A comparative study of dark matter flow & hydrodynamic turbulence and its applications

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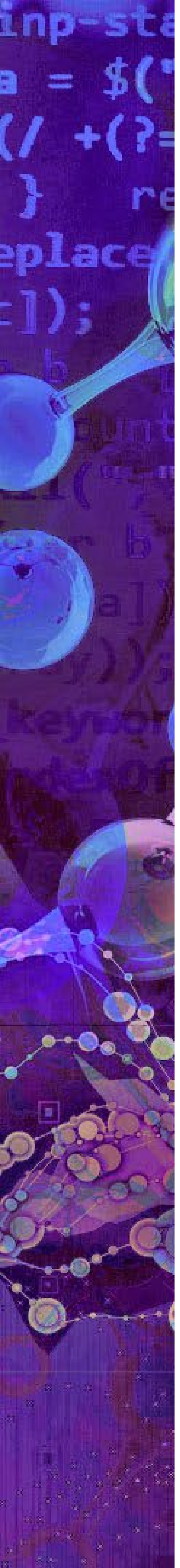
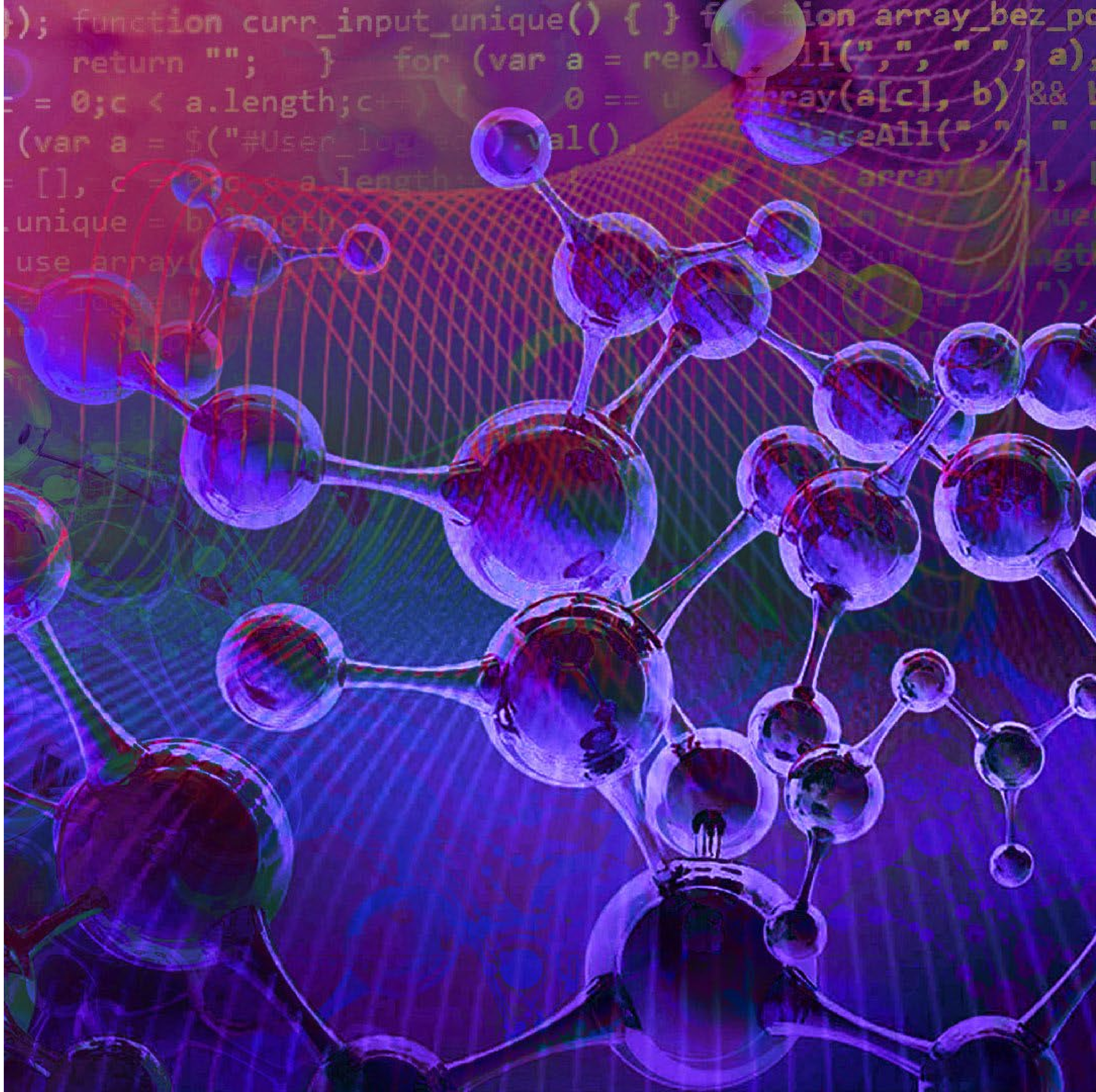




Preface

Dark matter, if exists, accounts for five times as much as ordinary baryonic matter. Therefore, dark matter flow might possess the widest presence in our universe. The other form of flow, hydrodynamic turbulence in air and water, is without doubt the most familiar flow in our daily life. During the pandemic, we have found time to think about and put together a systematic comparison for the connections and differences between two types of flow, both of which are typical non-equilibrium systems.

The goal of this presentation is to leverage this comparison for a better understanding of the nature of dark matter and its flow behavior on all scales. Science should be open. All comments are welcome.

Thank you!

Data repository and relevant publications

Structural (halo-based) approach:

0.	**Data https://dx.doi.org/10.5281/zenodo.6541230**
1.	Inverse mass cascade in dark matter flow and effects on halo mass functions https://doi.org/10.48550/arXiv.2109.09985
2.	Inverse mass cascade in dark matter flow and effects on halo deformation, energy, size, and density profiles https://doi.org/10.48550/arXiv.2109.12244
3.	Inverse energy cascade in self-gravitating collisionless dark matter flow and effects of halo shape https://doi.org/10.48550/arXiv.2110.13885
4.	The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos https://doi.org/10.48550/arXiv.2201.12665
5.	Two-body collapse model for gravitational collapse of dark matter and generalized stable clustering hypothesis for pairwise velocity https://doi.org/10.48550/arXiv.2110.05784
6.	Evolution of energy, momentum, and spin parameter in dark matter flow and integral constants of motion https://doi.org/10.48550/arXiv.2202.04054
7.	The maximum entropy distributions of velocity, speed, and energy from statistical mechanics of dark matter flow https://doi.org/10.48550/arXiv.2110.03126
8.	Halo mass functions from maximum entropy distributions in collisionless dark matter flow https://doi.org/10.48550/arXiv.2110.09676

Statistics (correlation-based) approach:

0.	**Data https://dx.doi.org/10.5281/zenodo.6569898**
1.	The statistical theory of dark matter flow for velocity, density, and potential fields https://doi.org/10.48550/arXiv.2202.00910
2.	The statistical theory of dark matter flow and high order kinematic and dynamic relations for velocity and density correlations https://doi.org/10.48550/arXiv.2202.02991
3.	The scale and redshift variation of density and velocity distributions in dark matter flow and two-thirds law for pairwise velocity https://doi.org/10.48550/arXiv.2202.06515
4.	Dark matter particle mass and properties from two-thirds law and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2202.07240
5.	The origin of MOND acceleration and deep-MOND from acceleration fluctuation and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.05606
6.	The baryonic-to-halo mass relation from mass and energy cascade in dark matter flow https://doi.org/10.48550/arXiv.2203.06899

Structural (halo-based) approach for dark matter flow

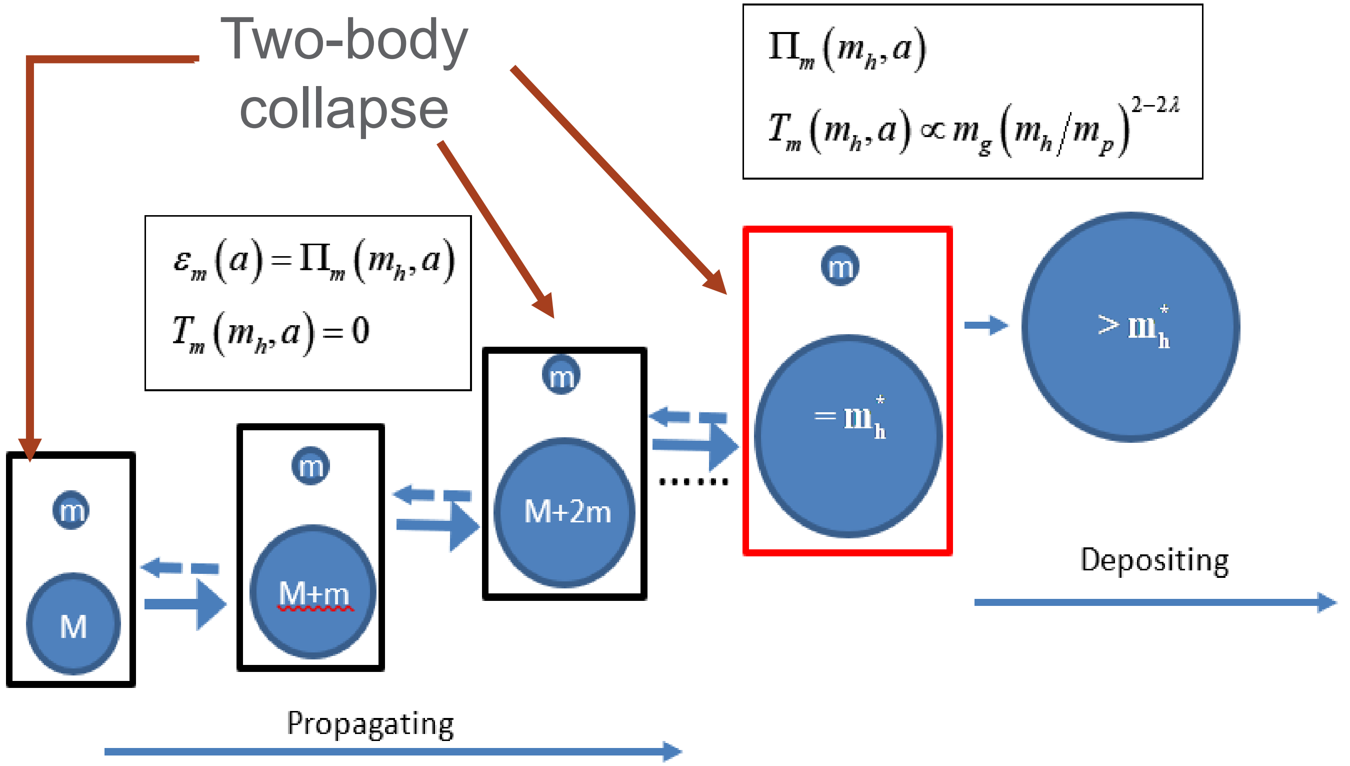
Pacific Northwest NATIONAL LABORATORY

Two-body collapse model (TBCM): an elementary step of mass cascade and GSCH for pairwise velocity

Xu Z., 2021, arXiv:2110.05784v1 [astro-ph.CO]
https://doi.org/10.48550/arXiv.2110.05784

Introduction: TBCM as an elementary step of inverse mass cascade

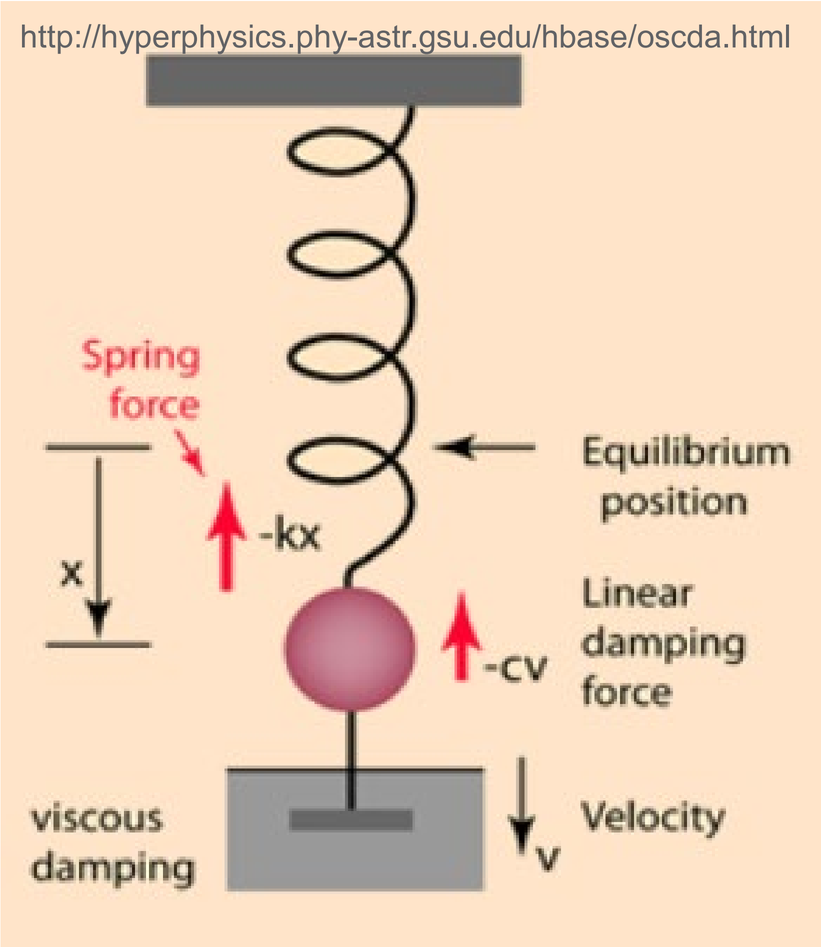
- Analytical tools are invaluable.
- Solutions are extremely difficult to find due to the highly non-linear nature of collapse.
- Two examples: the spherical collapse model (SCM) and stable clustering hypothesis (SCH).
- For an infinitesimal interval, mass cascade should involve the merging of two and only two substructures.
- Two-body problem in static background is known: Kepler's laws.
- Goal: solutions for two-body in expanding background and relations with SCM and SCH



Two-body collapse in expanding background is an elementary step of mass cascade.

- Goal: Prove SCH and Generalized SCH for moments of pairwise velocity.

Introduction: Damped harmonic oscillator as a fundamental model in dynamics



$$\ddot{\mathbf{r}} + (c/m)\dot{\mathbf{r}} + (k/m)\mathbf{r} = 0$$

damping — spring force

Competition

Define a critical ratio to quantify competition: $$\beta_s = \frac{(c/2m)^2}{(k/m)} = 1$$

Critical damping: $$c_s = 2\sqrt{km}$$

Energy evolution: $$\frac{dE}{dt} + \left(\frac{2c}{m}\right)K = 0$$

E: total energy (potential + kinetic)
K: kinetic energy

- Damped harmonic oscillator is a fundamental model in dynamics that is extremely insightful.
- There exist a critical damping c_s. For $c<c_s$, spring force is dominant (underdamped); For $c>c_s$, damping is dominant (overdamped).

- Does two-body collapse model play a similar role as harmonic oscillator?
- Overdamped and underdamped in gravitational collapse?
- Insights into the energy/momentum evolution?

Equations of motion in comoving and transformed systems

Equations of motion in a comoving system with expanding background → Equation of motion in a transformed system with fixed damping in static background

$$\frac{d^2\mathbf{x}_i}{dt^2} + 2H\frac{d\mathbf{x}_i}{dt} = -\frac{Gm_p}{a^3}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^3}$$

Potential with an arbitrary exponent of ***n*** for particle-particle interacting

$$V_p(r) = -G_n m_p^2 / r^{-n}$$

$$\frac{d^2\mathbf{x}_i}{dt^2} + 2H\frac{d\mathbf{x}_i}{dt} = \frac{nG_n m_p}{a^3}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}$$

Introduce a new transformed time scale **s**

$$ds/dt = a^p$$

- If p=-2, *s* is the time variable for integration in N-body simulation.
- Transformed system: fixed damping and no scale factor *a*;

$$\frac{d^2\mathbf{x}_i}{ds^2} + \frac{d\mathbf{x}_i}{ds}(p+2)a^{-p}H = \frac{nG_n m_p}{a^{3+2p}}\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}}$$

$$p = -3/2$$

Matter dominant

$$\dot{H} = -3H^2/2$$

$$H^2 = 8\pi G\bar{\rho}_y(a)/3$$

$$H_0^2 = H^2 a^3$$

$$\frac{d^2\mathbf{x}_i}{ds^2} + \frac{H_0}{2}\frac{d\mathbf{x}_i}{ds} = nG_n m_p\sum_{j\neq i}^{N}\frac{\mathbf{x}_i-\mathbf{x}_j}{\left|\mathbf{x}_i-\mathbf{x}_j\right|^{2-n}} = \frac{\mathbf{F}_i}{m_p}$$

Peculiar velocity in comoving**:**

$$\mathbf{u}_i = a\frac{d\mathbf{x}_i}{dt} = \frac{d\mathbf{r}_i}{dt} - H\mathbf{r}_i = a^{-1/2}\mathbf{v}_i$$

Velocity in time scale **s:**

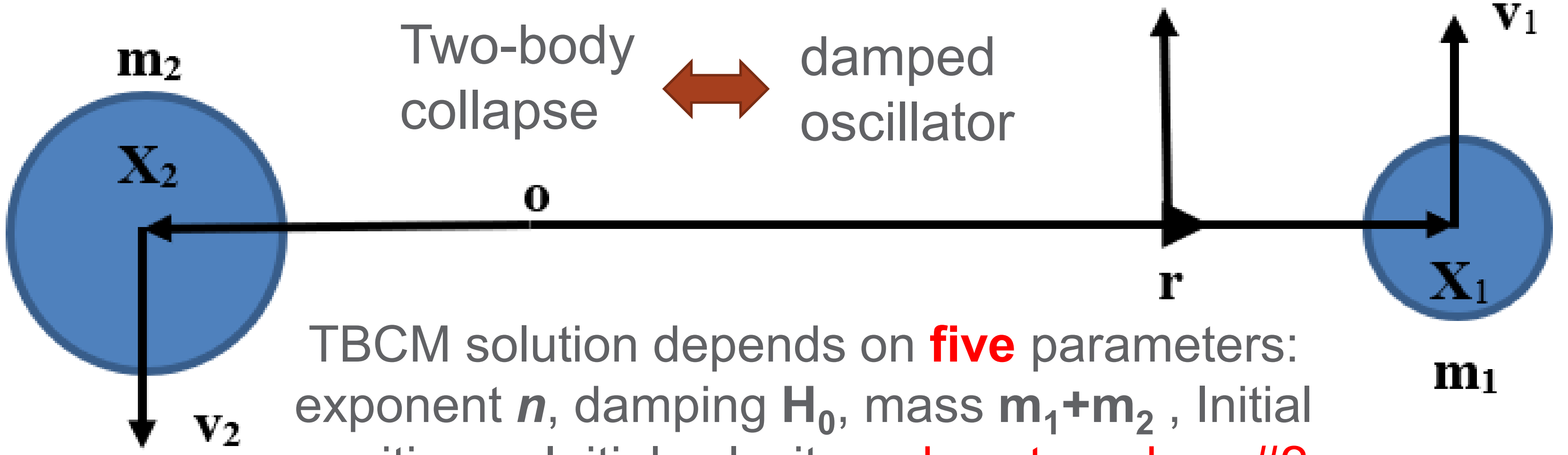
$$\mathbf{v}_i = \frac{d\mathbf{x}_i}{ds} = a^{3/2}\frac{d\mathbf{x}_i}{dt} = a^{1/2}\mathbf{u}_i$$

Formulation of a TBCM model in transformed system

Reduce to equations of motion for two-body:

$$\ddot{\mathbf{x}}_1 + \frac{H_0}{2}\dot{\mathbf{x}}_1 = \frac{nG_n m_2}{(2r)^{1-n}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$$

$$\ddot{\mathbf{x}}_2 + \frac{H_0}{2}\dot{\mathbf{x}}_2 = -\frac{nG_n m_1}{(2r)^{1-n}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$$



TBCM solution depends on **five** parameters: exponent ***n***, damping $\mathbf{H_0}$, mass $\mathbf{m_1+m_2}$, Initial position $\mathbf{r_i}$, Initial velocity $\mathbf{v_i}$; how to reduce #?

Displacement vector **r**:

$$\mathbf{r} = (\mathbf{x}_1 - \mathbf{x}_2)/2$$

Standard damped oscillator Eq.:

$$\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} + (k/m)\mathbf{r} = 0$$

Reduce to Eq. of motion for vector **r**:

$$\ddot{\mathbf{r}} + \frac{H_0}{2}\dot{\mathbf{r}} = \frac{nG_n(m_1+m_2)}{2(2r)^{1-n}} \cdot \frac{\mathbf{r}}{|\mathbf{r}|}$$

Similar equation as damped oscillator in dynamics

Equation of motion for radius function r (magnitude of **r**): **(similar to spherical collapse model)**

$$\ddot{r} + \frac{H_0}{2}\dot{r} - \frac{nG_n(m_1+m_2)}{2(2r)^{1-n}} = \frac{(r_i v_i)^2}{r^3}\exp(-H_0 s)$$

Expanding background or damping

Gravitational interaction

Angular momentum

Compute particle position and velocity:

$$\mathbf{x}_1 = \mu\mathbf{r} \qquad \mathbf{v}_1 = \mu\dot{\mathbf{r}} \qquad \mu = \frac{2m_2}{m_1+m_2}$$

$$\mathbf{x}_2 = -(2-\mu)\mathbf{r} \qquad \mathbf{v}_2 = -(2-\mu)\dot{\mathbf{r}}$$

Competition between three terms determines the collapse regimes

Formulation of a TBCM model in transformed system

Introduce frequency function F(s):

$$r(s)=(r_i v_i)^{1/2} F(s)\exp\left(-\frac{1}{4}H_0 s\right)$$

Equation of motion for r:

$$\ddot{r}+\frac{H_0}{2}\dot{r}-\frac{nG_n(m_1+m_2)}{2(2r)^{1-n}}=\frac{(r_i v_i)^2}{r^3}\exp(-H_0 s)$$

Equation for frequency function:

$$\frac{\partial^2 F}{\partial s^2}=\underbrace{\frac{H_0^2}{16}F(s)}_{1}-\underbrace{\gamma_s\left(\frac{v_i}{r_i}\right)^{1+n/2} F^{n-1}(s)\exp\left(-\frac{n-2}{4}H_0 s\right)}_{2}+\underbrace{F^{-3}(s)}_{3}$$

$$\left.\frac{\partial F}{\partial s}\right|_{s=0}=\frac{H_0}{4}\left(\frac{r_i}{v_i}\right)^{1/2} \qquad F(s=0)=\left(\frac{r_i}{v_i}\right)^{1/2}$$

$$\gamma_s=\left(\frac{v_{ri}}{v_i}\right)^2$$

stable orbital speed (virial theorem):

$$v_{ri}=\sqrt{\frac{-nG_n r_i}{(2r_i)^{1-n}}\frac{m_1+m_2}{2}}$$

Frequency ω: $\omega\equiv\omega(s)$

Frequency function F(s): $F(s)\equiv(\omega+s\dot{\omega})^{-1/2}=\left(\frac{\partial(\omega s)}{\partial s}\right)^{-1/2}$

Ratio γ_s reflects competition: gravity vs. angular momentum; System in initial virial equilibrium if $\gamma_s=1$;

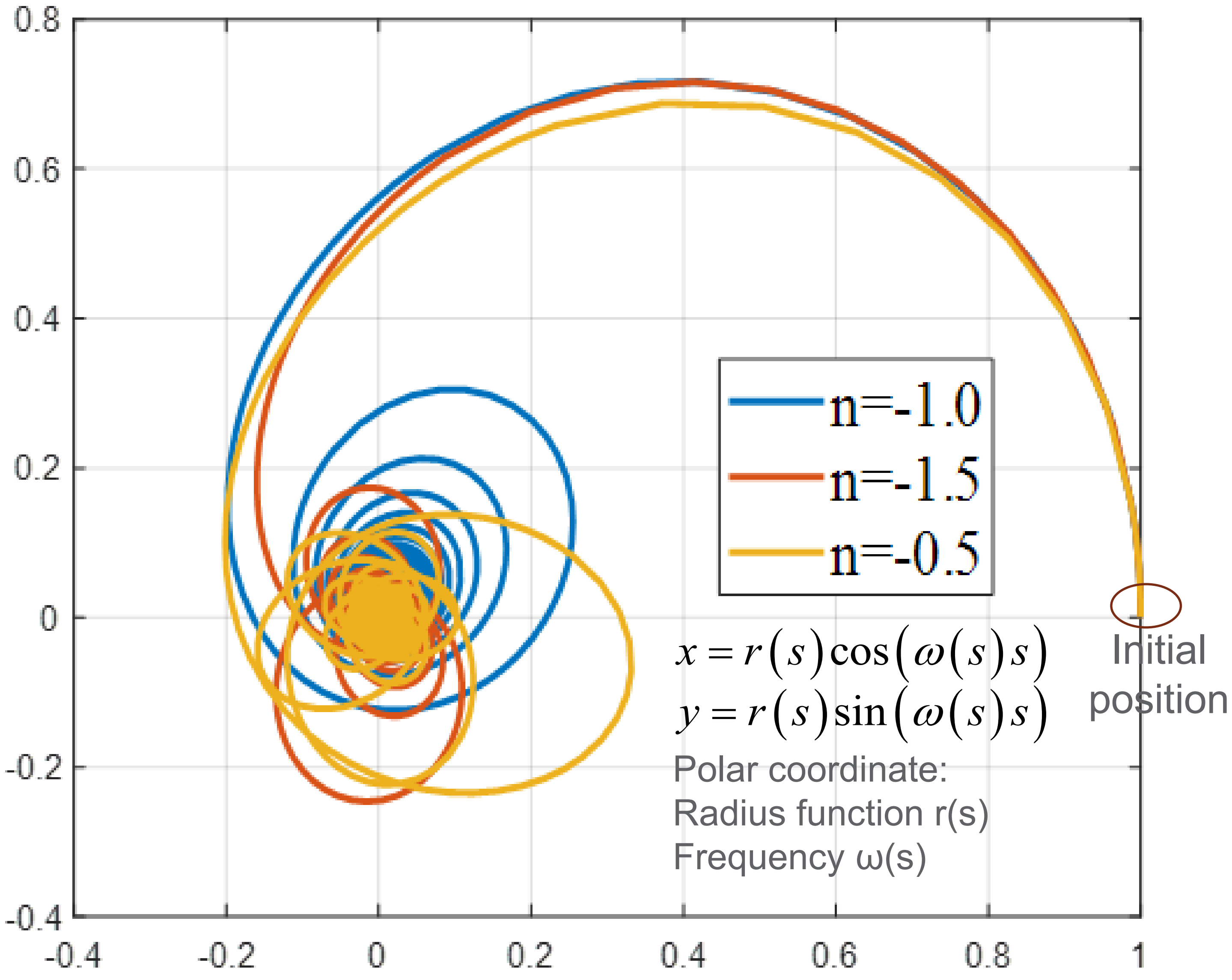
term 2 (gravitational force) = term 3 (angular momentum) leads to mean solutions:

$$F_m(s)=\gamma_s^{-1/(2+n)}\left(\frac{r_i}{v_i}\right)^{1/2}\exp\left(-\frac{2-n}{2+n}\cdot\frac{H_0 s}{4}\right)$$

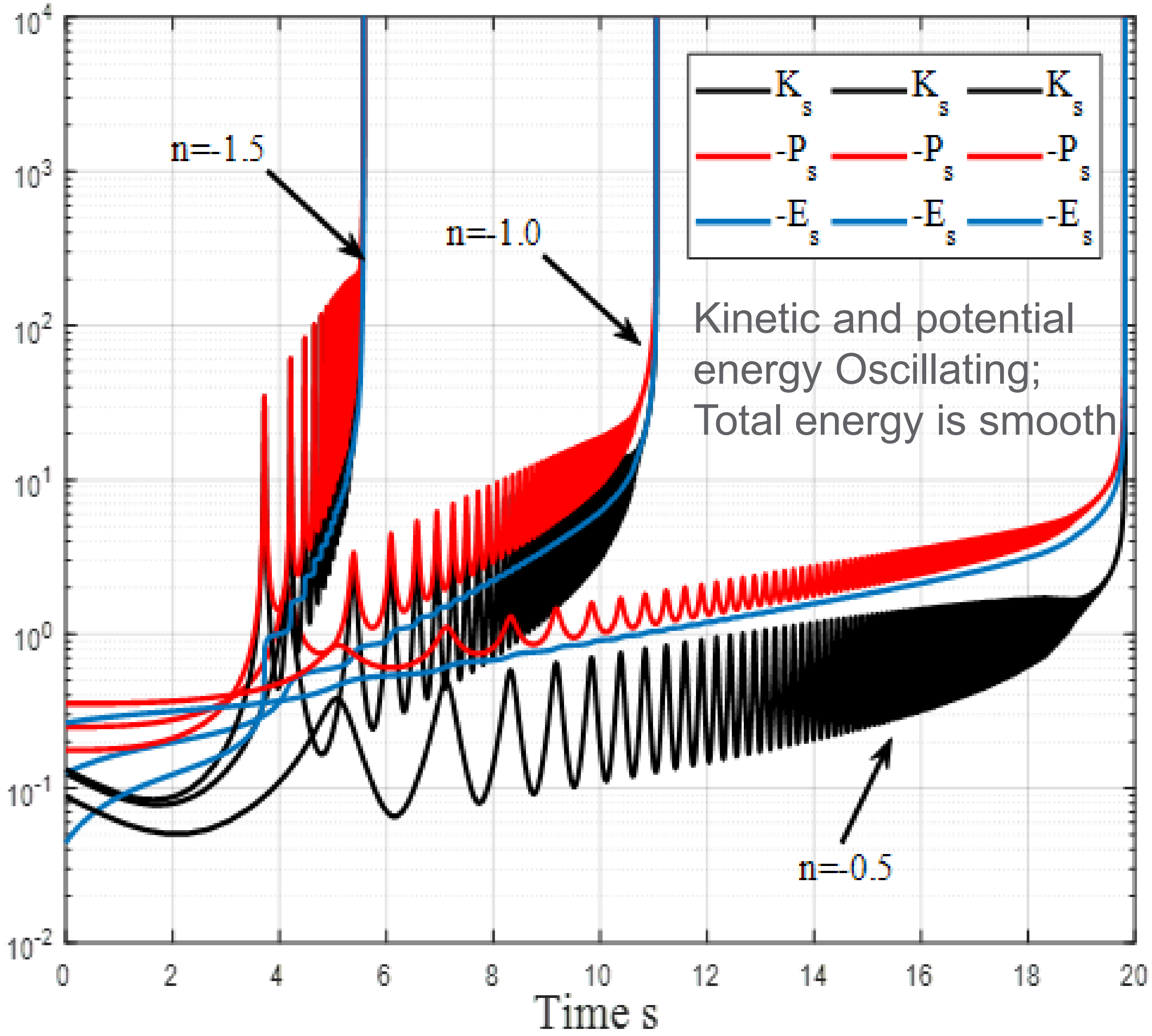
$$\omega_m(s)=\frac{1}{2\lambda_s s}\frac{2+n}{2-n}\gamma_s^{2/(2+n)}\exp\left(\frac{2-n}{2+n}\cdot\frac{H_0 s}{2}\right)$$

$$r_m(s)=\gamma_s^{-1/(2+n)} r_i \exp\left(-\frac{H_0 s}{2+n}\right)$$

Examples of numerical solutions

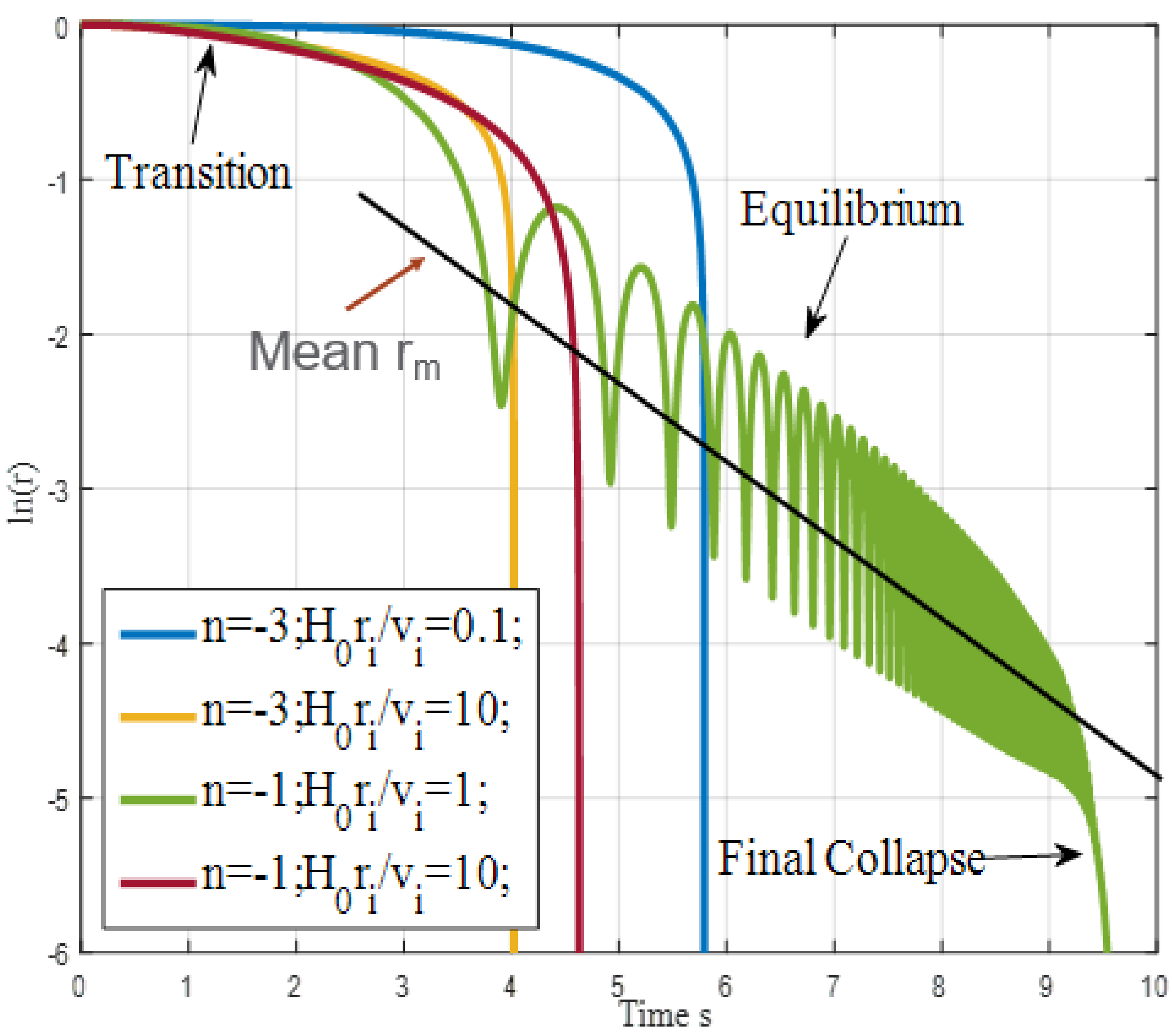


Trajectory of the motion of displacement vector **r**

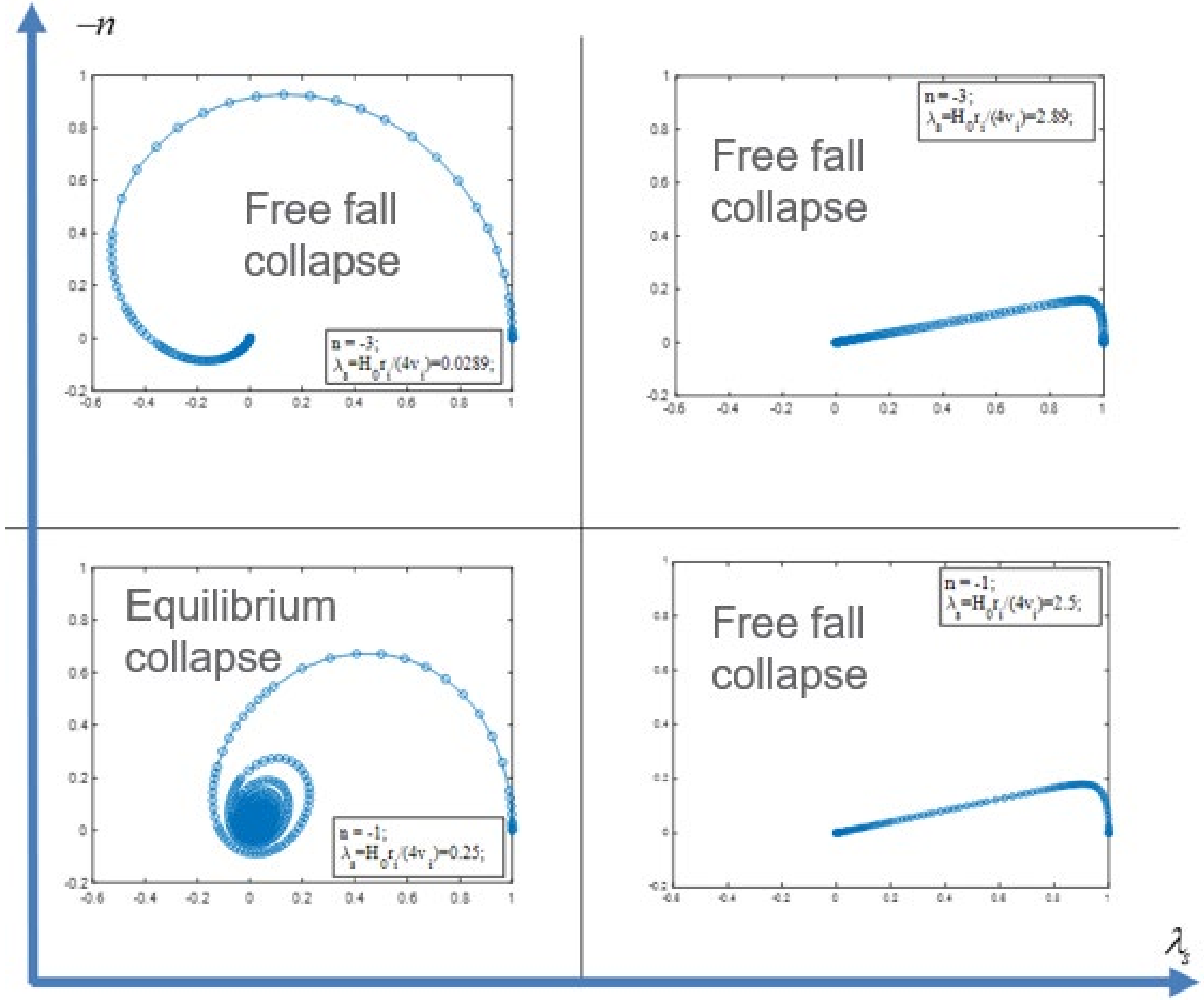


Time evolution of system kinetic, potential and total energy

Two-body collapse: free fall or equilibrium?



Variation of radius **r** with time **s** exbibits two different collapse. Equilibrium collapse involves a mean and fluctuation.



Depending on the competition between three forces, two types of collapse can be identified. $\lambda_s = \frac{H_0 r_i}{4 v_i}$

TBCM model in the simplest form and perturbative solutions for equilibrium collapse

Decompose frequency function F(s) into the mean and amplitude and substitute to equation for F(s):

$$F(s) = F_m(s) F_a(\omega_m s) = F_m(s) F_a(x)$$

mean ↗ ↓ ↖ amplitude

The simplest form of TBCM for amplitude function F_a:

$$\frac{\partial^2 F_a(x)}{\partial x^2} = \underbrace{\frac{2n}{(2-n)^2}\frac{F_a(x)}{x^2}}_{1} - \underbrace{F_a^{n-1}(x)}_{2} + \underbrace{F_a^{-3}(x)}_{3} \qquad x = \omega_m(s)s$$

$$F_a(x_0) = \gamma_s^{1/(2+n)} \qquad \left.\frac{\partial F_a}{\partial x}\right|_{x=x_0} = \frac{\beta_s \gamma_s^{-1/(2+n)}}{2+n} \qquad x_0 = \frac{2\gamma_s^{2/(2+n)}}{\beta_s}\frac{2+n}{2-n}$$

Mean solutions:

$$F_m(s) = \gamma_s^{-1/(2+n)}\left(\frac{r_i}{v_i}\right)^{1/2} \exp\left(-\frac{2-n}{2+n}\cdot\frac{H_0 s}{4}\right)$$

$$\omega_m(s) = \frac{2}{\beta_s s}\frac{2+n}{2-n}\gamma_s^{2/(2+n)} \exp\left(\frac{2-n}{2+n}\cdot\frac{H_0 s}{2}\right)$$

For long-range interaction n>-2, the competition between terms 2 and 3 leads to an oscillatory solution vibrating around the mean value F_a=1

Solution now only depends on **three** parameters:

- ratio γ_s reflects competition: gravity vs. angular momentum
- ratio β_s reflects competition: damping (or expanding background) vs. angular momentum
- exponent ***n***

$$\gamma_s = (v_{ri}/v_i)^2$$

$$\beta_s = H_0 r_i / v_i$$

Stable orbital speed:

$$v_{ri} = \sqrt{\frac{-nG_n r_i}{(2r_i)^{1-n}}\frac{m_1+m_2}{2}}$$

Classifying two-body collapse

$$\gamma_s = \left(v_{ri}/v_i\right)^2 \qquad \beta_s = H_0 r_i / v_i$$

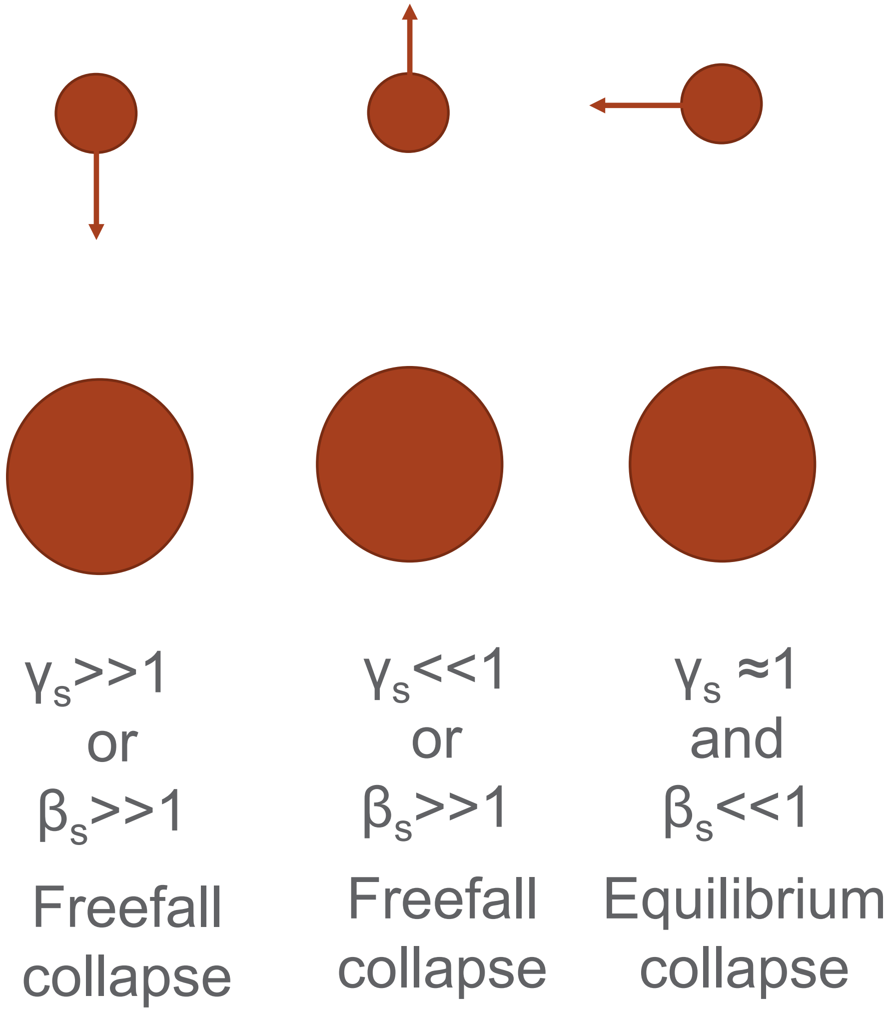
Freefall collapse :

- Short-range interaction with exponent $n<-2$
- $\gamma_s >>1$: gravity is dominant over angular momentum
- $\beta_s >>1$: damping is dominant
- $\gamma_s <<1$: There is a turnaround before free fall

Equilibrium collapse :

- $\gamma_s \approx 1$ and $\beta_s <<1$: stable orbit (angular momentum comparable with gravity) with week damping
- $\beta_s = 0$: Standard two-body problem in static background

Equilibrium collapse has an oscillatory motion with a much longer time to fully collapse than free fall collapse!



Solutions of free fall collapse and free fall time

Zero initial speed (no angular momentum):

$$v_i = 0 \Rightarrow \gamma_s = \left(v_{ri}/v_i\right)^2 \to \infty$$

Free fall time in static background:

$$s_{ce} = \frac{\pi r_i^{3/2}}{\sqrt{G\left(m_1 + m_2\right)}}$$

$$\lambda_{si} = \frac{H_0 r_i}{4 v_{ri}} \Leftarrow$$ Competition between damping and gravity

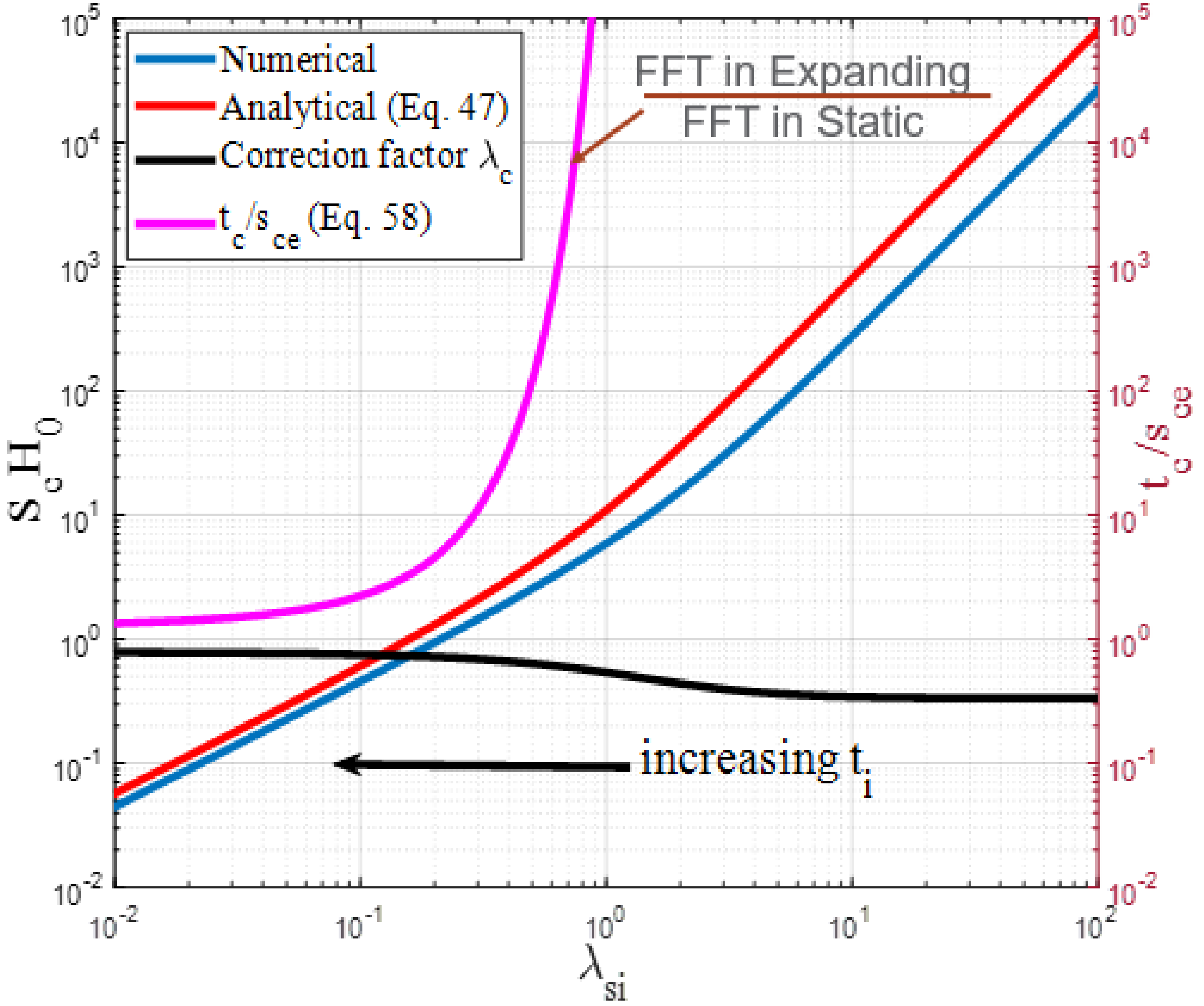
For small λ_{si} (weak damping):

$$s_c \approx s_{c1} = 4\sqrt{2}\frac{\lambda_{si}}{H_0} = \sqrt{\frac{2^{3-n} r_i^{2-n}}{-nG_n\left(m_1 + m_2\right)}}$$

For large λ_{si} (strong damping):

$$s_c \approx s_{c2} = 8\frac{\lambda_{si}^2}{H_0} = \frac{H_0 2^{1-n} r_i^{2-n}}{-nG_n\left(m_1 + m_2\right)}$$

- Due to damping, free fall time of two-body in expanding background is greater than the free fall time of same two-body in static background.



The earlier collapse starts (the smaller t_i), the greater the free fall time (H is decreasing)

Perturbative solutions for equilibrium collapse

Solve: $$\frac{\partial^2 F_a(x)}{\partial x^2} = \frac{2n}{(2-n)^2}\frac{F_a(x)}{x^2} - F_a^{n-1}(x) + F_a^{-3}(x)$$

Frequency function:

mean — fluctuation

$$F(s) = \left(\frac{r_i}{v_i}\right)^{1/2} \exp\left(-\frac{2-n}{2+n}\cdot\frac{H_0 s}{4}\right)\left\{1+\frac{\beta_s}{(2+n)^{3/2}}\sin(\theta_s)\right\}$$

Angle function:

$$\theta_s(s) = \frac{2\sqrt{2+n}}{\beta_s}\frac{2+n}{2-n}\left[\exp\left(\frac{2-n}{2+n}\cdot\frac{H_0 s}{2}\right)-1\right]$$

Radius function:

$$r(s) = r_i \exp\left(-\frac{H_0 s}{2+n}\right)\left\{1+\frac{\beta_s}{(2+n)^{3/2}}\sin(\theta_s)\right\}$$

Radial velocity:

$$\dot{r} = \frac{\partial r(s)}{\partial s} = \frac{H_0 r_i}{(2+n)}\exp\left(-\frac{nH_0 s}{2(2+n)}\right)\cos(\theta_s) - \frac{H_0 r}{2+n}$$

Specific kinetic energy:

$$K_s \approx \frac{2m_1 m_2 v_i^2}{(m_1+m_2)^2}\exp\left(\frac{-nH_0 s}{2+n}\right)\left[1-\frac{2\beta_s}{(2+n)^{3/2}}\sin\theta_s\right]$$

Specific potential energy:

$$P_s \approx \frac{2m_1 m_2 v_i^2}{(m_1+m_2)^2}\exp\left(\frac{-nH_0 s}{2+n}\right)\left[\frac{2}{n}+\frac{2\beta_s}{(2+n)^{3/2}}\sin\theta_s\right]$$

Specific total energy (fluctuation cancelled):

$$E_s = K_s + P_s = \frac{-(2+n)m_1 m_2}{(m_1+m_2)}\frac{G_n r_i}{(2r_i)^{1-n}}\exp\left(\frac{-nH_0 s}{2+n}\right)$$

Radial momentum: $$G_s = \frac{4m_1 m_2}{(m_1+m_2)^2}\mathbf{r}\cdot\mathbf{v} = \frac{4m_1 m_2}{(m_1+m_2)^2}\dot{r}r$$

Angular momentum: $$\mathbf{H}_s = \frac{4m_1 m_2}{(m_1+m_2)^2} r^2 F^{-2}(s)\hat{\mathbf{z}} = \frac{4m_1 m_2 v_i r_i}{(m_1+m_2)^2}\exp\left(-\frac{1}{2}H_0 s\right)\hat{\mathbf{z}}$$

Mean energy satisfying virial theorem: $2\langle K_s\rangle - n\langle P_s\rangle = 0$

All have exponential evolution in time scale s!

Critical values of β_s (analogue of critical damping) and critical halo density

Equilibrium collapse :

- $\gamma_s \approx 1$ and $\beta_s <<1$: stable orbit with week damping
- $\beta_s = 0$: Standard two-body problem in static background

$$\gamma_s = \left(\frac{v_{ri}}{v_i}\right)^2 \approx 1 \Rightarrow \beta_s = \frac{H_0 r_i}{v_i} \approx \frac{H_0 r_i}{v_{ri}} = \frac{Radial}{Circular} \ll 1$$

angular momentum comparable with gravity

Weak damping
Also see angle of incidence

First critical value for existence of equilibrium collapse with oscillator solution:

$$\frac{\beta_s}{(2+n)^{3/2}} \leq 1 \Rightarrow \beta_{s1} = (2+n)^{3/2}$$

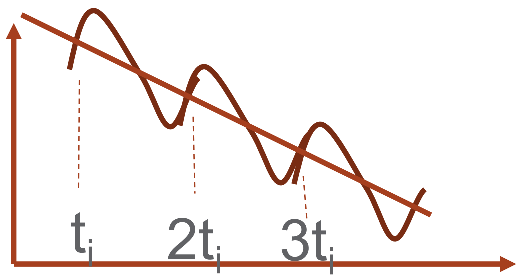
Second critical value for equilibrium collapse with oscillator solution:

Radius function:

$$r(s) = r_i \exp\left(-\frac{H_0 s}{2+n}\right)\left\{1 + \frac{\beta_s}{(2+n)^{3/2}} \sin(\theta_s)\right\}$$

Angle function:

$$\theta_s(s) = \frac{2(2+n)^{3/2}}{\beta_s(2-n)}\left[\left(\frac{t}{t_i}\right)^{\frac{2-n}{3(2+n)}} - 1\right]$$



$$\sin\left[\theta_s(t = kt_i)\right] = 0 \Rightarrow \beta_{s2} = \frac{(2+n)^{3/2}}{(2-n)\pi}$$

$$n = \frac{2-6m}{1+3m}$$

$$m = 1, 2, \ldots \infty$$

$$n = -1, -10/7, -8/5 \ldots -2$$

Critical halo density:

$$\Delta_c = 2/\beta_{s2}^2 = 18\pi^2$$

Evolution in comoving system for two-body angular velocity, spin parameter and angle of incidence

Evolution in transformed system with time scale *s* can be equivalently transformed back to original comoving system:

$$ds/dt = a^{-3/2} \rightarrow s = t_0 \ln\left(t/t_i\right)$$

$$\exp\left(\tau H_0 s\right) \rightarrow \left(a/a_i\right)^{\tau}$$

Exponential evolution in time scale s → Power-law evolution in time t

Two-body kinetic energy:

$$K_s \approx \frac{2m_1 m_2 v_i^2}{\left(m_1+m_2\right)^2} \exp\left(\frac{-nH_0 s}{2+n}\right)\left[1-\frac{2\beta_s}{\left(2+n\right)^{3/2}}\sin\theta_s\right]$$

Kinetic energy in terms of angular velocity:

$$\frac{1}{2}\left(m_1\mu^2+m_2\left(2-\mu\right)^2\right)\omega_s^2 r^2 = \left(m_1+m_2\right)K_s$$

$$\omega_s \approx \frac{v_i}{r_i}\exp\left[\frac{2-n}{2\left(2+n\right)}H_0 s\right] \rightarrow \omega_t = \frac{r_i^{3/2}}{\beta_s} H r_m^{-3/2}$$

Two-body spin parameter:

$$\lambda_p = \frac{\left|\mathbf{H}_s\right|\left|E_s\right|^{1/2}}{G\left(m_1+m_2\right)} = \frac{\sqrt{2}}{2}\frac{\left(m_1 m_2\right)^{3/2}}{\left(m_1+m_2\right)^3} = \frac{\sqrt{2}}{16} \approx 0.0884$$

Evolution of two-body angle of incidence:

$$\cot\left(\theta_{\mathbf{vr}}\right) = \frac{\mathbf{r}\cdot\mathbf{v}}{\left|\mathbf{r}\times\mathbf{v}\right|} = \frac{G_s}{\left|\mathbf{H}_s\right|} = -\frac{\beta_s}{\left(2+n\right)}\left(\frac{a}{a_i}\right)^{-\frac{3}{2}}$$

Kinetic energy for large halos with an infinitesimal lifetime:

$$dK_h = \frac{K_s\left(s=0\right)}{a} = \frac{2M}{M^2}dM\cdot\alpha_s\frac{GM}{ar_i} \rightarrow \sigma_v^2 = \frac{2}{3}K_h = \frac{GM}{2r_h}$$

Angular velocity in co-moving system dependent on halo size r_m, larger halo has smaller angular velocity

Prove stable clustering hypothesis (SCH) and derive generalized SCH

Peculiar pairwise velocity:

$$\Delta u_L(2r) = \left(u_L' - u_L\right)$$

$$a^{1/2}\Delta u_L = 2\frac{\mathbf{r}\cdot\mathbf{v}_1}{r} = 2\frac{G_s(s)}{r} = 2\dot{r}$$

[See two-body virial quantity for radial flow]

$$\Delta u_L = -2Har + 2\beta_s u_i \cos\theta_s$$

$$\langle \Delta u_L \rangle = -2Har + 2\langle \beta_s u_i \cos\theta_s \rangle$$

$$\langle \Delta u_L \rangle = -2Har = -2a^{-1/2}H_0 r$$

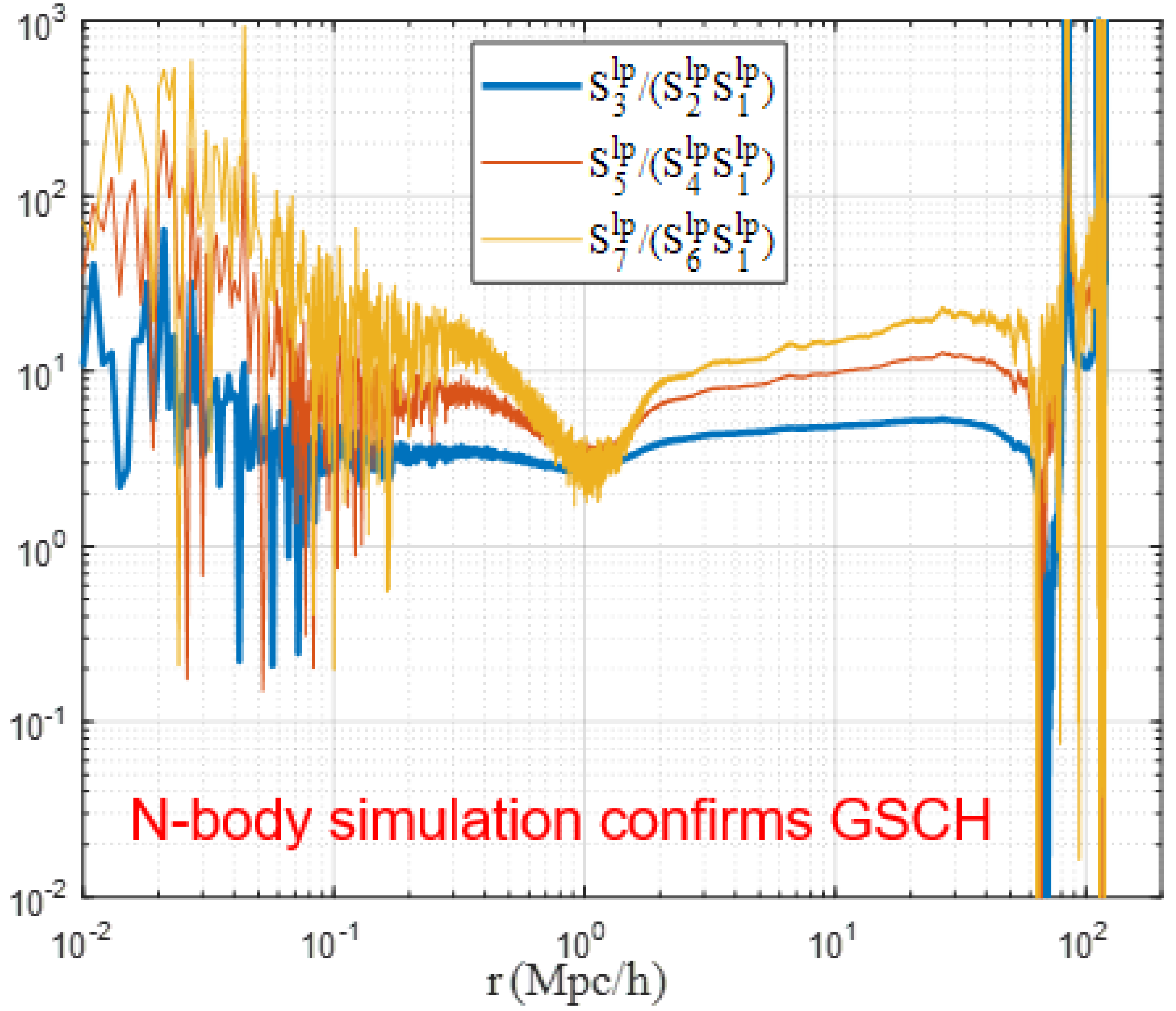
Stable clustering hypothesis (SCH) proved

$$\langle \Delta u_L^2 \rangle (r \to 0) = 4\langle \beta_s^2 u_i^2 \cos^2\theta_s \rangle > 0$$

Non-zero pairwise dispersion, a feature of collisionless flow

$$\langle \Delta u_L^{2m+1} \rangle = (2m+1)\langle \Delta u_L^{2m} \rangle \langle \Delta u_L \rangle = (2m+1)\langle \Delta u_L^{2m} \rangle (-2Har)$$

Generalized stable clustering hypothesis (GSCH)



Connections with spherical collapse model (SCM)

- Spherical collapse model (SCM) solves the motion of spherical shells. Many important insights can be obtained from SCM.
- There are fundamental connections between two-body collapse model (TBCM) and SCM.
- The original SCM describe exactly a two-body collapse with one-dimensional radial motion only and zero angular momentum.
- TBCM model describes a spherical collapse model with a non-zero angular momentum and non-radial orbits
- Both models predict a critical halo density ratio $\Delta=18\pi^2$, while TBCM can predict freefall and equilibrium collapse and SCH and GSCH.

$$\frac{d^2R}{dt^2}=-\frac{GM}{R^2}$$

Equation of motion for SCM in physical coordinate

$$\frac{\partial^2 r}{\partial s^2}+\frac{H_0}{2}\frac{\partial r}{\partial s}+\frac{GM}{2(2r)^2}=\underbrace{\frac{H_0^2}{2}r}_{1}$$

Equation of motion for SCM in comoving system

Term 1: due to the absence of a uniform background density

$$\frac{\partial^2 r}{\partial s^2}+\frac{H_0}{2}\frac{\partial r}{\partial s}+\frac{GM}{2(2r)^2}=\underbrace{\frac{(r_i v_i)^2}{r^3}\exp(-H_0 s)}_{2}$$

Equation of motion for two-body collapse model (TBCM)

Term 2: angular momentum

Summary and keywords

Harmonic oscillator	Transformed system	Free fall time
Critical damping	Two-body collapse	Expanding background
Stable clustering	Generalized SCH	Spherical collapse model
Equilibrium collapse	Freefall collapse	Critical halo density

- Formulate two-body collapse model (TBCM) that plays the same role as harmonic oscillator for fundamental understanding of gravitational collapse
- Propose the competition between gravity, expanding background, and angular momentum and classify collapse into: 1) freefall collapse for weak angular momentum; and 2) equilibrium collapse for weak damping
- Identify two critical values, $\beta_{s1}=1$ for free fall collapse and $\beta_{s2}=1/(3\pi)$ for equilibrium collapse, that quantifies the competition between damping and gravity
- Predict a critical halo density ratio of $18\pi^2$, same as the spherical collapse model.
- Prove the stable clustering hypothesis (SCH), i.e. mean pairwise velocity proportional to the separation r.
- Develop a generalized stable clustering hypothesis (GSCH) for higher order moments of pairwise velocity.