# A Note on Properties of Fermat Number

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**Abstract.** In this paper, we provide some properties of fermat number.

**Keywords.** Fermat Number; Divisor; Congruence

The $n$-th fermat number $F_n$ is defined as

$$F_n = 2^{2^n} + 1. \tag{1}$$

We will provide some properties of fermat numbers.

**Proposition 1.** *Let* $n \in \mathbb{N}$. *Then,*

$$F_n \equiv 1 \pmod{4}. \tag{2}$$

*Proof.* By (1), it is

$$F_n = 2^{2^n} + 1 \tag{3}$$
$$F_n - 1 = 2^{2^n} \tag{4}$$

Since $n \in \mathbb{N}$, $2^{2^1} = 4 \mid F_n - 1$.

$$\therefore F_n \equiv 1 \pmod{4} \tag{5}$$

This completes the proof. $\square$

**Proposition 2.** *Let* $n \in \mathbb{N}$. *Let* $k \in \mathbb{N}$, $k \leq 2^n - 1$, $k \equiv 1 \pmod{2}$. *Then,*

$$3 \cdot 2^k \mid F_n + 2^k - 1. \tag{6}$$

*Proof.* By (1), it is

$$F_n = 2^{2^n} + 1 \tag{7}$$
$$F_n + 2^k - 1 = 2^{2^n} + 2^k = 2^k(2^{2^n - k} + 1) \tag{8}$$
$$= 2^k(2^{2^n - k} + 1^{2^n - k}) \tag{9}$$

Since $k \in \mathbb{N}$, $k \leq 2^n - 1$ and $2^n - k \equiv 1 (\mathrm{mod}\ 2)$ ($\because k \equiv 1 \pmod 2$) so $2 + 1 \mid 2^{2^n - k} + 1^{2^n - k}$.

Therefore, it is

$$2^k \mid F_n + 2^k - 1,\ 3 \mid F_n + 2^k - 1 \tag{10}$$

thus

$$\therefore 3 \cdot 2^k \mid F_n + 2^k - 1. \tag{11}$$

This completes the proof. □

**Proposition 3.** *Let $n, k \in \mathbb{N}$. Then,*

$$2(2 + k) \mid F_n + 2k^{2^n - 1} - 1. \tag{12}$$

*Proof.* By (1), it is

$$F_n = 2^{2^n} + 1 \tag{13}$$

$$F_n + 2k^{2^n - 1} - 1 = 2^{2^n} + 2k^{2^n - 1} \tag{14}$$

$$= 2(2^{2^n - 1} + k^{2^n - 1}) \tag{15}$$

Since $n, k \in \mathbb{N}$ and $2^n - 1 \equiv 1 \pmod 2$ so 2 + k — $2^{2^n - 1} + k^{2^n - 1}$.

Therefore, it is

$$2^k \mid F_n + 2^k - 1,\ 3 \mid F_n + 2^k - 1 \tag{16}$$

thus

$$\therefore 3 \cdot 2^k \mid F_n + 2^k - 1. \tag{17}$$

This completes the proof. □

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# Reference

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