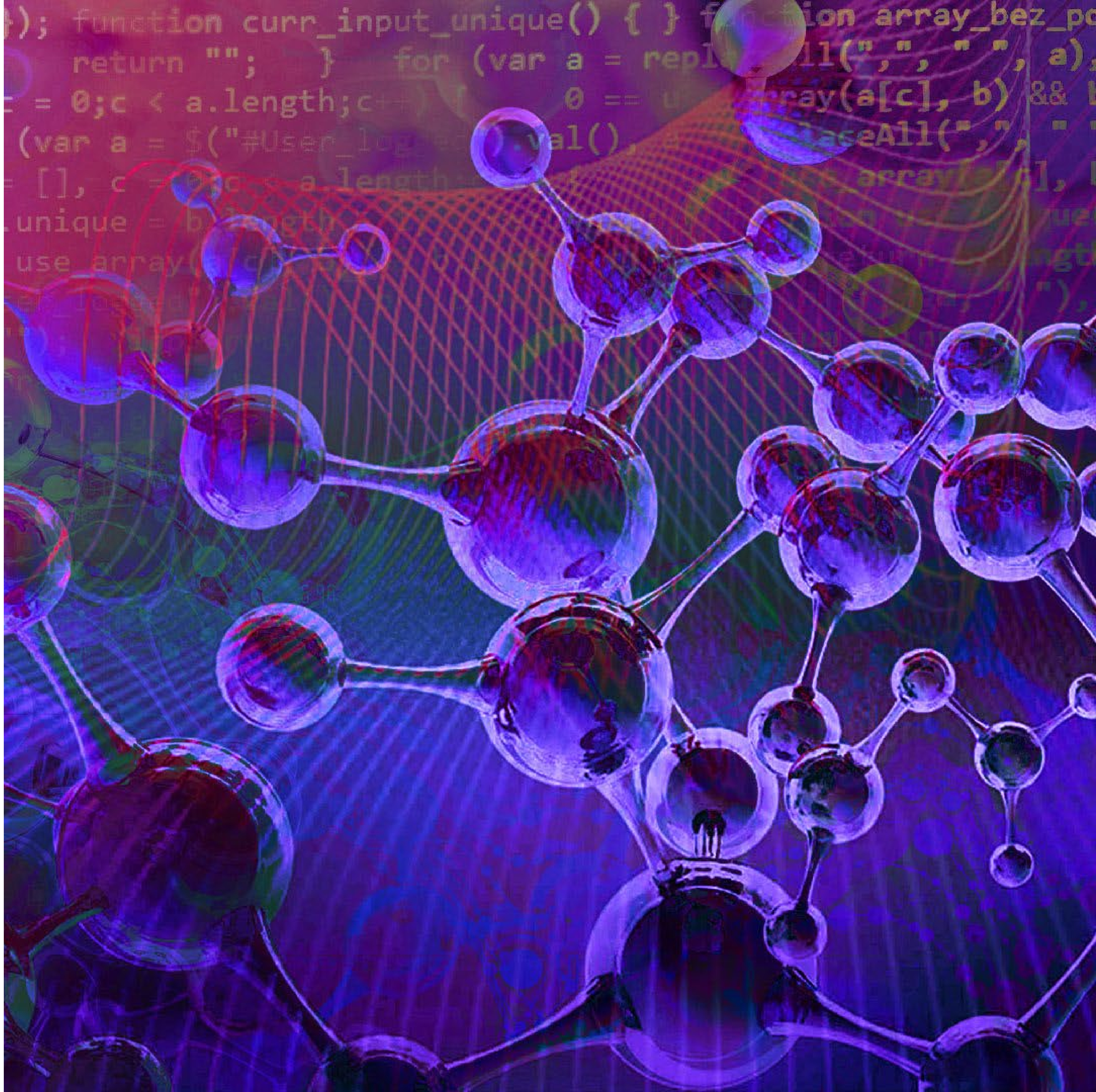

Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!

Data repository and relevant publications

Structural (halo-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6541230
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	Data https://dx.doi.org/10.5281/zenodo.6569898
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

Structural (halo-based) approach for dark matter flow

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Halo mass functions from maximum entropy distributions in collisionless dark matter flow

arXiv:2110.09676 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2110.09676

Introduction

- Halo mass function, the most fundamental quantity
- Conventional Mass function from nonlinear collapse
  - Press-Schechter (PS) formalism
    - Threshold overdensity from spherical collapse
  - Extended PS using an excursion set approach
    - Overdensity as a random walk process
  - ST model
    - Ellipsoidal collapse model gives a mass-dependent overdensity threshold
- Mass function from mass cascade in dark matter flow
  - Double-λ mass function
    - Assume two different halo geometry parameter λ for different size of halos.
- The mass/energy cascade as an intermediate statistically steady state for non-equilibrium systems to continuously maximize system entropy.

$$\nu = \delta_c^2 / \sigma_\delta^2(m_h) \qquad \delta_c = 1.6865$$

$$f_{PS}(\nu) = \frac{1}{\sqrt{2\pi}\sqrt{\nu}} e^{-\nu/2} \qquad \int_0^\infty f(\nu) d\nu = 1$$

$$f_{ST}(\nu) = A\sqrt{\frac{2q}{\pi}}\left(1 + \frac{1}{(q\nu)^p}\right)\frac{1}{2\sqrt{\nu}} e^{-q\nu/2}$$

$A = 0.32 \quad q = 0.75 \quad p = 0.3$

$A = 0.5 \quad q = 1.0 \quad p = 0 \rightarrow f_{PS}(\nu)$

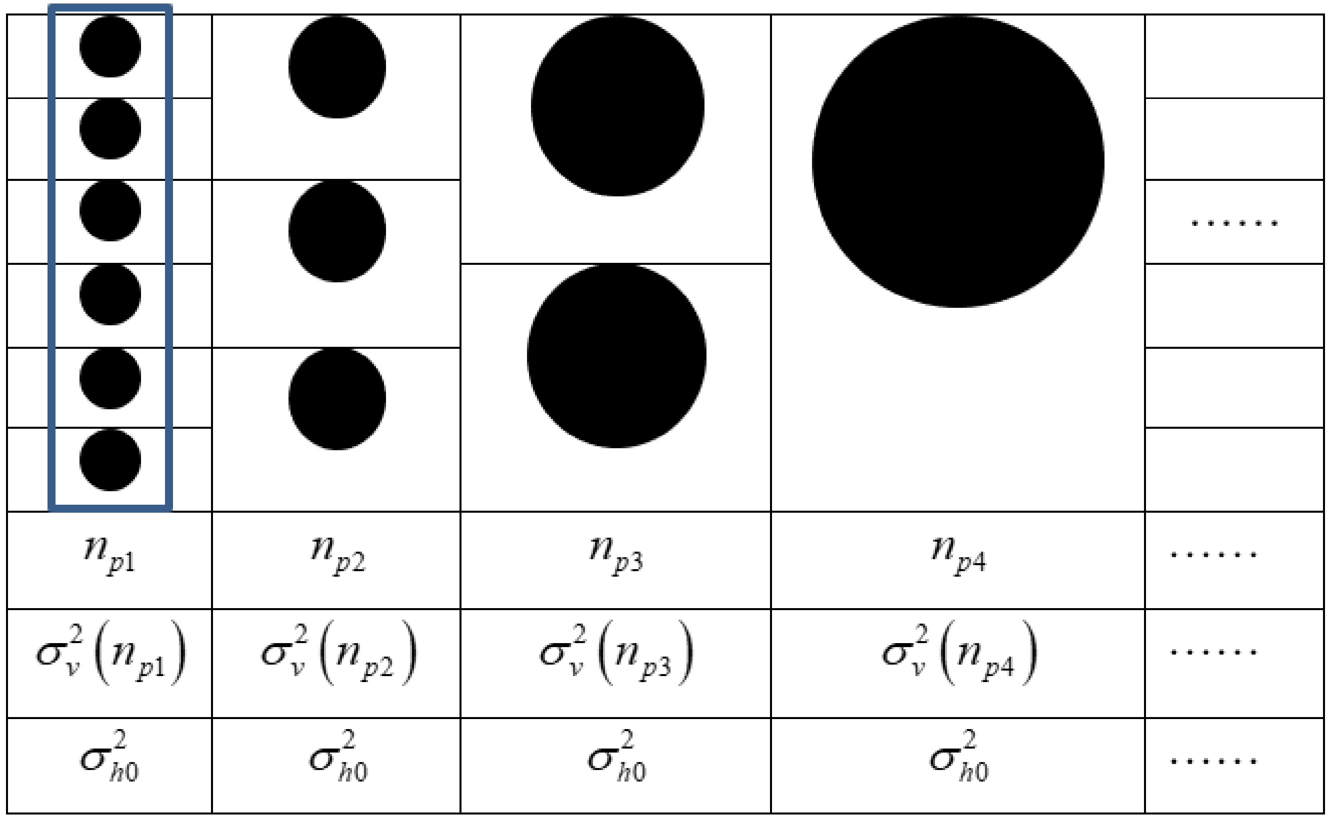
$$f_{D\lambda}(\nu) = \frac{\left(2\sqrt{\eta_0}\right)^{-q}}{\Gamma(q/2)} \nu^{q/2-1} \exp\left(-\frac{\nu}{4\eta_0}\right)$$

$\eta_0 = 0.76 \quad q = 0.556$

$\eta_0 = 0.5 \quad q = 1 \rightarrow f_{PS}(\nu)$

Are there or what are the connections between halo mass function and maximum entropy??

Maximum entropy distributions



Symbol	Physical meaning
X(v)	Distribution of one-dimensional particle velocity *v*
Z(v)	Distribution of particle speed v
E(ε)	Distribution of particle energy ε
$H(\sigma_v^2)$	Distribution of particle virial dispersion σ_v^2 (halo mass function)
$J(\sigma_v^2)$	Distribution of halos with virial dispersion σ_v^2
$P(v^2)$	Distribution of square of one-dimensional particle velocity v

- Long-range and collisionless nature
- Identify all halos of different sizes at given *z*
- Group halos according to halo size n_p

$$n_p \equiv n_p\left(\sigma_v^2\right) \qquad \left\langle\sigma_v^2\right\rangle = \int_0^\infty H\left(\sigma_v^2\right)\sigma_v^2 d\sigma_v^2$$

$$V(r) \propto r^n \qquad \text{n=-1 for standard gravity}$$

$$\left\langle\sigma_h^2\right\rangle \equiv \bar{\sigma}_h^2 = \int_0^\infty H\left(\sigma_v^2\right)\sigma_h^2 d\sigma_v^2 \quad \left\langle\sigma^2\right\rangle = \left\langle\sigma_v^2\right\rangle + \left\langle\sigma_h^2\right\rangle = \sigma_0^2$$

Relations between maximum entropy distributions

$$X(v)=\int_0^{\infty}\frac{1}{\sqrt{2\pi\sigma}}e^{-v^2/2\sigma^2}H\left(\sigma_v^2\right)d\sigma_v^2$$

$$\int_{-\infty}^{\infty}X(v)e^{-vt}dv=\int_0^{\infty}H\left(\sigma_v^2\right)e^{\sigma^2t^2/2}d\sigma_v^2$$

$$P\left(x=v^2\right)=\int_0^{\infty}\frac{1}{\sqrt{2\pi x\sigma}}e^{-x/2\sigma^2}H\left(\sigma_v^2\right)d\sigma_v^2$$

$$\int_0^{\infty}P(x)e^{-xt}dx=\int_0^{\infty}H\left(\sigma_v^2\right)\frac{1}{\sqrt{1+2\sigma^2t}}d\sigma_v^2$$

$$H\left(\sigma_v^2\right)=J\left(\sigma_v^2\right)n_p\left(\sigma_v^2\right)/\bar{N}$$

$$\bar{N}=\int_0^{\infty}J\left(\sigma_v^2\right)n_p\left(\sigma_v^2\right)d\sigma_v^2$$

Average number of particles per halo

The X distribution for maximum entropy principle:

$$X(v)=\frac{1}{2\alpha v_0}\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{K_1(\alpha)} \Rightarrow P(x)=\frac{e^{-\sqrt{\alpha^2+x/v_0^2}}}{2\alpha v_0K_1(\alpha)\sqrt{x}}$$

$$\int_0^{\infty}H\left(\sigma_v^2\right)e^{-\sigma^2t}d\sigma_v^2=\frac{K_1\left(\alpha\sqrt{1+2v_0^2t}\right)}{K_1(\alpha)\sqrt{1+2v_0^2t}}$$

$$v=\left(\frac{m_h}{m_h^*}\right)^{2/3}=\frac{\sigma_v^2(m_h)}{\sigma_v^2(m_h^*)}=\frac{\sigma_v^2(m_h)}{\bar{\sigma}_h^2}$$

Introduce dimensionless variable

$$f(v)=H\left(v\bar{\sigma}_h^2\right)\bar{\sigma}_h^2$$

Halo mass function is intrinsically related to H, and hence X, the maximum entropy distribution

Parameters and distributions for some typical potential exponents n

Laplacian or exponential

	n	β	α	v_0^2	$\langle \sigma_h^2 \rangle$	$\langle \sigma_v^2 \rangle$	$X(v)$	$H(x=\sigma_v^2)$	$P(x=v^2)$
	0	1	0	$\frac{\sigma_0^2}{2}$	0	σ_0^2	$\frac{e^{-\sqrt{2}v/\sigma_0}}{\sqrt{2}\sigma_0}$	$\frac{e^{-x/\sigma_0^2}}{\sigma_0^2}$	$\frac{e^{-\sqrt{2x}/\sigma_0}}{\sigma_0\sqrt{2x}}$
Long range interaction	-1	$\frac{3}{2}$	$\frac{K_1(\alpha)}{K_2(\alpha)} = \frac{\langle \sigma_h^2 \rangle}{\sigma_0^2}$	$\frac{\sigma_0^2 K_1(\alpha)}{\alpha K_2(\alpha)}$	$\sim \frac{\sigma_0^2}{2}$	$\sim \frac{\sigma_0^2}{2}$	$\frac{e^{-\sqrt{\alpha^2+(v/v_0)^2}}}{2\alpha v_0 K_1(\alpha)}$		$\frac{e^{-\sqrt{\alpha^2+x/v_0^2}}}{2\alpha v_0 K_1(\alpha)\sqrt{x}}$
Short range interaction	-2	3	∞	0	σ_0^2	0	$\frac{e^{-v^2/2\sigma_0^2}}{\sqrt{2\pi}\sigma_0}$	$\delta(x)$	$\frac{e^{-x/2\sigma_0^2}}{\sigma_0\sqrt{2\pi x}}$

X distribution

Gaussian

Integral transformations between distributions:

$$\int_{-\infty}^{\infty} X(v) e^{-vt} dv = \int_0^{\infty} H(\sigma_v^2) e^{\sigma^2 t^2/2} d\sigma_v^2$$

$$\int_0^{\infty} P(x) e^{-xt} dx = \int_0^{\infty} H(\sigma_v^2) \frac{1}{\sqrt{1+2\sigma^2 t}} d\sigma_v^2$$

H and J Distributions for large halos

We first consider an extreme case, large halos with $\sigma_h^2 << \sigma_v^2$:

Halo group temperature → $\sigma_h^2 \to 0$ and $\sigma^2 \approx \sigma_v^2$ ← Halo temperature

From integral transformations between distributions:

$$\int_0^\infty H\left(\sigma_v^2\right)e^{-\sigma^2 t}d\sigma_v^2 = \frac{K_1\left(\alpha\sqrt{1+2v_0^2 t}\right)}{K_1\left(\alpha\right)\sqrt{1+2v_0^2 t}}$$

With $\sigma^2 = \sigma_v^2$ ↓ H distribution for large halos:

$$H_\infty\left(\sigma_v^2\right) = \frac{1}{2\alpha v_0^2 K_1\left(\alpha\right)}\cdot\exp\left[-\frac{\alpha}{2}\left(\frac{\sigma_v^2}{\alpha v_0^2}+\frac{\alpha v_0^2}{\sigma_v^2}\right)\right]$$

Dimensionless H distribution for large halos: ↓

$$f_{H_\infty}\left(\nu\right) = \frac{1}{2\gamma K_1\left(\alpha\right)}\cdot\exp\left[-\frac{\alpha}{2}\left(\frac{\nu}{\gamma}+\frac{\gamma}{\nu}\right)\right] \qquad \gamma = \frac{\alpha v_0^2}{\bar{\sigma}_h^2}$$

$$H\left(\sigma_v^2\right) = J\left(\sigma_v^2\right)n_p\left(\sigma_v^2\right)/\bar{N}$$

J distribution for large halos: ↓

$$J_\infty\left(\sigma_v^2\right) = \frac{1}{2\alpha v_0^2 K_{\beta-1}\left(\alpha\right)}\left(\frac{\alpha v_0^2}{\sigma_v^2}\right)^\beta \exp\left[-\frac{\alpha}{2}\left(\frac{\sigma_v^2}{\alpha v_0^2}+\frac{\alpha v_0^2}{\sigma_v^2}\right)\right]$$

Halo size: $n_p\left(\sigma_v^2\right) = \bar{N}\frac{K_{\beta-1}\left(\alpha\right)}{K_1\left(\alpha\right)}\left(\frac{\sigma_v^2}{\alpha v_0^2}\right)^\beta$

$\beta = 3/\left(3+n\right)$ $\qquad \beta = 3/2$ for $n=-1$

Interestingly, H_∞ distribution can be obtained directly using the maximum entropy principle without resorting to X distribution (Next slides)

H_∞ and J_∞ Distributions from maximum entropy principle

Following the maximum entropy principle for velocity distrution:

H_∞ distribution is a maximum entropy distribution satisfying three constraints:

$$\int_0^\infty H_\infty\left(\sigma_v^2\right)d\sigma_v^2 = 1$$

$$\int_0^\infty H_\infty\left(\sigma_v^2\right)\sigma_v^2 d\sigma_v^2 = \left\langle\sigma_v^2\right\rangle$$

$$\int_0^\infty J_\infty\left(\sigma_v^2\right)d\sigma_v^2 = \int_0^\infty \frac{H_\infty\left(\sigma_v^2\right)}{\mu\left(\sigma_v^2/v_0^2\right)^\beta}d\sigma_v^2 = 1$$

Write down the entropy functional with Lagrangian multiplier:

$$\begin{aligned}S\left[H_\infty\left(\sigma_v^2\right)\right] = &-\int_0^\infty H_\infty\left(\sigma_v^2\right)\ln H_\infty\left(\sigma_v^2\right)d\sigma_v^2\\ &+\lambda_1\left(\int_0^\infty H_\infty\left(\sigma_v^2\right)d\sigma_v^2 - 1\right)\\ &+\lambda_2\left(\int_0^\infty H_\infty\left(\sigma_v^2\right)\sigma_v^2 d\sigma_v^2 - \left\langle\sigma_v^2\right\rangle\right)\\ &+\lambda_3\left(\int_0^\infty \frac{H_\infty\left(\sigma_v^2\right)}{\mu\left(\sigma_v^2/v_0^2\right)^\beta}d\sigma_v^2 - 1\right)\end{aligned}$$

Taking the variation of the entropy functional with respect to distribution *H*:

$$H_\infty\left(\sigma_v^2\right) = e^{\lambda_1 - 1}\exp\left(\lambda_2\sigma_v^2 + \frac{\lambda_3}{\mu}\left(\frac{v_0^2}{\sigma_v^2}\right)^\beta\right)$$

$$H_\infty\left(\sigma_v^2\right) = \frac{1}{2\alpha v_0^2 K_1(\alpha)}\cdot\exp\left[-\frac{\alpha}{2}\left(\frac{\sigma_v^2}{\alpha v_0^2} + \frac{\alpha v_0^2}{\sigma_v^2}\right)\right]$$

Modeling halo virial dispersion and halo velocity dispersion

To solve H distribution using integral transformation:

$$\int_0^\infty H\left(\sigma_v^2\right)e^{-\sigma^2 t}d\sigma_v^2 = \frac{K_1\left(\alpha\sqrt{1+2v_0^2 t}\right)}{K_1\left(\alpha\right)\sqrt{1+2v_0^2 t}}$$

We need model for velocity dispersion σ^2:

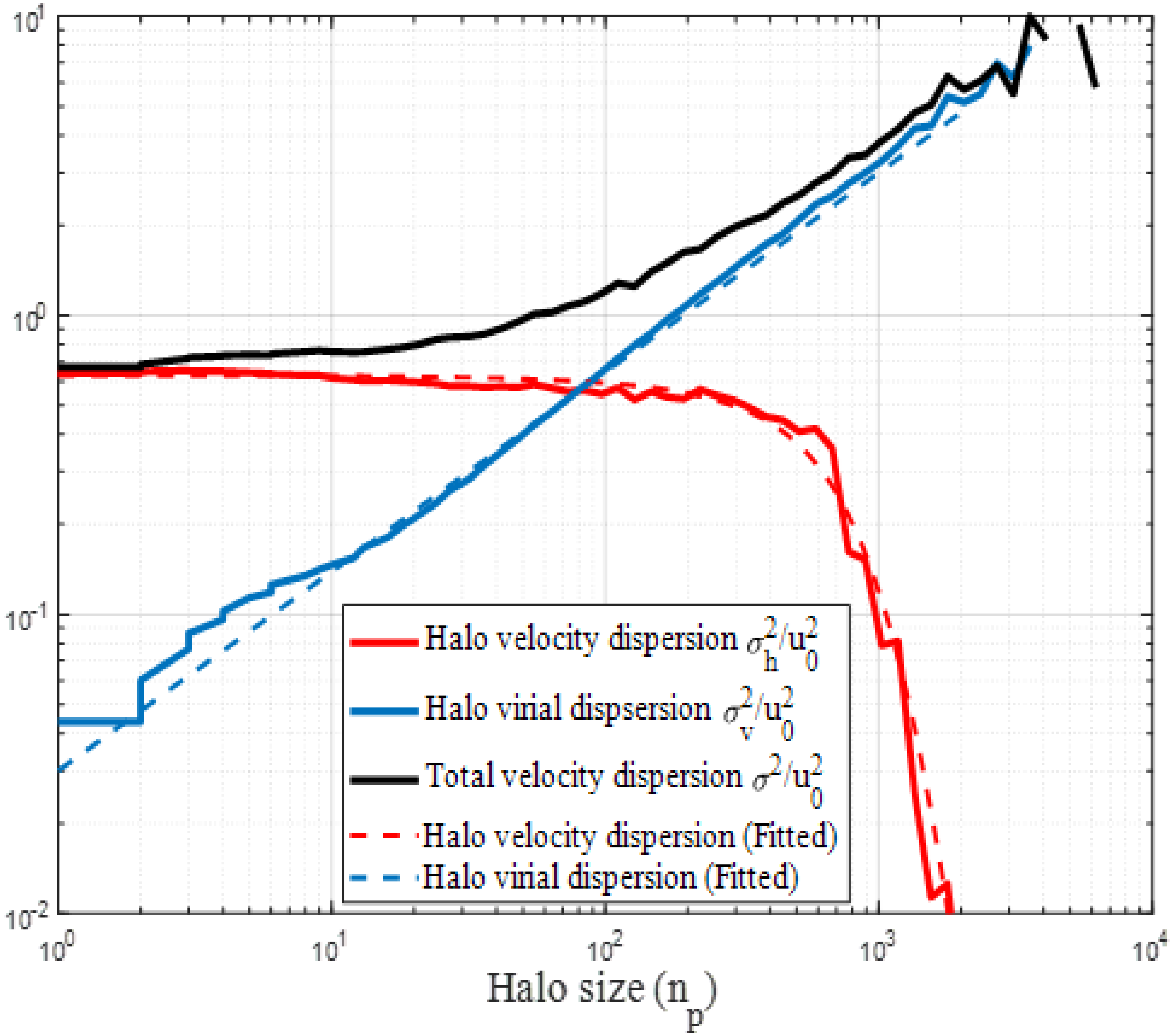
$$\sigma^2 = \sigma_v^2 + \sigma_h^2$$

Model for halo virial dispersion (halo temperature):

$$\sigma_v^2\left(m_h\right) = 0.03 n_p^{2/3} u_0^2 = 0.03\left(m_h/m_p\right)^{2/3} u_0^2$$

Model for halo velocity dispersion (halo group temperature):

$$\sigma_h^2\left(m_h\right) = 0.375\left[1-\tanh\left(\frac{m_h/m_p - 500}{600}\right)\right]u_0^2$$



H Distribution for small halos

We consider another extreme case, small halos with $\sigma_v^2 << \sigma_h^2$:

Halo group temperature → $\sigma_v^2 \to 0$ and $\sigma^2 \approx \sigma_h^2$ ← Halo temperature

If approximate the virial dispersion with v^2 $v^2 \approx \sigma_v^2$

$$H\left(x=\sigma_v^2\right) \approx P\left(x=v^2\right) = \frac{e^{-\alpha}}{2\alpha v_0 K_1(\alpha)\sqrt{x}} \exp\left(-\frac{x}{2\alpha v_0^2}\right)$$

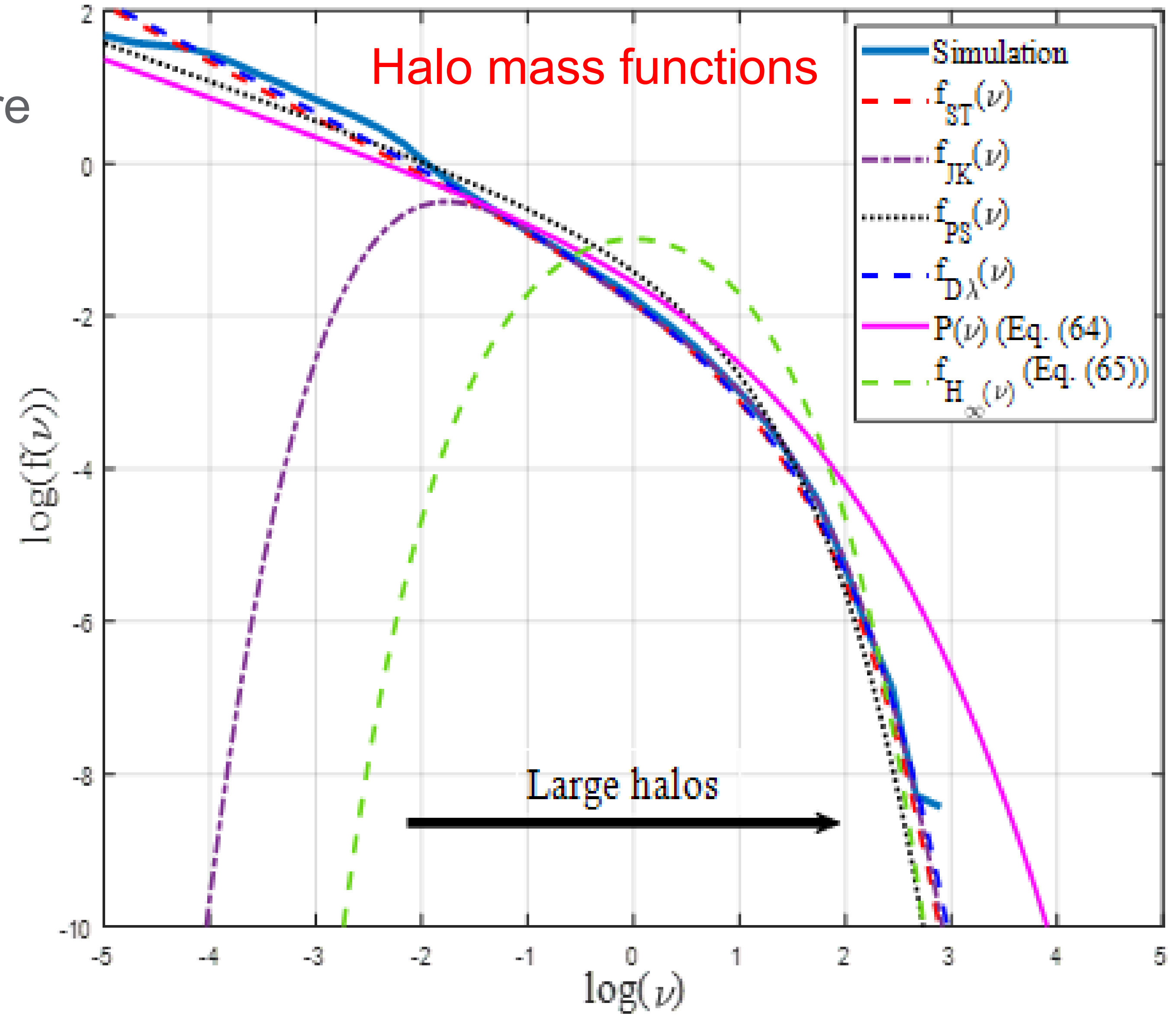
H distribution for small halos:

$$H_s\left(\sigma_v^2\right) = \frac{1}{\sqrt{2\pi\alpha v_0^2 \sigma_v^2}} \exp\left(-\frac{\sigma_v^2}{2\alpha v_0^2}\right)$$

PS mass function for γ=1

Dimensionless mass function for small halos:

$$f_{H_s}(\nu) = \frac{1}{\sqrt{2\pi\gamma\nu}} \exp\left(-\frac{\nu}{2\gamma}\right) \qquad \gamma = \frac{\alpha v_0^2}{\bar{\sigma}_h^2}$$



Halo mass function from maximum entropy distributions

From integral transformations between distributions:

H distribution from maximum entropy distribution should satisfy:

$$\int_0^\infty H\left(\sigma_v^2\right)e^{-\sigma^2 t}d\sigma_v^2 = \frac{K_1\left(\alpha\sqrt{1+2v_0^2 t}\right)}{K_1\left(\alpha\right)\sqrt{1+2v_0^2 t}}$$

Relation between dimensionless halo mass function and H distribution:

$$f\left(\nu\right) = H\left(\nu\bar{\sigma}_h^2\right)\bar{\sigma}_h^2$$

Dimensionless maximum entropy halo mass function:

$$\int_0^\infty f_{ME}\left(\nu\right)e^{-(\nu+\nu_h)t}d\nu = \frac{K_1\left(\alpha\sqrt{1+2\gamma t/\alpha}\right)}{K_1\left(\alpha\right)\sqrt{1+2\gamma t/\alpha}}$$

$\nu_h = \sigma_h^2/\bar{\sigma}_h^2$ and $\bar{\sigma}_h^2\left(a\right) = \sigma_v^2\left(m_h^*, a\right)$

Laplace transform of halo mass functions:

$$\int_0^\infty f_{PS}\left(\nu\right)e^{-\nu t}d\nu = \frac{1}{\sqrt{1+2t}}$$

$$\int_0^\infty f_{ST}\left(\nu\right)e^{-\nu t}d\nu = \frac{\sqrt{q}}{\sqrt{q+2t}}\frac{\sqrt{\pi}+\Gamma\left(1/2-p\right)\left(1/2+t/q\right)^p}{\sqrt{\pi}+2^{-p}\Gamma\left(1/2-p\right)}$$

$$\int_0^\infty f_{D\lambda}\left(\nu\right)e^{-\nu t}d\nu = \frac{1}{\left(1+4\eta_0 t\right)^{q/2}}$$

Moments of halo mass functions:

$$\int_0^\infty f_{PS}\left(\nu\right)\nu^n d\nu = 2^n\frac{\Gamma\left(1/2+n\right)}{\sqrt{\pi}}$$

$$\int_0^\infty f_{ST}\left(\nu\right)\nu^n d\nu = \left(\frac{2}{q}\right)^2\frac{\Gamma\left(1/2+n\right)+2^{-p}\Gamma\left(1/2+n-p\right)}{\Gamma\left(1/2\right)+2^{-p}\Gamma\left(1/2-p\right)}$$

$$\int_0^\infty f_{D\lambda}\left(\nu\right)\nu^n d\nu = \frac{\left(4\eta_0\right)^n\Gamma\left(q/2+n\right)}{\Gamma\left(q/2\right)}$$

Halo mass function from maximum entropy distributions

Equation for maximum entropy halo mass function:

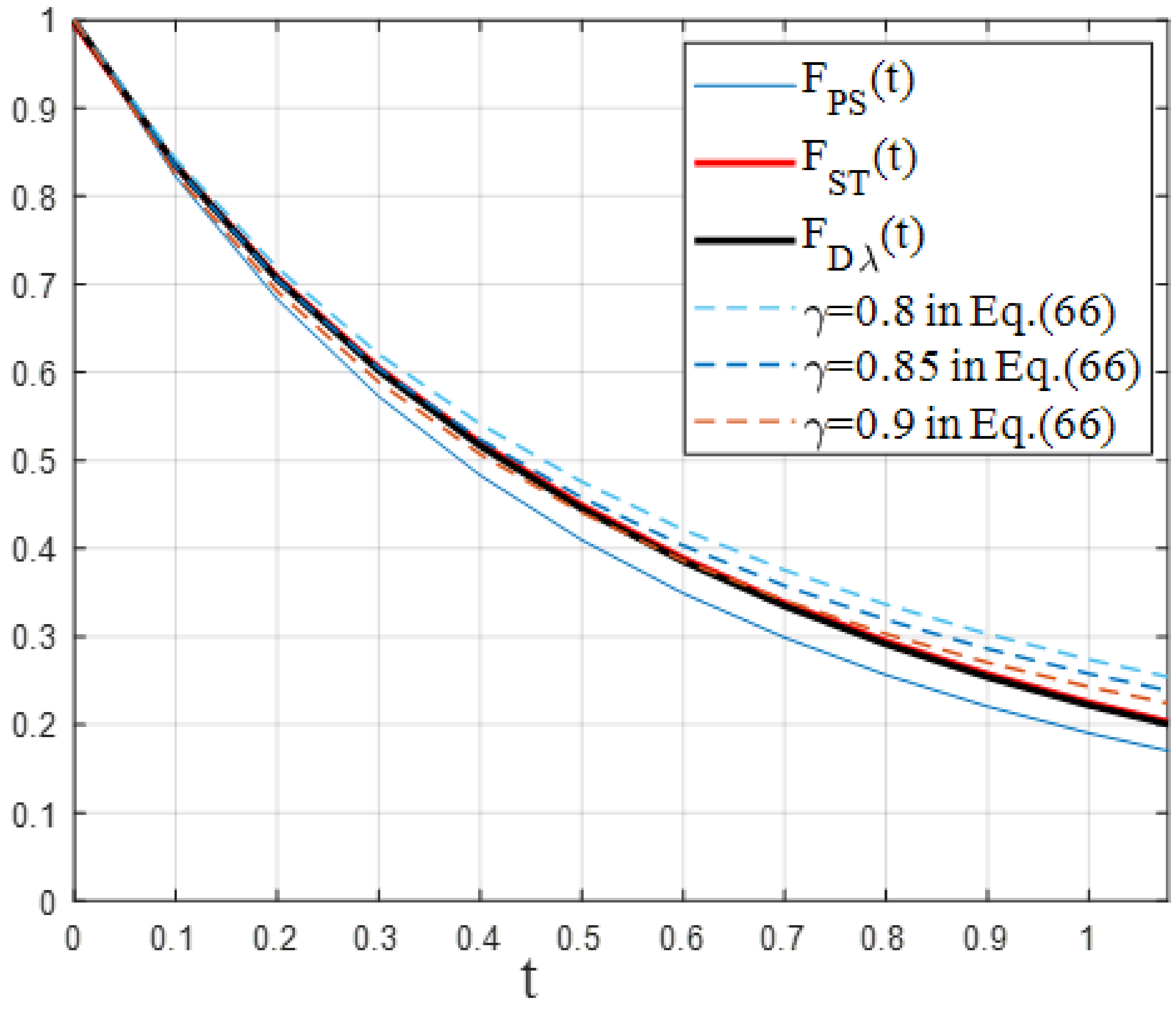
$$\int_0^{\infty} f_{ME}(\nu) e^{-(\nu+\nu_h)t} d\nu = \frac{K_1\left(\alpha\sqrt{1+2\gamma t/\alpha}\right)}{K_1(\alpha)\sqrt{1+2\gamma t/\alpha}}$$

No analytical solutions can be found. Instead Introduce a transformed function F_x to compare different halo mass functions:

$$F_X(t) = \int_0^{\infty} f_X(\nu) e^{-(\nu+\nu_h)t} d\nu$$

Subscript X is the abbreviation of the mass function model, PS, ST, Dλ and ME.

- ST and Dλ almost coincide with each other.
- Both agree better with the ME than the PS mass function.
- Halo mass function can be an intrinsic distribution to maximize system entropy.



Summary and keywords

Maximum entropy	Velocity distribution	Spherical collapse
Halo mass function	Energy distribution	H and P distributions

- Halo mass function is a fundamental quantity for structure formation and evolution.
- Conventional halo mass functions are based on simplified spherical/elliptical collapse models
- The H distribution for particle virial dispersion is essentially the halo mass function that can be related to X distribution that maximizes system entropy.
- The H distribution for large halos is also a maximum entropy distribution.
- For small halos, H approximates the distribution of square velocity (P) and recovers the Press-Schechter mass function.
- Halo mass function can be interpreted as an intrinsic distribution to maximize the system entropy