

Online Companion: Embedding Dependencies Between Wind Farms in Uncertainty-Aware Optimal Power Flow

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This manuscript serves as an electronic companion to [1]. Section 1 defines the Wasserstein probability metric. Sections 3 and 4 provide the final optimization model under the metric-based ambiguity set \mathcal{M}_1 and the copula-based ambiguity set \mathcal{M}_2 , respectively. The real-time operational problems for the out-of-sample simulation, in both the DC and AC formulations, are given in Section 5. Finally, Section 6 gives the economic and technical input data of our case study built upon the IEEE reliability test system.

1 Wasserstein probability metric

The distance between two distributions, namely \mathbb{Q}_1 and \mathbb{Q}_2 , is assessed via the Wasserstein metric, whose mathematical formulation is given in Definition 1.1. This metric can be seen as an optimal transportation problem that aims to minimize the cost of transporting the probability mass from \mathbb{Q}_1 to \mathbb{Q}_2 .

Definition 1.1 (*Wasserstein metric [2]*) *The Wasserstein metric $d : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}^+$ is defined as*

$$d(\mathbb{Q}_1, \mathbb{Q}_2) = \left\{ \begin{array}{l} \min_{\Pi \in \Pi(\mathbb{Q}_1, \mathbb{Q}_2)} \int_{\Xi^2} \|\tilde{\xi}_1 - \tilde{\xi}_2\| \Pi(d\tilde{\xi}_1, d\tilde{\xi}_2) \\ \text{s.t. } \Pi \text{ is a joint distribution of } \tilde{\xi}_1 \text{ and } \tilde{\xi}_2 \\ \text{with marginals } \mathbb{Q}_1 \text{ and } \mathbb{Q}_2, \text{ respectively} \end{array} \right\}. \quad (1)$$

The objective function in (1) represents the transportation cost of moving the probability mass from \mathbb{Q}_1 to \mathbb{Q}_2 . The chosen cost for moving each data sample is the norm $\|\tilde{\xi}_1 - \tilde{\xi}_2\|$, defined as the Wasserstein distance. The joint distribution Π , which is the optimization variable, refers to the optimal transportation plan. Based on this definition, the Wasserstein metric-based ambiguity set collects the closest distributions from an empirical one $\hat{\mathbb{Q}}_N = \frac{1}{N} \sum_{i=1}^N \delta_{\tilde{\xi}_i}$, that is typically constituted of N observed samples, each assigned with probability $\frac{1}{N}$. We thereby mathematically define the Wasserstein ambiguity set as follows:

$$\mathcal{M}_1 = \left\{ \mathbb{Q} \in \mathcal{F} \mid d(\mathbb{Q}, \hat{\mathbb{Q}}_N) \leq \theta_1 \right\}. \quad (2)$$

2 Proof of first assertion in Theorem 1

We depart from the worst-case expectation problem

$$\max_{\mathbb{Q} \in \mathcal{M}_2} \mathbb{E}^{\mathbb{Q}} \left[a(x)^\top \tilde{\xi} + b(x) \right], \quad (3)$$

where \mathbb{Q} is the variable distribution function and $a(x)^\top \tilde{\xi} + b(x)$ represents the linear objective function, which depends on outer decision variables $x \in \mathcal{X}$. The problem seeks the distribution within the ambiguity set \mathcal{M}_2 , defined by (4), which maximizes the objective function in expectation. This problem can be developed as follows:

$$\max_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[a(x)^\top \tilde{\xi} + b(x) \right] \quad (4a)$$

$$\text{s.t. } d_W \left(\mathbb{Q}, \widehat{\mathbb{Q}}_N \right) \leq \theta_1 \quad (4b)$$

$$d_W \left(\mathbb{C}, \widehat{\mathbb{C}}_N \right) \leq \theta_2. \quad (4c)$$

From now on, for the sake of clarity, we assume that distribution \mathbb{Q} is discrete. Note that it is straightforward to extend the proof for continuous distribution functions. Note also that this assumption will be relaxed in the following developments, making the final result valid for continuous distributions as well. Given a set of fixed samples ξ_l (e.g., a mesh of the support of uncertainty) with $l \in \{1, \dots, L\}$, where L is an arbitrarily large number, the variable distribution \mathbb{Q} is defined by

$$\mathbb{Q} = \sum_{l=1}^L \mathbb{Q}_l \delta_{\xi_l}, \quad (5)$$

where \mathbb{Q}_l is the variable probability affected to sample ξ_l , and δ_{ξ_l} is the Dirac distribution centered on ξ_l . We are now able to reformulate (4) into

$$\max_{\mathbb{Q}_l} \sum_{l=1}^L \Psi(x, \xi_l) \mathbb{Q}_l \quad (6a)$$

$$\text{s.t. } \left\{ \begin{array}{l} \min_{\pi_{li}^{(1)} \geq 0} \sum_{l=1}^L \sum_{i=1}^N d \left(\widehat{\xi}_i, \xi_l \right) \pi_{li}^{(1)} \\ \text{s.t. } \sum_{l=1}^L \pi_{li}^{(1)} = \left(\widehat{\mathbb{Q}}_N \right)_i \quad \forall i \in \{1, \dots, N\} \\ \sum_{i=1}^N \pi_{li}^{(1)} = \mathbb{Q}_l \quad \forall l \in \{1, \dots, L\} \end{array} \right\} \leq \theta_1 \quad (6b)$$

$$\left\{ \begin{array}{l} \min_{\pi_{li}^{(2)} \geq 0} \sum_{l=1}^L \sum_{i=1}^N d_F \left(\widehat{\xi}_i, \xi_l \right) \pi_{li}^{(2)} \\ \text{s.t. } \sum_{l=1}^L \pi_{li}^{(2)} = \left(\widehat{\mathbb{C}}_N \right)_i \quad \forall i \in \{1, \dots, N\} \\ \sum_{i=1}^N \pi_{li}^{(2)} = \mathbb{C}_l \quad \forall l \in \{1, \dots, L\} \end{array} \right\} \leq \theta_2, \quad (6c)$$

where the expectation in the objective function has been developed for distribution \mathbb{Q} , and the Wasserstein distances $d_W \left(\mathbb{Q}, \widehat{\mathbb{Q}}_N \right)$ and $d_W \left(\mathbb{C}, \widehat{\mathbb{C}}_N \right)$ have been reformulated according to their mathematical definitions. The variables $\pi_{li}^{(1)} \in \mathbb{R}^{L \times N}$ and $\pi_{li}^{(2)} \in \mathbb{R}^{L \times N}$ represent the optimal transportation plan between distributions $\widehat{\mathbb{Q}}_N$ and \mathbb{Q} , and $\widehat{\mathbb{C}}_N$ and \mathbb{C} , respectively. The distance functions $d \left(\widehat{\xi}_i, \xi_l \right)$ and $d_F \left(\widehat{\xi}_i, \xi_l \right)$ will be defined later on within the proof.

We reformulate (6) via three steps. First, we observe that the probabilities $(\widehat{\mathbb{Q}}_N)_i = (\widehat{\mathbb{C}}_N)_i = \frac{1}{N} \forall i \in \{1, \dots, N\}$ and $\mathbb{Q}_l = \mathbb{C}_l \forall l \in \{1, \dots, L\}$. Next, we are able to drop the minimization operators since they appear in a “less-or-equal-to” constraint and to add the variables $\pi_{li}^{(1)}$ and $\pi_{li}^{(2)}$ to the set of overall variables. Finally, we drop variables \mathbb{Q}_l by (i) substituting its expression in the third equation of (6b) into the objective function, and (ii) adding the constraint $\sum_{i=1}^N \pi_{li}^{(1)} = \sum_{i=1}^N \pi_{li}^{(2)} \forall l \in \{1, \dots, L\}$, which translates $\mathbb{Q}_l = \mathbb{C}_l \forall l \in \{1, \dots, L\}$. We end up in

$$\max_{\pi_{li}^{(1)}, \pi_{li}^{(2)}} \sum_{l=1}^L \sum_{i=1}^N \Psi(x, \xi_l) \pi_{li}^{(1)} \quad (7a)$$

$$\text{s.t.} \quad \sum_{l=1}^L \sum_{i=1}^N d(\widehat{\xi}_i, \xi_l) \pi_{li}^{(1)} \leq \theta_1 \quad : \alpha \quad (7b)$$

$$\sum_{l=1}^L \pi_{li}^{(1)} = \frac{1}{N} \quad \forall i \in \{1, \dots, N\} \quad : y_i \quad (7c)$$

$$\sum_{l=1}^L \sum_{i=1}^N d_F(\widehat{\xi}_i, \xi_l) \pi_{li}^{(2)} \leq \theta_2 \quad : \beta \quad (7d)$$

$$\sum_{l=1}^L \pi_{li}^{(2)} = \frac{1}{N} \quad \forall i \in \{1, \dots, N\} \quad : t_i \quad (7e)$$

$$\sum_{i=1}^N \pi_{li}^{(1)} = \sum_{i=1}^N \pi_{li}^{(2)} \quad \forall l \in \{1, \dots, L\} \quad : \sigma_l, \quad (7f)$$

where $\{\alpha, \beta, y_i, t_i, \sigma_l\}$ is the set of dual variables. Next, we dualize (7), yielding

$$\min_{\alpha \geq 0, \beta \geq 0, y_i, t_i, \sigma_l} \alpha \theta_1 + \beta \theta_2 + \frac{1}{N} \sum_{i=1}^N y_i + \frac{1}{N} \sum_{i=1}^N t_i \quad (8a)$$

$$\text{s.t.} \quad y_i \geq \Psi(x, \xi_l) - \alpha d(\widehat{\xi}_i, \xi_l) + \sigma_l \quad \forall i, \forall l \quad (8b)$$

$$t_i \geq -\beta d_F(\widehat{\xi}_i, \xi_l) - \sigma_l \quad \forall i, \forall l. \quad (8c)$$

The constraints in (8) can be equivalently reformulated using maximization operators:

$$\min_{\alpha \geq 0, \beta \geq 0, y_i, t_i, \sigma_l} \alpha \theta_1 + \beta \theta_2 + \frac{1}{N} \sum_{i=1}^N y_i + \frac{1}{N} \sum_{i=1}^N t_i \quad (9a)$$

$$\text{s.t.} \quad y_i \geq \max_{\xi_l, \sigma_l} \Psi(x, \xi_l) - \alpha d(\widehat{\xi}_i, \xi_l) + \sigma_l \quad \forall i \quad (9b)$$

$$t_i \geq \max_{\xi_l, \sigma_l} -\beta d_F(\widehat{\xi}_i, \xi_l) - \sigma_l \quad \forall i. \quad (9c)$$

The maximization operators in (9b) and (9c) can be moved to the objective function, while dropping auxiliary variables y_i and t_i . The summation and maximization operators are merged such that the variable σ_l simplifies. Furthermore, we relax the assumption on discrete set of sample ξ_l by introducing a continuous support of uncertainty $\Xi = \{\xi \in \Xi \mid C\xi \leq d\}$. Finally, for the ease of developments, we re-introduce auxiliary variable y_i , such that (9) is equivalent to

$$\min_{\alpha, \beta \geq 0, y_i} \alpha \theta_1 + \beta \theta_2 + \frac{1}{N} \sum_{i=1}^N y_i \quad (10a)$$

$$\text{s.t. } y_i \geq \max_{\xi \in \Xi} a(x)^\top \xi + b(x) - \alpha d(\widehat{\xi}_i, \xi) - \beta d_F(\widehat{\xi}_i, \xi) \quad \forall i. \quad (10b)$$

which completes the proof.

3 DR-OPF under metric-based ambiguity set \mathcal{M}_1

3.1 Meshed transmission systems

$$\min_{\substack{g, \bar{r}, \underline{r}, V, \lambda, \sigma_i, \gamma_i, \\ \tau_f, \lambda_f, \sigma_{f,i}, \gamma_{f,i,1}, \gamma_{f,i,2}, \\ \bar{\tau}_g, \bar{\lambda}_g, \bar{\sigma}_{g,i}, \bar{\gamma}_{g,i,1}, \bar{\gamma}_{g,i,2}, \underline{\tau}_g, \underline{\lambda}_g, \underline{\sigma}_{g,i}, \underline{\gamma}_{g,i,1}, \underline{\gamma}_{g,i,2}}} c^\top g + \bar{c}^\top \bar{r} + \underline{c}^\top \underline{r} + \lambda \rho + \frac{1}{N} \sum_{i=1}^N \sigma_i \quad (11a)$$

s.t.

$$g + \bar{r} \leq p^{\max} \quad (11b)$$

$$g - \underline{r} \geq 0 \quad (11c)$$

$$0 \leq \underline{r} \leq r^{\max}; 0 \leq \bar{r} \leq r^{\max} \quad (11d)$$

$$\mathbf{1}^\top g + \mathbf{1}^\top W \mu - \mathbf{1}^\top d = 0 \quad (11e)$$

$$\mathbf{1}^\top V + \mathbf{1}^\top W = 0 \quad (11f)$$

$$\begin{cases} c^\top V \hat{\xi}_i + \gamma_i^\top (h - Q \hat{\xi}_i) \leq \sigma_i & \forall i \in \{1, \dots, N\} \\ \|Q^\top \gamma_i - c^\top V\|_* \leq \lambda & \forall i \in \{1, \dots, N\} \\ \gamma_i \geq 0 & \forall i \in \{1, \dots, N\} \end{cases} \quad (11g)$$

$$\left. \begin{cases} \bar{\tau}_g + \frac{1}{\epsilon} \left(\bar{\lambda}_g \rho + \frac{1}{N} \sum_{i=1}^N \bar{\sigma}_{g,i} \right) \leq 0 \\ V_g \hat{\xi}_i - \bar{r}_g - \bar{\tau}_g + \bar{\gamma}_{g,i,1}^\top (h - Q \hat{\xi}_i) \leq \bar{\sigma}_{g,i} & \forall i \in \{1, \dots, N\} \\ \bar{\gamma}_{g,i,2}^\top (h - Q \hat{\xi}_i) \leq \bar{\sigma}_{g,i} & \forall i \in \{1, \dots, N\} \\ \|Q^\top \bar{\gamma}_{g,i,1} - V_g\|_* \leq \bar{\lambda}_g & \forall i \in \{1, \dots, N\} \\ \|Q^\top \bar{\gamma}_{g,i,2}\|_* \leq \bar{\lambda}_g & \forall i \in \{1, \dots, N\} \\ \bar{\gamma}_{g,i,1} \geq 0; \bar{\gamma}_{g,i,2} \geq 0 & \forall i \in \{1, \dots, N\} \end{cases} \right\} \forall g \in \mathcal{G} \quad (11h)$$

$$\left. \begin{cases} \underline{\tau}_g + \frac{1}{\epsilon} \left(\underline{\lambda}_g \rho + \frac{1}{N} \sum_{i=1}^N \underline{\sigma}_{g,i} \right) \leq 0 \\ -V_g \hat{\xi}_i - \underline{r}_g - \underline{\tau}_g + \underline{\gamma}_{g,i,1}^\top (h - Q \hat{\xi}_i) \leq \underline{\sigma}_{g,i} & \forall i \in \{1, \dots, N\} \\ \underline{\gamma}_{g,i,2}^\top (h - Q \hat{\xi}_i) \leq \underline{\sigma}_{g,i} & \forall i \in \{1, \dots, N\} \\ \|Q^\top \underline{\gamma}_{g,i,1} + V_g\|_* \leq \underline{\lambda}_g & \forall i \in \{1, \dots, N\} \\ \|Q^\top \underline{\gamma}_{g,i,2}\|_* \leq \underline{\lambda}_g & \forall i \in \{1, \dots, N\} \\ \underline{\gamma}_{g,i,1} \geq 0; \underline{\gamma}_{g,i,2} \geq 0 & \forall i \in \{1, \dots, N\} \end{cases} \right\} \forall g \in \mathcal{G} \quad (11i)$$

$$\left. \begin{cases} \tau_f + \frac{1}{\epsilon} \left(\lambda_f \rho + \frac{1}{N} \sum_{i=1}^N \sigma_{f,i} \right) \leq 0 \\ \left(T_f^P (g + V \hat{\xi}) + T_f^W (\mu + \tilde{\xi}) - T_f^D (d) - f_f^{\max} \right) - \tau_f + \gamma_{f,i,1}^\top (h - Q \hat{\xi}_i) \leq \sigma_{f,i} & \forall i \in \{1, \dots, N\} \\ \gamma_{f,i,2}^\top (h - Q \hat{\xi}_i) \leq \sigma_{f,i} & \forall i \in \{1, \dots, N\} \\ \|Q^\top \gamma_{f,i,1} - (T_f^G V + T_f^W W)\|_* \leq \lambda_f & \forall i \in \{1, \dots, N\} \\ \|Q^\top \gamma_{f,i,2}\|_* \leq \lambda_f & \forall i \in \{1, \dots, N\} \\ \gamma_i \geq 0 & \forall i \in \{1, \dots, N\} \end{cases} \right\} \forall f \in \mathcal{F}. \quad (11j)$$

3.2 Radial distribution systems

$$\min_{g^\dagger, f^\dagger, V, u} \sum_{i \in \mathcal{N}} c_i g_i^P + \lambda \rho + \frac{1}{N} \sum_{i=1}^N \sigma_i \quad (12a)$$

$$\text{s.t. } g_0^\dagger = \sum_{i \in \mathcal{D}_0} (d_i^\dagger - g_i^\dagger), \dagger \in \{P, Q\}, \quad (12b)$$

$$u_0 = 1, \quad (12c)$$

$$f_l^\dagger = d_l^\dagger - g_l^\dagger - \mu_l^\dagger + \sum_{i \in \mathcal{D}_l} (d_i^\dagger - g_i^\dagger - \mu_i^\dagger), \dagger \in \{P, Q\}, \forall l \in \mathcal{L}, \quad (12d)$$

$$u_i = u_0 - 2 \sum_{l \in \mathcal{R}_i} (f_l^P R_l + f_l^Q X_l), \forall i \in \mathcal{L}, \quad (12e)$$

$$(f_l^P)^2 + (f_l^Q)^2 \leq \bar{f}_l^2, \forall l \in \mathcal{L}, \quad (12f)$$

$$\sum_{i \in \mathcal{N}} V_{i,w} + 1 = 0 \quad \forall w \in \mathcal{W}, \quad (12g)$$

$$\left\{ \begin{array}{ll} \sum_{i \in \mathcal{N}} c_i V_i \widehat{\xi}^P + \gamma_i^\top (h - Q \widehat{\xi}_i) \leq \sigma_i & \forall i \in \{1, \dots, N\} \\ \|Q^\top \gamma_i - c^\top V\|_* \leq \lambda & \forall i \in \{1, \dots, N\} \\ \gamma_i \geq 0 & \forall i \in \{1, \dots, N\} \end{array} \right. \quad (12h)$$

$$\left\{ \begin{array}{ll} \bar{\tau}_g + \frac{1}{\epsilon} \left(\bar{\lambda}_g \rho + \frac{1}{N} \sum_{i=1}^N \bar{\sigma}_{g,i} \right) \leq 0 \\ g_g^\dagger + V_g \widehat{\xi}_i^\dagger - \bar{g}_g^\dagger - \bar{\tau}_g + \bar{\gamma}_{g,i,1}^\top (h - Q \widehat{\xi}_i) \leq \bar{\sigma}_{g,i} & \forall i \in \{1, \dots, N\} \\ \bar{\gamma}_{g,i,2}^\top (h - Q \widehat{\xi}_i) \leq \bar{\sigma}_{g,i} & \forall i \in \{1, \dots, N\} \\ \|Q^\top \bar{\gamma}_{g,i,1} - V_g\|_* \leq \bar{\lambda}_g & \forall i \in \{1, \dots, N\} \\ \|Q^\top \bar{\gamma}_{g,i,2}\|_* \leq \bar{\lambda}_g & \forall i \in \{1, \dots, N\} \\ \bar{\gamma}_{g,i,1} \geq 0; \bar{\gamma}_{g,i,2} \geq 0 & \forall i \in \{1, \dots, N\} \end{array} \right\} \dagger \in \{P, Q\}, \forall g \in \mathcal{G}, \quad (12i)$$

$$\left\{ \begin{array}{ll} \underline{\tau}_g + \frac{1}{\epsilon} \left(\underline{\lambda}_g \rho + \frac{1}{N} \sum_{i=1}^N \underline{\sigma}_{g,i} \right) \leq 0 \\ -g_g^\dagger - V_g \widehat{\xi}_i^\dagger - \underline{g}_g^\dagger - \underline{\tau}_g + \underline{\gamma}_{g,i,1}^\top (h - Q \widehat{\xi}_i) \leq \underline{\sigma}_{g,i} & \forall i \in \{1, \dots, N\} \\ \underline{\gamma}_{g,i,2}^\top (h - Q \widehat{\xi}_i) \leq \underline{\sigma}_{g,i} & \forall i \in \{1, \dots, N\} \\ \|Q^\top \underline{\gamma}_{g,i,1} + V_g\|_* \leq \underline{\lambda}_g & \forall i \in \{1, \dots, N\} \\ \|Q^\top \underline{\gamma}_{g,i,2}\|_* \leq \underline{\lambda}_g & \forall i \in \{1, \dots, N\} \\ \underline{\gamma}_{g,i,1} \geq 0; \underline{\gamma}_{g,i,2} \geq 0 & \forall i \in \{1, \dots, N\} \end{array} \right\} \dagger \in \{P, Q\}, \forall g \in \mathcal{G}, \quad (12j)$$

$$u_i - 2 \sum_{l \in \mathcal{R}_i} R_l \left(-V_l \widehat{\xi}_i^P - \widehat{\xi}_{l,i}^P + \sum_{i \in \mathcal{D}_l} (-V_i \widehat{\xi}_i^P - \widehat{\xi}_{l,i}^P) \right) \quad \forall i \in \{1, \dots, N\} \quad (12k)$$

$$+ X_l \left(-V_l \widehat{\xi}_i^Q - \widehat{\xi}_{l,i}^Q + \sum_{i \in \mathcal{D}_l} (-V_i \widehat{\xi}_i^Q - \widehat{\xi}_{l,i}^Q) \right) \leq \bar{v}_i^2$$

$$-u_i + 2 \sum_{l \in \mathcal{R}_i} R_l \left(-V_l \widehat{\xi}_i^P - \widehat{\xi}_{l,i}^P + \sum_{i \in \mathcal{D}_l} (-V_i \widehat{\xi}_i^P - \widehat{\xi}_{l,i}^P) \right) \quad \forall i \in \{1, \dots, N\} \quad (12l)$$

$$+ X_l \left(-V_l \widehat{\xi}_i^Q - \widehat{\xi}_{l,i}^Q + \sum_{i \in \mathcal{D}_l} (-V_i \widehat{\xi}_i^Q - \widehat{\xi}_{l,i}^Q) \right) \leq -\underline{v}_i^2$$

4 DR-OPF under copula-based ambiguity set \mathcal{M}_2

4.1 Meshed transmission systems

$$\begin{aligned}
 & \min_{g, \bar{r}, \underline{r}, V, \alpha, \beta, \sigma_i, \zeta_{ki}^{(1)}, \zeta_{ki}^{(2)}, \mu_{wi}^{(0)}, \mu_{kij}^{(1)}, \mu_{kij}^{(2)}, \mu_{kij}^{(3)}, \mu_{kij}^{(4)}, \mu_{kij}^{(5)}, \mu_{kij}^{(6)}, \mu_{kij}^{(7)}, \mu_{kij}^{(8)}, \mu_{kij}^{(9)}, \mu_{kij}^{(10)}, \mu_{kij}^{(11)}, \bar{\alpha}_g, \bar{\beta}_g, \bar{r}_g, \bar{\sigma}_{gi}, \bar{\zeta}_{gki}^{(1)}, \bar{\zeta}_{gki}^{(2)}, \bar{\mu}_{gki}^{(0)}, \bar{\mu}_{gkij}^{(1)}, \bar{\mu}_{gkij}^{(2)}, \bar{\mu}_{gkij}^{(3)}, \bar{\mu}_{gkij}^{(4)}, \bar{\mu}_{gkij}^{(5)}, \bar{\mu}_{gkij}^{(6)}, \bar{\mu}_{gkij}^{(7)}, \bar{\mu}_{gkij}^{(8)}, \bar{\mu}_{gkij}^{(9)}, \bar{\mu}_{gkij}^{(10)}, \bar{\mu}_{gkij}^{(11)}, \underline{\alpha}_g, \underline{\beta}_g, \underline{r}_g, \underline{\sigma}_{gi}, \underline{\zeta}_{gki}^{(1)}, \underline{\zeta}_{gki}^{(2)}, \underline{\mu}_{gki}^{(0)}, \underline{\mu}_{gkij}^{(1)}, \underline{\mu}_{gkij}^{(2)}, \underline{\mu}_{gkij}^{(3)}, \underline{\mu}_{gkij}^{(4)}, \underline{\mu}_{gkij}^{(5)}, \underline{\mu}_{gkij}^{(6)}, \underline{\mu}_{gkij}^{(7)}, \underline{\mu}_{gkij}^{(8)}, \underline{\mu}_{gkij}^{(9)}, \underline{\mu}_{gkij}^{(10)}, \underline{\mu}_{gkij}^{(11)}, \alpha_f, \beta_f, \tau_f, \sigma_{fi}, \zeta_{fki}^{(1)}, \zeta_{fki}^{(2)}, \mu_{fki}^{(0)}, \mu_{fkij}^{(1)}, \mu_{fkij}^{(2)}, \mu_{fkij}^{(3)}, \mu_{fkij}^{(4)}, \mu_{fkij}^{(5)}, \mu_{fkij}^{(6)}, \mu_{fkij}^{(7)}, \mu_{fkij}^{(8)}, \mu_{fkij}^{(9)}, \mu_{fkij}^{(10)}, \mu_{fkij}^{(11)}, \lambda_{fkij}} c^\top g + \bar{c}^\top \bar{r} + \underline{c}^\top \underline{r} + \alpha \theta_1 \\
 & \quad + \beta \theta_2 + \frac{1}{N} \sum_{i=1}^N \sigma_i \tag{13a}
 \end{aligned}$$

s.t.

$$\begin{aligned}
 & g + \bar{r} \leq p^{\max} \tag{13b} \\
 & g - \underline{r} \geq 0 \tag{13c} \\
 & 0 \leq \underline{r} \leq r^{\max}; 0 \leq \bar{r} \leq r^{\max} \tag{13d} \\
 & \mathbf{1}^\top g + \mathbf{1}^\top W \mu - \mathbf{1}^\top d = 0 \tag{13e} \\
 & \mathbf{1}^\top V + \mathbf{1}^\top W = 0 \tag{13f}
 \end{aligned}$$

$$\left\{ \begin{aligned}
 & -\zeta_i^{(1)\top} \hat{\xi}_i - \zeta_i^{(2)\top} \hat{F}_i + \mu^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\mu_{kji}^{(3)} \bar{\zeta}_k^{\max} - \mu_{kji}^{(4)} \bar{\zeta}_k^{\min} + \mu_{kji}^{(6)} + \frac{1}{N} \mu_{kji}^{(7)} + \mu_{kji}^{(9)} \bar{V} \bar{\zeta}_k^{\max} - \mu_{kji}^{(10)} \bar{V} \bar{\zeta}_k^{\min} \right) \leq \sigma_i \quad \forall i \\
 & V_k^\top c + \zeta_{ki}^{(1)} - Q_k^\top \mu_i^{(0)} + \sum_{j=1}^N \left(\mu_{kji}^{(3)} + \mu_{kji}^{(4)} + \bar{V} \mu_{kji}^{(10)} - \bar{V} \mu_{kji}^{(9)} \right) = 0 \quad \forall k, i \\
 & \frac{1}{N} \zeta_{ik}^{(2)} - \mu_{kji}^{(1)} \hat{\zeta}_{kj} - \mu_{kji}^{(2)} \bar{\zeta}_k^{\min} - \mu_{kji}^{(3)} \bar{\zeta}_k^{\max} + \mu_{kji}^{(4)} \bar{\zeta}_k^{\min} + \mu_{kji}^{(5)} \bar{\zeta}_k^{\max} - \mu_{kji}^{(6)} + \frac{1}{N} \lambda_k \leq 0 \quad \forall k, j, i \\
 & \mu_{kji}^{(1)} + \mu_{kji}^{(2)} + \mu_{kji}^{(3)} - \mu_{kji}^{(4)} - \mu_{kji}^{(5)} = 0 \quad \forall k, j, i \\
 & -\mu_{kji}^{(7)} + \mu_{kji}^{(8)} + \mu_{kji}^{(9)} - \mu_{kji}^{(10)} - \mu_{kji}^{(11)} = 0 \quad \forall k, j, i \\
 & \mu_{kji}^{(7)} - \lambda_k = 0 \quad \forall k, j, i \\
 & \mu_{kji}^{(7)} - \mu_{kji}^{(8)} \epsilon_k^{\min} - \mu_{kji}^{(9)} \epsilon_k^{\max} + \mu_{kji}^{(10)} \epsilon_k^{\min} + \mu_{kji}^{(11)} \epsilon_k^{\max} \leq 0 \quad \forall k, j, i \\
 & \|\zeta_i^{(1)}\|_* \leq \alpha \quad \forall i \\
 & \|\zeta_i^{(2)}\|_* \leq \beta \quad \forall i.
 \end{aligned} \right. \tag{13g}$$

$$\left\{ \begin{aligned}
 & \bar{r}_g + \frac{1}{\epsilon} \left(\bar{\alpha}_g \theta_1 + \bar{\beta}_g \theta_2 + \frac{1}{N} \sum_{i=1}^N \bar{\sigma}_{gi} \right) \leq 0 \\
 & -\bar{r}_g - \bar{r}_g - \bar{\zeta}_{1gi}^{(1)\top} \hat{\xi}_i - \bar{\zeta}_{1gi}^{(2)\top} \hat{F}_i + \bar{\mu}_{1gi}^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\bar{\mu}_{1kij}^{(3)\top} \bar{\zeta}_k^{\max} - \bar{\mu}_{1kij}^{(4)\top} \bar{\zeta}_k^{\min} + \bar{\mu}_{1kij}^{(6)\top} + \frac{1}{N} \bar{\mu}_{1kij}^{(7)\top} + \bar{\mu}_{1kij}^{(9)\top} \bar{V} \bar{\zeta}_k^{\max} - \bar{\mu}_{1kij}^{(10)\top} \bar{V} \bar{\zeta}_k^{\min} \right) \leq \bar{\sigma}_{gi} \quad \forall i \\
 & -\bar{\zeta}_{2gi}^{(1)\top} \hat{\xi}_i - \bar{\zeta}_{2gi}^{(2)\top} \hat{F}_i + \bar{\mu}_{2gi}^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\bar{\mu}_{2kij}^{(3)\top} \bar{\zeta}_k^{\max} - \bar{\mu}_{2kij}^{(4)\top} \bar{\zeta}_k^{\min} + \bar{\mu}_{2kij}^{(6)\top} + \frac{1}{N} \bar{\mu}_{2kij}^{(7)\top} + \bar{\mu}_{2kij}^{(9)\top} \bar{V} \bar{\zeta}_k^{\max} - \bar{\mu}_{2kij}^{(10)\top} \bar{V} \bar{\zeta}_k^{\min} \right) \leq \bar{\sigma}_{gi} \quad \forall i \\
 & V_g k + \bar{\zeta}_{1gk}^{(1)} - Q_k^\top \bar{\mu}_{1gi}^{(0)} + \sum_{j=1}^N \left(\bar{\mu}_{1gkij}^{(4)} + \bar{\mu}_{1gkij}^{(3)} + \bar{V} \bar{\mu}_{1gkij}^{(10)} - \bar{V} \bar{\mu}_{1gkij}^{(9)} \right) = 0 \quad \forall k, i \\
 & \bar{\zeta}_{2gk}^{(1)} - Q_k^\top \bar{\mu}_{2gi}^{(0)} + \sum_{j=1}^N \left(\bar{\mu}_{2gkij}^{(4)} + \bar{\mu}_{2gkij}^{(3)} + \bar{V} \bar{\mu}_{2gkij}^{(10)} - \bar{V} \bar{\mu}_{2gkij}^{(9)} \right) = 0 \quad \forall k, i \\
 & \frac{1}{N} \bar{\zeta}_{vgk}^{(2)} - \bar{\mu}_{vkji}^{(1)} \hat{\zeta}_{kj} - \bar{\mu}_{vkji}^{(2)} \bar{\zeta}_k^{\min} - \bar{\mu}_{vkji}^{(3)} \bar{\zeta}_k^{\max} + \bar{\mu}_{vkji}^{(4)} \bar{\zeta}_k^{\min} + \bar{\mu}_{vkji}^{(5)} \bar{\zeta}_k^{\max} - \bar{\mu}_{vkji}^{(6)} + \frac{1}{N} \bar{\lambda}_{vgk} \leq 0 \quad \forall v, k, j, i \\
 & \bar{\mu}_{vgkij}^{(1)} + \bar{\mu}_{vgkij}^{(2)} + \bar{\mu}_{vgkij}^{(3)} - \bar{\mu}_{vgkij}^{(4)} - \bar{\mu}_{vgkij}^{(5)} = 0 \quad \forall v, k, j, i \\
 & -\bar{\mu}_{vgkij}^{(7)} + \bar{\mu}_{vgkij}^{(8)} + \bar{\mu}_{vgkij}^{(9)} - \bar{\mu}_{vgkij}^{(10)} - \bar{\mu}_{vgkij}^{(11)} = 0 \quad \forall v, k, j, i \\
 & \bar{\mu}_{vgkij}^{(7)} - \bar{\lambda}_{vgk} = 0 \quad \forall v, k, j, i \\
 & \bar{\mu}_{vgkij}^{(7)} - \bar{\mu}_{vgkij}^{(8)} \epsilon_k^{\min} - \bar{\mu}_{vgkij}^{(9)} \epsilon_k^{\max} + \bar{\mu}_{vgkij}^{(10)} \epsilon_k^{\min} + \bar{\mu}_{vgkij}^{(11)} \epsilon_k^{\max} \leq 0 \quad \forall v, k, j, i \\
 & \|\bar{\zeta}_{vgi}^{(1)}\|_* \leq \bar{\alpha}_g \quad \forall v, i \\
 & \|\bar{\zeta}_{vgi}^{(2)}\|_* \leq \bar{\beta}_g \quad \forall v, i
 \end{aligned} \right. \tag{13h}$$

$$\left. \begin{aligned}
& \tau_g + \frac{1}{\epsilon} \left(\alpha_g \theta_1 + \beta_g \theta_2 + \frac{1}{N} \sum_{i=1}^N \sigma_{gi} \right) \leq 0 \\
& -\tau_g - \tau_g - \zeta_{1gi}^{(1)\top} \widehat{\xi}_i - \zeta_{1gi}^{(2)\top} \widehat{F}_i + \mu_{1gi}^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\mu_{1kgij}^{(3)\top} \widehat{\xi}_k^{\max} - \mu_{1kgij}^{(4)\top} \widehat{\xi}_k^{\min} + \mu_{1kgij}^{(6)\top} + \frac{1}{N} \mu_{1kgij}^{(7)\top} + \mu_{1kgij}^{(9)\top} \overline{V} \widehat{\xi}_k^{\max} - \mu_{1kgij}^{(10)\top} \overline{V} \widehat{\xi}_k^{\min} \right) \leq \sigma_{gi} \quad \forall i \\
& -\zeta_{2gi}^{(1)\top} \widehat{\xi}_i - \zeta_{2gi}^{(2)\top} \widehat{F}_i + \mu_{2gi}^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\mu_{2kgij}^{(3)\top} \widehat{\xi}_k^{\max} - \mu_{2kgij}^{(4)\top} \widehat{\xi}_k^{\min} + \mu_{2kgij}^{(6)\top} + \frac{1}{N} \mu_{2kgij}^{(7)\top} + \mu_{2kgij}^{(9)\top} \overline{V} \widehat{\xi}_k^{\max} - \mu_{2kgij}^{(10)\top} \overline{V} \widehat{\xi}_k^{\min} \right) \leq \sigma_{gi} \quad \forall i \\
& -V_{gk} + \zeta_{1gki}^{(1)} - Q_k^\top \mu_{1gi}^{(0)} + \sum_{j=1}^N \left(\mu_{1gkij}^{(4)} + \mu_{1gkij}^{(3)} + \overline{V} \mu_{1gkij}^{(10)} - \overline{V} \mu_{1gkij}^{(9)} \right) = 0 \quad \forall k, i \\
& \zeta_{2gki}^{(1)} - Q_k^\top \mu_{2gi}^{(0)} + \sum_{j=1}^N \left(\mu_{2gkij}^{(4)} + \mu_{2gkij}^{(3)} + \overline{V} \mu_{2gkij}^{(10)} - \overline{V} \mu_{2gkij}^{(9)} \right) = 0 \quad \forall k, i \\
& \frac{1}{N} \zeta_{v g k i}^{(2)} - \mu_{v k j i}^{(1)} \widehat{\xi}_{k j} - \mu_{v k j i}^{(2)} \widehat{\xi}_k^{\min} - \mu_{v k j i}^{(3)} \widehat{\xi}_k^{\max} + \mu_{v k j i}^{(4)} \widehat{\xi}_k^{\min} + \mu_{v k j i}^{(5)} \widehat{\xi}_k^{\max} - \mu_{v k j i}^{(6)} + \frac{1}{N} \lambda_{v g k i} \leq 0 \quad \forall v, k, j, i \\
& \mu_{v g k i j}^{(1)} + \mu_{v g k i j}^{(2)} + \mu_{v g k i j}^{(3)} - \mu_{v g k i j}^{(4)} - \mu_{v g k i j}^{(5)} = 0 \quad \forall v, k, j, i \\
& -\mu_{v g k j i}^{(7)} + \mu_{v g k j i}^{(8)} + \mu_{v g k j i}^{(9)} - \mu_{v g k j i}^{(10)} - \mu_{v g k j i}^{(11)} = 0 \quad \forall v, k, j, i \\
& \mu_{v g k j i}^{(7)} - \lambda_{v g k i} = 0 \quad \forall v, k, j, i \\
& \mu_{v g k j i}^{(7)} - \mu_{v g k j i}^{(8)} \xi_k^{\min} - \mu_{v g k j i}^{(9)} \xi_k^{\max} + \mu_{v g k j i}^{(10)} \xi_k^{\min} + \mu_{v g k j i}^{(11)} \xi_k^{\max} \leq 0 \quad \forall v, k, j, i \\
& \|\zeta_{v g i}^{(1)}\|_* \leq \alpha_g \quad \forall v, i \\
& \|\zeta_{v g i}^{(2)}\|_* \leq \beta_g \quad \forall v, i
\end{aligned} \right\} \forall g \in \mathcal{G} \tag{13i}$$

$$\left. \begin{aligned}
& \tau_f + \frac{1}{\epsilon} \left(\alpha_f \theta_1 + \beta_f \theta_2 + \frac{1}{N} \sum_{i=1}^N \sigma_{fi} \right) \leq 0 \\
& -f_f^{\max} + T_f^P g + T_f^W W \mu - T_f^D d - \tau_f - \zeta_{1fi}^{(1)\top} \widehat{\xi}_i - \zeta_{1fi}^{(2)\top} \widehat{F}_i + \mu_{1fi}^{(0)\top} h \\
& \quad + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\mu_{1k f i j}^{(3)\top} \widehat{\xi}_k^{\max} - \mu_{1k f i j}^{(4)\top} \widehat{\xi}_k^{\min} + \mu_{1k f i j}^{(6)\top} + \frac{1}{N} \mu_{1k f i j}^{(7)\top} + \mu_{1k f i j}^{(9)\top} \overline{V} \widehat{\xi}_k^{\max} - \mu_{1k f i j}^{(10)\top} \overline{V} \widehat{\xi}_k^{\min} \right) \leq \sigma_{fi} \quad \forall i \\
& -\zeta_{2fi}^{(1)\top} \widehat{\xi}_i - \zeta_{2fi}^{(2)\top} \widehat{F}_i + \mu_{2fi}^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\mu_{2k f i j}^{(3)\top} \widehat{\xi}_k^{\max} - \mu_{2k f i j}^{(4)\top} \widehat{\xi}_k^{\min} + \mu_{2k f i j}^{(6)\top} + \frac{1}{N} \mu_{2k f i j}^{(7)\top} + \mu_{2k f i j}^{(9)\top} \overline{V} \widehat{\xi}_k^{\max} - \mu_{2k f i j}^{(10)\top} \overline{V} \widehat{\xi}_k^{\min} \right) \leq \sigma_{fi} \quad \forall i \\
& T_f^P V + T_f^W W + \zeta_{1fki}^{(1)} - Q_k^\top \mu_{1fi}^{(0)} + \sum_{j=1}^N \left(\mu_{1f k i j}^{(4)} + \mu_{1f k i j}^{(3)} + \overline{V} \mu_{1f k i j}^{(10)} - \overline{V} \mu_{1f k i j}^{(9)} \right) = 0 \quad \forall k, i \\
& \zeta_{2fki}^{(1)} - Q_k^\top \mu_{2fi}^{(0)} + \sum_{j=1}^N \left(\mu_{2f k i j}^{(4)} + \mu_{2f k i j}^{(3)} + \overline{V} \mu_{2f k i j}^{(10)} - \overline{V} \mu_{2f k i j}^{(9)} \right) = 0 \quad \forall k, i \\
& \frac{1}{N} \lambda_{v f k i}^{(2)} - \mu_{v f k j i}^{(1)} \widehat{\xi}_{k j} - \mu_{v f k j i}^{(2)} \widehat{\xi}_k^{\min} - \mu_{v f k j i}^{(3)} \widehat{\xi}_k^{\max} + \mu_{v f k j i}^{(4)} \widehat{\xi}_k^{\min} + \mu_{v f k j i}^{(5)} \widehat{\xi}_k^{\max} - \mu_{v f k j i}^{(6)} + \frac{1}{N} \lambda_{v f k i} \leq 0 \quad \forall v, k, j, i \\
& \mu_{v f k i j}^{(1)} + \mu_{v f k i j}^{(2)} + \mu_{v f k i j}^{(3)} - \mu_{v f k i j}^{(4)} - \mu_{v f k i j}^{(5)} = 0 \quad \forall v, k, j, i \\
& -\mu_{v f k j i}^{(7)} + \mu_{v f k j i}^{(8)} + \mu_{v f k j i}^{(9)} - \mu_{v f k j i}^{(10)} - \mu_{v f k j i}^{(11)} = 0 \quad \forall v, k, j, i \\
& \mu_{v f k j i}^{(7)} - \lambda_{v f k i} = 0 \quad \forall v, k, j, i \\
& \mu_{v f k j i}^{(7)} - \mu_{v f k j i}^{(8)} \xi_k^{\min} - \mu_{v f k j i}^{(9)} \xi_k^{\max} + \mu_{v f k j i}^{(10)} \xi_k^{\min} + \mu_{v f k j i}^{(11)} \xi_k^{\max} \leq 0 \quad \forall v, k, j, i \\
& \|\zeta_{v f i}^{(1)}\|_* \leq \alpha_f \quad \forall v, i \\
& \|\zeta_{v f i}^{(2)}\|_* \leq \beta_f \quad \forall v, i
\end{aligned} \right\} \forall f \in \mathcal{F}$$

4.2 Radial distribution systems

$$\min_{g^\dagger, f^\dagger, V, u} \sum_{i \in \mathcal{N}} c_i g_i^P + \alpha \theta_1 + \beta \theta_2 + \frac{1}{N} \sum_{i=1}^N \sigma_i \tag{14a}$$

s.t.

$$g_0^\dagger = \sum_{i \in \mathcal{D}_0} (d_i^\dagger - g_i^\dagger), \quad \dagger \in \{P, Q\}, \tag{14b}$$

$$u_0 = 1, \tag{14c}$$

$$f_l^\dagger = d_l^\dagger - g_l^\dagger - \mu_l^\dagger + \sum_{i \in \mathcal{D}_l} (d_i^\dagger - g_i^\dagger - \mu_i^\dagger), \quad \dagger \in \{P, Q\}, \forall l \in \mathcal{L}, \tag{14d}$$

$$u_i = u_0 - 2 \sum_{l \in \mathcal{R}_i} (f_l^P R_l + f_l^Q X_l), \quad \forall i \in \mathcal{L}, \tag{14e}$$

$$(f_l^P)^2 + (f_l^Q)^2 \leq \bar{f}_l^2, \quad \forall l \in \mathcal{L}, \tag{14f}$$

$$\sum_{i \in \mathcal{N}} V_i, w + 1 = 0 \quad \forall w \in \mathcal{W}, \tag{14g}$$

$$\left\{ \begin{array}{l}
-\zeta_i^{(1)\top} \widehat{\xi}_i - \zeta_i^{(2)\top} \widehat{F}_i + \mu^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\mu_{kji}^{(3)} \widehat{\xi}_k^{\max} - \mu_{kji}^{(4)} \widehat{\xi}_k^{\min} + \mu_{kji}^{(6)} + \frac{1}{N} \mu_{kji}^{(7)} + \mu_{kji}^{(9)} \overline{V} \xi_k^{\max} - \mu_{kji}^{(10)} \overline{V} \xi_k^{\min} \right) \leq \sigma_i \quad \forall i \\
V_k^\top c + \zeta_{ki}^{(1)} - Q_k^\top \mu_i^{(0)} + \sum_{j=1}^N \left(\mu_{kji}^{(3)} + \mu_{kji}^{(4)} + \overline{V} \mu_{kji}^{(10)} - \overline{V} \mu_{kji}^{(9)} \right) = 0 \quad \forall k, i \\
\frac{1}{N} \zeta_{ik}^{(2)} - \mu_{kji}^{(1)} \widehat{\xi}_{kj} - \mu_{kji}^{(2)} \widehat{\xi}_k^{\min} - \mu_{kji}^{(3)} \widehat{\xi}_k^{\max} + \mu_{kji}^{(4)} \widehat{\xi}_k^{\min} + \mu_{kji}^{(5)} \widehat{\xi}_k^{\max} - \mu_{kji}^{(6)} + \frac{1}{N} \lambda_k \leq 0 \quad \forall k, j, i \\
\mu_{kji}^{(1)} + \mu_{kji}^{(2)} + \mu_{kji}^{(3)} - \mu_{kji}^{(4)} - \mu_{kji}^{(5)} = 0 \quad \forall k, j, i \\
-\mu_{kji}^{(7)} + \mu_{kji}^{(8)} + \mu_{kji}^{(9)} - \mu_{kji}^{(10)} - \mu_{kji}^{(11)} = 0 \quad \forall k, j, i \\
\mu_{kji}^{(7)} - \lambda_k = 0 \quad \forall k, j, i \\
\mu_{kji}^{(7)} - \mu_{kji}^{(8)} \xi_k^{\min} - \mu_{kji}^{(9)} \xi_k^{\max} + \mu_{kji}^{(10)} \xi_k^{\min} + \mu_{kji}^{(11)} \xi_k^{\max} \leq 0 \quad \forall k, j, i \\
\|\zeta_i^{(1)}\|_* \leq \alpha \quad \forall i \\
\|\zeta_i^{(2)}\|_* \leq \beta \quad \forall i.
\end{array} \right. \quad (14h)$$

$$\left\{ \begin{array}{l}
\overline{\tau}_g + \frac{1}{\epsilon} \left(\overline{\alpha}_g \theta_1 + \overline{\beta}_g \theta_2 + \frac{1}{N} \sum_{i=1}^N \overline{\sigma}_{gi} \right) \leq 0 \\
g_g^\dagger - \overline{g}_g^\dagger - \overline{\tau}_g - \overline{\zeta}_{1gi}^{(1)\top} \widehat{\xi}_i - \overline{\zeta}_{1gi}^{(2)\top} \widehat{F}_i + \overline{\mu}_{1gi}^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\overline{\mu}_{1kgij}^{(3)} \widehat{\xi}_k^{\max} - \overline{\mu}_{1kgij}^{(4)} \widehat{\xi}_k^{\min} + \overline{\mu}_{1kgij}^{(6)} + \frac{1}{N} \overline{\mu}_{1kgij}^{(7)} + \overline{\mu}_{1kgij}^{(9)} \overline{V} \xi_k^{\max} - \overline{\mu}_{1kgij}^{(10)} \overline{V} \xi_k^{\min} \right) \leq \overline{\sigma}_{gi} \quad \forall i \\
-\overline{\zeta}_{2gi}^{(1)\top} \widehat{\xi}_i - \overline{\zeta}_{2gi}^{(2)\top} \widehat{F}_i + \overline{\mu}_{2gi}^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\overline{\mu}_{2kgij}^{(3)} \widehat{\xi}_k^{\max} - \overline{\mu}_{2kgij}^{(4)} \widehat{\xi}_k^{\min} + \overline{\mu}_{2kgij}^{(6)} + \frac{1}{N} \overline{\mu}_{2kgij}^{(7)} + \overline{\mu}_{2kgij}^{(9)} \overline{V} \xi_k^{\max} - \overline{\mu}_{2kgij}^{(10)} \overline{V} \xi_k^{\min} \right) \leq \overline{\sigma}_{gi} \quad \forall i \\
V_{gk} + \overline{\zeta}_{1gki}^{(1)} - Q_k^\top \overline{\mu}_{1gi}^{(0)} + \sum_{j=1}^N \left(\overline{\mu}_{1gkij}^{(4)} + \overline{\mu}_{1gkij}^{(3)} + \overline{V} \overline{\mu}_{1gkij}^{(10)} - \overline{V} \overline{\mu}_{1gkij}^{(9)} \right) = 0 \quad \forall k, i \\
\overline{\zeta}_{2gki}^{(1)} - Q_k^\top \overline{\mu}_{2gi}^{(0)} + \sum_{j=1}^N \left(\overline{\mu}_{2gkij}^{(4)} + \overline{\mu}_{2gkij}^{(3)} + \overline{V} \overline{\mu}_{2gkij}^{(10)} - \overline{V} \overline{\mu}_{2gkij}^{(9)} \right) = 0 \quad \forall k, i \\
\frac{1}{N} \overline{\zeta}_{v gki}^{(2)} - \overline{\mu}_{v kji}^{(1)} \widehat{\xi}_{kj} - \overline{\mu}_{v kji}^{(2)} \widehat{\xi}_k^{\min} - \overline{\mu}_{v kji}^{(3)} \widehat{\xi}_k^{\max} + \overline{\mu}_{v kji}^{(4)} \widehat{\xi}_k^{\min} + \overline{\mu}_{v kji}^{(5)} \widehat{\xi}_k^{\max} - \overline{\mu}_{v kji}^{(6)} + \frac{1}{N} \overline{\lambda}_{v gki} \leq 0 \quad \forall v, k, j, i \\
\overline{\mu}_{v gkij}^{(1)} + \overline{\mu}_{v gkij}^{(2)} + \overline{\mu}_{v gkij}^{(3)} - \overline{\mu}_{v gkij}^{(4)} - \overline{\mu}_{v gkij}^{(5)} = 0 \quad \forall v, k, j, i \\
-\overline{\mu}_{v gkji}^{(7)} + \overline{\mu}_{v gkji}^{(8)} + \overline{\mu}_{v gkji}^{(9)} - \overline{\mu}_{v gkji}^{(10)} - \overline{\mu}_{v gkji}^{(11)} = 0 \quad \forall v, k, j, i \\
\overline{\mu}_{v gkji}^{(7)} - \overline{\lambda}_{v gki} = 0 \quad \forall v, k, j, i \\
\overline{\mu}_{v gkji}^{(7)} - \overline{\mu}_{v gkji}^{(8)} \xi_k^{\min} - \overline{\mu}_{v gkji}^{(9)} \xi_k^{\max} + \overline{\mu}_{v gkji}^{(10)} \xi_k^{\min} + \overline{\mu}_{v gkji}^{(11)} \xi_k^{\max} \leq 0 \quad \forall v, k, j, i \\
\|\overline{\zeta}_{v gi}^{(1)}\|_* \leq \overline{\alpha}_g \quad \forall v, i \\
\|\overline{\zeta}_{v gi}^{(2)}\|_* \leq \overline{\beta}_g \quad \forall v, i
\end{array} \right. \quad \left. \begin{array}{l}
\dagger \in \{P, Q\}, \\
\forall g \in \mathcal{G},
\end{array} \right. \quad (14i)$$

$$\left\{ \begin{array}{l}
\tau_g + \frac{1}{\epsilon} \left(\alpha_g \theta_1 + \beta_g \theta_2 + \frac{1}{N} \sum_{i=1}^N \sigma_{gi} \right) \leq 0 \\
-g_g^\dagger - \overline{g}_g^\dagger - \tau_g - \zeta_{1gi}^{(1)\top} \widehat{\xi}_i - \zeta_{1gi}^{(2)\top} \widehat{F}_i + \mu_{1gi}^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\mu_{1kgij}^{(3)} \widehat{\xi}_k^{\max} - \mu_{1kgij}^{(4)} \widehat{\xi}_k^{\min} + \mu_{1kgij}^{(6)} + \frac{1}{N} \mu_{1kgij}^{(7)} + \mu_{1kgij}^{(9)} \overline{V} \xi_k^{\max} - \mu_{1kgij}^{(10)} \overline{V} \xi_k^{\min} \right) \leq \sigma_{gi} \quad \forall i \\
-\zeta_{2gi}^{(1)\top} \widehat{\xi}_i - \zeta_{2gi}^{(2)\top} \widehat{F}_i + \mu_{2gi}^{(0)\top} h + \sum_{k=1}^{|\mathcal{W}|} \sum_{j=1}^N \left(\mu_{2kgij}^{(3)} \widehat{\xi}_k^{\max} - \mu_{2kgij}^{(4)} \widehat{\xi}_k^{\min} + \mu_{2kgij}^{(6)} + \frac{1}{N} \mu_{2kgij}^{(7)} + \mu_{2kgij}^{(9)} \overline{V} \xi_k^{\max} - \mu_{2kgij}^{(10)} \overline{V} \xi_k^{\min} \right) \leq \sigma_{gi} \quad \forall i \\
-V_{gk} + \zeta_{1gki}^{(1)} - Q_k^\top \mu_{1gi}^{(0)} + \sum_{j=1}^N \left(\mu_{1gkij}^{(4)} + \mu_{1gkij}^{(3)} + \overline{V} \mu_{1gkij}^{(10)} - \overline{V} \mu_{1gkij}^{(9)} \right) = 0 \quad \forall k, i \\
\zeta_{2gki}^{(1)} - Q_k^\top \mu_{2gi}^{(0)} + \sum_{j=1}^N \left(\mu_{2gkij}^{(4)} + \mu_{2gkij}^{(3)} + \overline{V} \mu_{2gkij}^{(10)} - \overline{V} \mu_{2gkij}^{(9)} \right) = 0 \quad \forall k, i \\
\frac{1}{N} \zeta_{v gki}^{(2)} - \mu_{v kji}^{(1)} \widehat{\xi}_{kj} - \mu_{v kji}^{(2)} \widehat{\xi}_k^{\min} - \mu_{v kji}^{(3)} \widehat{\xi}_k^{\max} + \mu_{v kji}^{(4)} \widehat{\xi}_k^{\min} + \mu_{v kji}^{(5)} \widehat{\xi}_k^{\max} - \mu_{v kji}^{(6)} + \frac{1}{N} \lambda_{v gki} \leq 0 \quad \forall v, k, j, i \\
\mu_{v gkij}^{(1)} + \mu_{v gkij}^{(2)} + \mu_{v gkij}^{(3)} - \mu_{v gkij}^{(4)} - \mu_{v gkij}^{(5)} = 0 \quad \forall v, k, j, i \\
-\mu_{v gkji}^{(7)} + \mu_{v gkji}^{(8)} + \mu_{v gkji}^{(9)} - \mu_{v gkji}^{(10)} - \mu_{v gkji}^{(11)} = 0 \quad \forall v, k, j, i \\
\mu_{v gkji}^{(7)} - \lambda_{v gki} = 0 \quad \forall v, k, j, i \\
\mu_{v gkji}^{(7)} - \mu_{v gkji}^{(8)} \xi_k^{\min} - \mu_{v gkji}^{(9)} \xi_k^{\max} + \mu_{v gkji}^{(10)} \xi_k^{\min} + \mu_{v gkji}^{(11)} \xi_k^{\max} \leq 0 \quad \forall v, k, j, i \\
\|\zeta_{v gi}^{(1)}\|_* \leq \alpha_g \quad \forall v, i \\
\|\zeta_{v gi}^{(2)}\|_* \leq \beta_g \quad \forall v, i
\end{array} \right. \quad \left. \begin{array}{l}
\dagger \in \{P, Q\}, \\
\forall g \in \mathcal{G},
\end{array} \right. \quad (14j)$$

$$u_i - 2 \sum_{l \in \mathcal{R}_i} R_l \left(-V_l \widehat{\xi}_i^P - \widehat{\xi}_{l,i}^P + \sum_{i \in \mathcal{D}_l} \left(-V_l \widehat{\xi}_i^P - \widehat{\xi}_{l,i}^P \right) \right) \\
+ X_l \left(-V_l \widehat{\xi}_i^Q - \widehat{\xi}_{l,i}^Q + \sum_{i \in \mathcal{D}_l} \left(-V_l \widehat{\xi}_i^Q - \widehat{\xi}_{l,i}^Q \right) \right) \leq \overline{v}_i^2 \quad \forall i \in \{1, \dots, N\} \quad (14k)$$

$$\begin{aligned}
& -u_i + 2 \sum_{l \in \mathcal{R}_i} R_l \left(-V_l \widehat{\xi}_i^P - \widehat{\xi}_{l,i}^P + \sum_{i \in \mathcal{D}_l} (-V_i \widehat{\xi}_i^P - \widehat{\xi}_{l,i}^P) \right) \\
& + X_l \left(-V_l \widehat{\xi}_i^Q - \widehat{\xi}_{l,i}^Q + \sum_{i \in \mathcal{D}_l} (-V_i \widehat{\xi}_i^Q - \widehat{\xi}_{l,i}^Q) \right) \leq -v_i^2
\end{aligned} \quad \forall i \in \{1, \dots, N\} \tag{141}$$

5 The real-time operational problem

5.1 The real-time operational problem for meshed transmission systems

In this section, we present the real-time operational optimization problem for the out-of-sample analysis, in case of meshed transmission systems. Given the realized uncertainty $\widehat{\xi}_j$ that refers to the wind power forecast error with respect to the day-ahead forecast μ , this problem writes as

$$\min_{V, \Delta d, \Delta w} c^\top V \widehat{\xi}_j + v_{\text{Shed}}^\top \Delta d \quad (15a)$$

$$\text{s.t. } 0 \leq \Delta d \leq d \quad (15b)$$

$$0 \leq \Delta w \leq W(\mu + \widehat{\xi}_j) \quad (15c)$$

$$-\underline{r} \leq V \widehat{\xi}_j \leq \bar{r} \quad (15d)$$

$$\mathbf{1}^\top V \widehat{\xi}_j + \mathbf{1}^\top W \widehat{\xi}_j + \mathbf{1}^\top \Delta d - \mathbf{1}^\top \Delta w = 0. \quad (15e)$$

$$\mathbf{T}_f^P (g + V \widehat{\xi}) + \mathbf{T}_f^W W (\mu + \widehat{\xi}) - \mathbf{T}_f^D (d - \Delta d) \leq f_f^{\max} \quad \forall f \in \mathfrak{F}, \quad (15f)$$

The objective function (15a) models the total recourse operational cost, incurred by the recourse production cost of conventional generating units (the first term) and the load shedding cost (the second term). The parameter $v_{\text{Shed}} \in \mathbb{R}^{|\mathcal{D}|}$ refers to the value of lost load. The wind spillage cost is assumed to be zero. Constraints (15b) and (15c) limit the wind power spillage $\Delta w \in \mathbb{R}^{|\mathcal{W}|}$ and the load shedding $\Delta d \in \mathbb{R}^{|\mathcal{D}|}$. The recourse production $V \widehat{\xi}_j$ of conventional generating units is limited in (15d) by the reserve capacities \underline{r} and \bar{r} procured in the day-ahead stage. The real-time power balance is ensured by (15e). Finally, the maximum capacity limits are ensured by (15f) for each transmission line $f \in \mathfrak{F}$.

5.2 The real-time operational problem for radial distribution systems

In this section, we present the real-time operational optimization problem for the out-of-sample analysis, in case of radial distribution systems. Given the realized uncertainty $\widehat{\xi}_j$ that refers to the wind power forecast error with respect to the day-ahead forecast μ , this problem writes as

$$\min_{V, \Delta d, \Delta w, \tilde{f}^\dagger, \tilde{u}} c^\top V \widehat{\xi}_j + v_{\text{Shed}}^\top \Delta d \quad (16a)$$

s.t.

$$0 \leq \Delta d \leq d \quad (16b)$$

$$0 \leq \Delta w \leq W(\mu + \widehat{\xi}_j) \quad (16c)$$

$$-\underline{g}_g^\dagger \leq g_g^\dagger + V_g \widehat{\xi}_j \leq \bar{g}_g^\dagger, \quad \forall g \in \mathcal{G} \quad (16d)$$

$$\mathbf{1}^\top V \widehat{\xi}_j + \mathbf{1}^\top \widehat{\xi}_j + \mathbf{1}^\top \Delta d - \mathbf{1}^\top \Delta w = 0. \quad (16e)$$

$$\tilde{f}_l^\dagger = f_l^\dagger - V_l \widehat{\xi}^\dagger - \tilde{\xi}_l^\dagger + \sum_{i \in \mathcal{D}_l} \left(-V_i \widehat{\xi}^\dagger - \tilde{\xi}_i^\dagger \right), \quad \dagger \in \{P, Q\}, \quad (16f)$$

$$\left(f_l^P \right)^2 + \left(f_l^Q \right)^2 \leq \bar{f}_l^2, \quad \forall l \in \mathcal{L}, \quad (16g)$$

$$\tilde{u}_i = u_0 - 2 \sum_{l \in \mathcal{R}_i} \left(\tilde{f}_l^P R_l + \tilde{f}_l^Q X_l \right) \quad (16h)$$

$$\underline{v}_i^2 \leq \tilde{u}_i \leq \bar{v}_i^2 \quad \forall i \in \mathcal{N}_0, \quad (16i)$$

The objective function (16a) models the total recourse operational cost, incurred by the recourse production cost of controllable generators (the first term) and the load shedding cost (the second term). Similar to (15), the parameter $v_{\text{Shed}} \in \mathbb{R}^{|\mathcal{D}|}$ refers to the value of lost load and the wind spillage cost is assumed to be zero. Constraints (16b) and (16c) limit the wind power

spillage $\Delta w \in \mathbb{R}^{|\mathcal{W}|}$ and the load shedding $\Delta d \in \mathbb{R}^{|\mathcal{D}|}$. The real-time generation $g_g^\dagger + V_g \widehat{\xi}_j$ of controllable generators is limited in (16d) by the minimum and maximum operating capacities \underline{g}_g^\dagger and \overline{g}_g^\dagger . The real-time power balance is ensured by (16e). The real-time power flows are calculated via (16f) and the maximum capacity limits is ensured by (16g) for each distribution line $l \in \mathcal{L}$. Finally, the real-time square nodal voltage is formulated as (16h) and is limited to the minimum and maximum square voltage restrictions in (16i).

6 Input data

6.1 The meshed transmission systems

We build our case study upon the IEEE reliability test system [3]. All economic data are available in [4] and the transmission system topology is showcased in Fig. 1. The input data are given by Table 1. It includes technical data of conventional generating units, including their production cost c_p in $\$/\text{MWh}$, upward reserve capacity procurement cost \bar{c}_p in $\$/\text{MW}$, downward reserve capacity procurement cost \underline{c}_p in $\$/\text{MW}$, capacity g_p^{\max} in MW, maximum upward regulation capability \bar{r}_p^{\max} in MW, and maximum downward regulation capability \underline{r}_p^{\max} in MW. In addition, Table 1 includes the day-ahead wind power forecast μ in MW and their installed capacity $W_{(w,w)}$ in MW. This table also gives the input data of 17 loads, including their consumption d in MW and the value of lost load v_{Shed} in $\$/\text{MWh}$. Finally, the generating units, the wind farms, and the loads are connected via a 24-node network and their respective connecting node number are referred in the Table 1. The lines are characterized by their connecting nodes, their per-unit inverse susceptance $1/B$ as well as their maximum line capacity f_f^{\max} in MW.

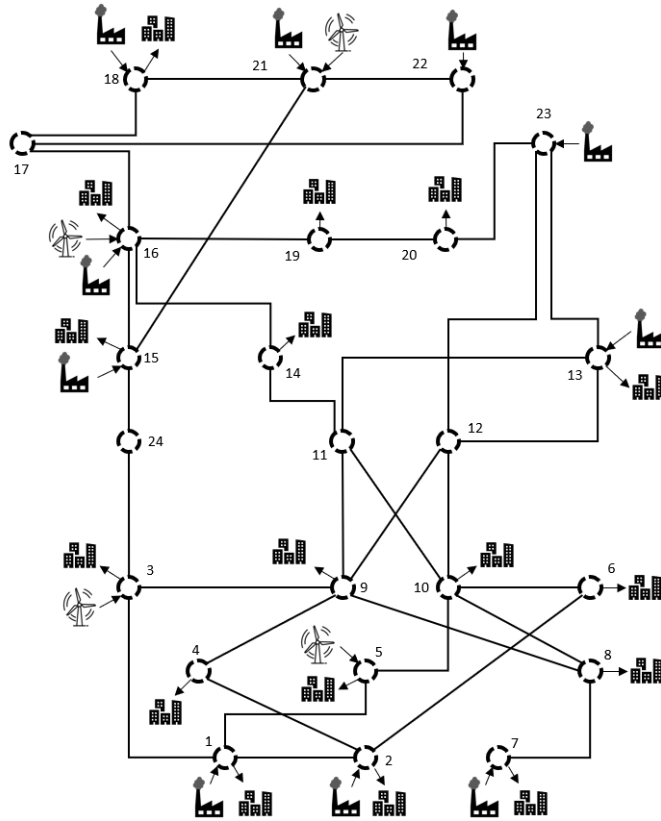


Fig. 1: The considered 24-node transmission system.

Table 1: Input Data

Generating units		1	2	3	4	5	6	7	8	9	10	11	12					
Node		1	2	7	13	15	15	16	18	21	22	23	23					
c_p [\$/MWh]		13.32	13.32	20.7	20.93	26.11	10.52	10.52	6.02	5.47	7	10.52	10.89					
\bar{c}_p [\$/MW]		1.68	1.68	3.30	4.07	1.89	5.48	5.48	4.98	5.53	8.00	3.45	5.11					
\underline{c}_p [\$/MW]		2.32	2.32	4.67	3.93	3.11	3.52	3.52	5.02	4.97	6.00	2.52	2.89					
P_p^{\max} [MW]		106.4	106.4	245	413.7	42	108.5	108.5	280	280	210	217	245					
\bar{P}_p [MW]		48	48	84	216	42	36	36	60	60	48	48	48					
\underline{P}_p [MW]		48	48	84	216	42	36	36	60	60	48	48	48					
Wind farms										1	2	3	4					
Node										3	5	16	21					
P_q^{\max} [MW]												500	500					
Day-ahead wind power forecast [MW]												120.54	115.52					
Loads		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Node		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
d [MW]		84	75	139	58	55	106	97	132	135	150	205	150	245	77	258	141	100
Y_{shed} [\$/MWh]		500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500	500
Lines: From node		1	1	1	2	2	3	3	4	5	6	7	8	8	9	9	10	10
To node		2	3	5	4	6	9	24	9	10	10	8	9	10	11	12	11	12
$1/B$ [pu]		0.0146	0.2253	0.0907	0.1356	0.205	0.1271	0.084	0.111	0.094	0.0642	0.0652	0.1762	0.1762	0.084	0.084	0.084	0.084
f_p^{\max} [MW]		175	175	350	175	175	175	400	175	350	175	350	175	175	400	400	400	400
Lines: From node		11	11	12	12	13	14	15	15	15	16	16	17	17	18	19	20	21
To node		13	14	13	23	23	16	16	21	24	17	19	18	22	21	20	23	22
$1/B$ [pu]		0.0488	0.0426	0.0488	0.0985	0.0884	0.0594	0.0172	0.0249	0.0529	0.0263	0.0234	0.0143	0.1069	0.0132	0.0203	0.0112	0.0692
f_f^{\max} [MW]		500	500	500	500	250	250	500	400	500	500	500	500	500	1000	1000	1000	500

6.2 The radial distribution system

We build our case study upon the distribution system described in [5]. The input data are given by Table 2, expressed in per-unit values, and the topology of the system is shown in Fig. 2. It includes technical data of distribution lines, including their resistance R , reactance X and maximum power capacity F_{\max} , but also active and reactive power demands at each node of the system P_d and Q_d . To supply these demands, two generators are connected to the system (in node 0 and node 11) with a maximum production capacity equal to 1 MW, and a production cost equal to 20 \$/MWh and 31 \$/MWh, respectively. In addition, two wind turbines, with capacity equal to 500 kW, are connected to nodes 7 and 14. The minimum and maximum square voltage restrictions \underline{v}_i^2 and \bar{v}_i^2 are respectively equal to 0.81 and 1.21 p.u.

Table 2: Input parameters for the 15-node distribution test system. All values are provided in per unit.

	R	X	F_{\max}	P_d	Q_d
1	0.001	0.12	2	0.7936	0.1855
2	0.0883	0.1262	0.256	0	0
3	0.1384	0.1978	0.256	0.0201	0.0084
4	0.0191	0.0273	0.256	0.0173	0.0043
5	0.0175	0.0251	0.256	0.0291	0.0073
6	0.0482	0.0689	0.256	0.0219	0.0055
7	0.0523	0.0747	0.256	-0.1969	0.0019
8	0.0407	0.0582	0.256	0.0235	0.0059
9	0.01	0.0143	0.256	0.0229	0.0142
10	0.0241	0.0345	0.256	0.0217	0.0065
11	0.0103	0.0148	0.256	0.0132	0.0033
12	0.001	0.12	1	0.6219	0.1291
13	0.1559	0.1119	0.204	0.0014	0.0008
14	0.0953	0.0684	0.204	0.0224	0.0083

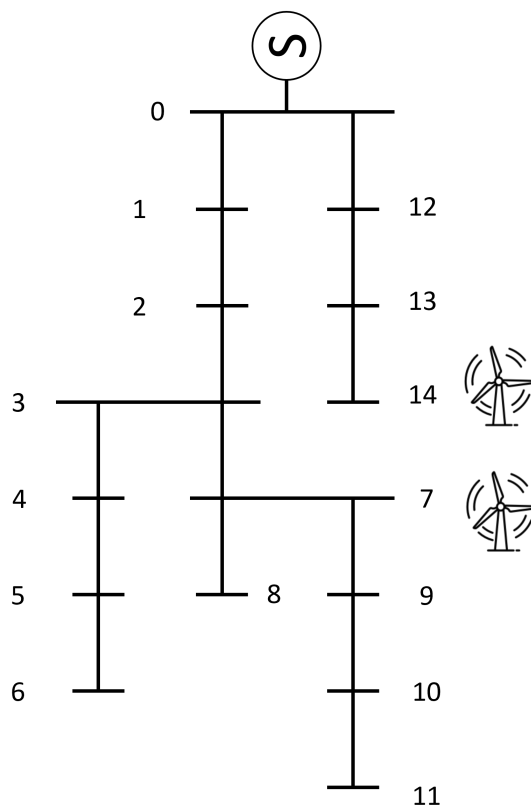


Fig. 2: The considered 15-node distribution system. The slack node is numbered 0. Two wind farms are connected in nodes 7 and 14.

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