



[white paper]

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# Proofs of Theorems in Topology

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## Abstract

We prove some theorems in topology using the fewest number of symbols at each step. Our purpose is pedagogical.

**keywords:** proofs, theorems, topology

*The most updated version of this white paper is available at*

<https://osf.io/wn24y/download>  
<https://zenodo.org/record/6622005>

## Introduction

1. [1–3]
2.  $\mathbb{N} = \{0, 1, 2, \dots\}$ ;  $\mathbb{N}^+ = \{1, 2, 3, \dots\}$
3.  $(A \vdash B) := (A \text{ proves } B)$
4.  $(A \vdash B \vdash C) := (A \text{ proves } B \text{ and } B \text{ proves } C)$
5.  $(A := B) \equiv (A \text{ is defined by } B)$

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6.  $(x =_\ell y) \equiv (\text{let } x \text{ be } y)$
7.  $(x R_\ell A) \equiv (\text{let } x R A); \quad R := \text{binary relation}$
8.  $\blacksquare := \text{an instance of the theorem was proved}$
9.  $\square := \text{qed}$
10.  $(\text{wlog } A) := \text{assume } A \text{ without loss of generality}$
11.  $(x \in A, B) \equiv ((x \in A) \wedge (x \in B))$
12.  $(X \subseteq A, B) \equiv ((X \subseteq A) \wedge (X \subseteq B))$
13.  $\exists_n := \text{there is a finite number}$
14.  $(? \vdash A) := \text{we will prove } A$

## Cauchy sequence bounded by a rational number

### 1. Theorem

$\forall (x_n) : (x_n) \equiv \text{Cauchy sequence of rational numbers} \rightarrow$   
 $\rightarrow (x_n) \equiv \text{bounded by } q \in \mathbb{Q}$

### 2. Definitions

3.  $(x_n) := \text{rational-valued sequence}$
4.  $(x_n) \equiv \text{Cauchy sequence if}$

$$\forall k \in \mathbb{N}^+, \exists K \in \mathbb{N}, m, n \in \mathbb{N}^+ : m \geq n > K \rightarrow |x_m - x_n| < \frac{1}{k}$$

5. Triangle Inequality:  $\forall x, y \in \mathbb{Q} : |x + y| \leq |x| + |y|$

### 6. Proof

7.  $(x_n) =_\ell$  Cauchy sequence of rational numbers
8.  $(5) \vdash |x_m| = |(x_m - x_n) + x_n| \leq |x_m - x_n| + |x_n|$
9.  $k =_\ell 1$
10.  $(4,7,9) \vdash m \geq n > K \rightarrow |x_m| \leq |x_m - x_n| + |x_n| < 1 + |x_n|$
11.  $m \geq n > K \rightarrow |x_m| < 1 + |x_n|$
12.  $n =_\ell K + 1$
13.  $m > K \rightarrow |x_m| < 1 + |x_{K+1}|$
14.  $M =_\ell \max\{|x_0|, |x_1|, \dots, |x_K|, |x_{K+1}|\}$
15.  $\forall m \in \mathbb{N}, |x_m| \leq M$  □

## Rationals and Reals have the Archimedean Property

### 1. Theorem

$\mathbb{Q}, \mathbb{R}$  has the Archimedean Property

### 2. Definitions

3.

$(\forall x \in F \exists n \in \mathbb{N} : n > x) \rightarrow (F \text{ has the Archimedean Property})$

4.  $F :=$  ordered field

5. (3) means that  $\mathbb{N}$  is *unbounded* in  $F$ .

6.  $(x_n) \equiv$  Cauchy sequence if

$$\forall k \in \mathbb{N}^+, \exists K \in \mathbb{N}, m, n \in \mathbb{N}^+ : m \geq n > K \rightarrow |x_m - x_n| < \frac{1}{k}$$

7.  $\mathbb{R} = \{[(x_n)] \mid (x_n) \equiv \text{Cauchy sequence of } q \in \mathbb{Q}\}$
8.  $\sim :=$  equivalence relation
9.  $(x \in S) \rightarrow [x] = \{y \in S \mid x \sim y\}$
10.  $[x] :=$  equivalence class of  $x$
11.  $(f : \mathbb{N} \rightarrow A) \equiv$  infinite sequence
12.  $(f : \mathbb{N} \rightarrow X, f(n) = x_n) \rightarrow (f := (x_n)_{n \in \mathbb{N}} := (x_n))$
13. Proof for  $\mathbb{Q}$
14.  $q \in \mathbb{Q}$  (arbitrary)
15.  $<$  is transitive
16. case 1:  $(15) \vdash q \leq 0, 0 < 1 \rightarrow q < 1$
17. case 2:  $q > 0, \exists a, b \in \mathbb{Z}^+ : q = \frac{a}{b}$
18.  $a + 1 - q = (a + 1) - \frac{a}{b} = \frac{a(b - 1) + b}{b}$
19.  $a, b \in \mathbb{Z}^+ \vdash b - 1 \geq 0 \vdash a(b - 1) \geq 0$
20.  $a(b - 1) + b \geq 0 + b > 0$
21.  $(a + 1) - \frac{a}{b} = \frac{a(b - 1) + b}{b} > 0$
22.  $(a + 1) > \frac{a}{b} = q$
23.  $(a, 1 \in \mathbb{N}; \mathbb{N} \text{ is closed under addition}) \vdash a + 1 \in \mathbb{N}$
24.  $(14, 23) \vdash \forall q \in \mathbb{Q} \exists n \in \mathbb{N} : n > q$  □
25. Proof (by contradiction) for  $\mathbb{R}$

26.  $(x_n) :=$  Cauchy sequence
27.  $(7) \vdash [(x_n)] \in \mathbb{R}$
28.  $[(i)] :=$  natural number  $(i \in \mathbb{N})$
29.  $[(x_n)] :=$  real number  $(n \in \mathbb{N})$
30. Suppose (toward contradiction):  $\forall i \in \mathbb{N}, [(i)] \leq [(x_n)]$ .
31.  $(30) \vdash \forall i \in \mathbb{N}, \exists K_i \in \mathbb{N} : n > K_i \rightarrow i \leq x_n$
32.  $(1,26) \vdash \forall n \in \mathbb{N}, x_n \leq M \in \mathbb{Q}$
33.  $i \in {}_\ell \mathbb{N}$  (arbitrary), choose  $K_i : n > K_i \rightarrow i \leq x_n$
34.  $i \leq x_n, x_n \leq M, \leq$  is transitive  $\vdash i \leq M \in \mathbb{Q}$
35.  $(34) \vdash \mathbb{N} \equiv$  bounded by  $M \in \mathbb{Q}$
36. (35) contradicts (24).
37.  $\forall x \in \mathbb{R} \exists n \in \mathbb{N} : n > x$  □

## Density Theorem

### 1. Theorem

$$(x, y \in \mathbb{R}, x < y) \rightarrow (\exists q \in \mathbb{Q} : x < q < y)$$

### 2. Definitions

3. WOP := Well Ordering Principle

4.

$$(\forall x \in F \exists n \in \mathbb{N} : n > x) \rightarrow (F \text{ has the Archimedean Property})$$

### 5. Lemma

$\mathbb{Q}, \mathbb{R}$  has the Archimedean Property

6. Proof
7.  $x, y \in \mathbb{R}$
8. Case 1:  $0 \leq x < y$
9.  $z = y - x = y + (-x)$
10.  $\mathbb{R}$  has the additive inverse property,  $\mathbb{R}$  is closed under addition  $\vdash z \in \mathbb{R}$
11.  $(8, 9) \vdash z > 0$
12.  $(4, 5) \vdash \exists n \in \mathbb{N} : n > \frac{1}{z}$
13.  $\frac{1}{n} < z$
14.  $(4, 5) \vdash \exists m \in \mathbb{N} : m > nx$
15.  $\frac{m}{n} > x$
16.  $\left\{ m \in \mathbb{N} : \frac{m}{n} > x \right\} \neq \emptyset$
17.  $(3) \vdash \left\{ m \in \mathbb{N} : \frac{m}{n} > x \right\}$  has a least element  $k$
18.  $x \geq 0, \quad n > 0, \quad k > 0$
19.  $k \equiv \text{least } \mathbb{N} : \frac{k}{n} > x$
20.  $k - 1 \in \mathbb{N}, \quad \frac{k - 1}{n} \leq x$
21.  $\frac{k}{n} - \frac{1}{n} \leq x$
22.  $(9, 13, 21) \vdash \frac{k}{n} \leq x + \frac{1}{n} < x + z = x + (y - x) = y$

$$23. (19,22) \vdash x < \frac{k}{n} < y$$

$$24. k, n \in \mathbb{N}; \quad \frac{k}{n} \in \mathbb{Q} \quad \blacksquare$$

$$25. \text{ Case 2: } x < 0 \text{ and } x < y$$

$$26. (4,5) \vdash \exists t \in \mathbb{N} : t > -x$$

$$27. (25,26) \vdash 0 < x + t < y + t$$

$$28. (8,23,24,27) \vdash \exists q \in \mathbb{Q} : x + t < q < y + t$$

$$29. x < q - t < y$$

$$30. t \in \mathbb{N}, \quad -t \in \mathbb{Z}$$

$$31. \mathbb{Z} \subseteq \mathbb{Q}, \quad -t \in \mathbb{Q}$$

$$32. q, -t \in \mathbb{Q}$$

$$33. \mathbb{Q} \equiv \text{closed under addition}$$

$$34. q' = q - t = q + (-t) \in \mathbb{Q}$$

$$35. \exists q' \in \mathbb{Q} : x < q' < y \quad \blacksquare$$

$$36. (24,35) \vdash (1) \quad \square$$

## Infinite interval open in $\mathbb{R}$

### 1. Theorem

$$a \in \mathbb{R} \rightarrow (a, \infty) \equiv \text{open in } \mathbb{R}$$

### 2. $(a, \infty) :=$ infinite open interval in $\mathbb{R}$

### 3. Definitions

4.

$$(a, b) = \{x \in X \mid a < x < b\}$$

5.

$$\mathcal{X} := \text{open set in } Y$$

$$(i) \quad \mathcal{X} \subseteq Y$$

$$(ii) \quad \forall x \in \mathcal{X}, \exists (a, b) : x \in (a, b) \subseteq \mathcal{X}$$

6.

$$\forall (a, b) : (a, b) \equiv \text{open in } \mathbb{R}$$

7.  $X, Y := \text{sets}$

8.  $(a, b) := \text{open interval in } \mathbb{R}$

9. Proof

$$10. \quad x \in_{\ell} (a, \infty), \quad b =_{\ell} x + 1.$$

$$11. \quad x \in (a, \infty) \vdash x > a$$

$$12. \quad b = x + 1 \vdash b - x = 1 > 0 \vdash b > x$$

$$13. \quad (11, 12) \vdash a < x < b \vdash x \in (a, b)$$

$$14. \quad (a, b) \subseteq (a, \infty)$$

$$15. \quad x \text{ arbitrary} \vdash \forall x \in \mathbb{R} : (a, x + 1) \subseteq (a, \infty)$$

$$16. \quad (6) \vdash (a, x + 1) \equiv \text{open in } \mathbb{R}$$

$$17. \quad (15, 16) \vdash (a, \infty) \equiv \text{open in } \mathbb{R}$$

□



# Empty set and $\mathbb{R}$ open in $\mathbb{R}$

## 1. Theorem

$$\emptyset, \mathbb{R} \equiv \text{open in } \mathbb{R}$$

## 2. Definitions

3.

$$(a, b) = \{x \in X \mid a < x < b\}$$

4.

$$\mathcal{X} := \text{open set in } Y$$

$$(i) \quad \mathcal{X} \subseteq Y$$

$$(ii) \quad \forall x \in \mathcal{X}, \exists (a, b) : x \in (a, b) \subseteq \mathcal{X}$$

5.

$$\forall (a, b) : (a, b) \equiv \text{open in } \mathbb{R}$$

6.  $X, Y := \text{sets}$

7.  $(a, b) := \text{open interval in } \mathbb{R}$

## 8. Proof

9.  $\emptyset \equiv \text{open in } \mathbb{R}$  (vacuously true)

□

10.  $x \in_{\ell} \mathbb{R}$  arbitrary

11.  $x \in (x - 1, x + 1) \vdash (x - 1, x + 1) \subseteq \mathbb{R}$

12.  $(a, b) := \text{open interval}$

13.  $(10, 11) \vdash \forall x \in \mathbb{R} : \exists (a, b), x \in (a, b) \subseteq \mathbb{R}$

14.  $(5, 13) \vdash \mathbb{R} \equiv \text{open in } \mathbb{R}$

□

# Subset of $\mathbb{R}$ , Open in $\mathbb{R}$

## 1. Theorem

$$(X \equiv \text{open in } \mathbb{R}) \leftrightarrow (\forall x \in X, \exists c > 0, c \in \mathbb{R} : (x - c, x + c) \subseteq X)$$

## 2. $X \subseteq \mathbb{R}$

## 3. $(x - c, x + c) :=$ open interval in $\mathbb{R}$

## 4. Definitions

## 5.

$$(a, b) = \{x \in X \mid a < x < b\}$$

## 6.

$$\mathcal{X} := \text{open set in } Y$$

$$(i) \mathcal{X} \subseteq Y$$

$$(ii) \forall x \in \mathcal{X}, \exists (a, b) : x \in (a, b) \subseteq \mathcal{X}$$

## 7.

$$\forall (a, b) : (a, b) \equiv \text{open in } \mathbb{R}$$

## 8. $X, Y :=$ sets

## 9. $(a, b) :=$ open interval in $\mathbb{R}$

## 10. Proof

## 11. $(\rightarrow)$

$$12. X \subseteq_{\ell} \mathbb{R}, \quad X =_{\ell} \text{open in } \mathbb{R}, \quad x \in_{\ell} \mathbb{R}.$$

$$13. (12) \vdash \exists (a, b) \subseteq X, \quad x \in (a, b)$$

$$14. c =_{\ell} \min\{x - a, b - x\}$$

$$15. (14) \vdash c \leq x - a \vdash c - x \leq x - a - x \vdash c - x \leq -a \vdash x - c \geq a$$

16.  $c \leq b - x$
17.  $x + c \leq x + b - x \vdash x + c \leq b$
18.  $(13) \vdash x \in (a, b) \vdash x > a, x < b \vdash x - a > 0, b - x > 0$
19.  $(14, 18) \vdash c > 0 \vdash x + c > x$
20.  $c > 0 \vdash -c < 0 \vdash x - c < x$
21.  $(15, 17, 19, 20) \vdash a \leq x - c < x < x + c \leq b$
22.  $(x - c, x + c) \subseteq (a, b) \subseteq X$
23.  $\subseteq$  is transitive
24.  $(22, 23) \vdash (x - c, x + c) \subseteq X \quad \blacksquare$
25.  $(\leftarrow)$
26.  $x \in \mathbb{R}, (x - c, x + c) \subseteq X, c > 0, (7) \vdash X \subseteq \mathbb{R}$  is open in  $\mathbb{R} \quad \blacksquare$
27.  $(24, 26) \vdash (1) \quad \square$

## Union open sets, open in $\mathbb{R}$

### 1. Theorem

$$A \cup B \equiv \text{open in } \mathbb{R}$$

### 2. $A, B :=$ open sets in $\mathbb{R}$

### 3. Definitions

4.

$$(a, b) = \{x \in X \mid a < x < b\}$$

5.

$$\mathcal{X} := \text{open set in } Y$$

- (i)  $\mathcal{X} \subseteq Y$
- (ii)  $\forall x \in \mathcal{X}, \exists (a, b) : x \in (a, b) \subseteq \mathcal{X}$
- 6.  $X, Y :=$  sets
- 7.  $(a, b) :=$  open interval in  $\mathbb{R}$
- 8. Proof
- 9.  $A, B =_\ell$  open in  $\mathbb{R}; \quad x \in_\ell A \cup B$
- 10.  $(x \in A) \vee (x \in B)$
- 11. wlog, assume  $x \in A$
- 12.  $A \equiv$  open in  $\mathbb{R} \vdash \exists (a, b) : x \in (a, b) \subseteq A$
- 13.  $A \subseteq A \cup B$
- 14.  $\subseteq$  is transitive
- 15.  $(a, b) \subseteq A \cup B$
- 16.  $A \cup B \equiv$  open in  $\mathbb{R}$  □

## Bounded Open Intervals

### 1. Theorem

$$(\forall X \subseteq \mathbb{R}, X \neq \emptyset, X := \text{open set in } \mathbb{R}) \rightarrow (X = \bigcup B)$$

### 2. $B :=$ set of bounded open intervals

### 3. Definitions

### 4.

$$(a, b) = \{x \in X \mid a < x < b\}$$

5.

$\mathcal{X} := \text{open set in } Y$

(i)  $\mathcal{X} \subseteq Y$

(ii)  $\forall x \in \mathcal{X}, \exists (a, b) : x \in (a, b) \subseteq \mathcal{X}$

6.  $X, Y := \text{sets}$

7.  $(a, b) := \text{open interval in } \mathbb{R}$

8. Proof

9.  $X \neq \emptyset$

10.  $X := \text{open in } \mathbb{R}$

11.  $(10) \vdash \forall x \in X : \exists (a_x, b_x) : x \in (a_x, b_x) \subseteq X$

12.  $Y =_{\ell} \{(a_x, b_x) \mid x \in X\}$

13.  $x \in_{\ell} X$  arbitrary

14.  $(11, 13) \vdash x \in (a_x, b_x)$

15.  $(12) \vdash (a_x, b_x) \in Y$

16.  $x \in (a_x, b_x) \in Y \vdash x \in \bigcup Y$

17.  $(13, 16) \vdash X \subseteq \bigcup Y \quad \blacksquare$

18.  $x \in_{\ell} \bigcup Y$  arbitrary

19.  $x \in \bigcup Y, Y = \{(a_x, b_x) \mid x \in X\} \vdash \exists z \in X : x \in (a_z, b_z)$

20.  $(10, 19) \vdash x \in (a_z, b_z) \subseteq X$

21.  $(18, 20) \vdash x \in \bigcup Y \rightarrow x \in X$

22.  $(18, 21) \vdash \bigcup Y \subseteq X \quad \blacksquare$

23.  $(17, 22) \vdash X = \bigcup Y$

□

# Intersection of open sets in $\mathbb{R}$ is open

## 1. Theorem

$$A \cap B \equiv \text{open in } \mathbb{R}$$

2.  $A, B :=$  open sets in  $\mathbb{R}$

## 3. Definitions

4.

$$(a, b) = \{x \in X \mid a < x < b\}$$

5.

$$\mathcal{X} := \text{open set in } Y$$

$$(i) \ \mathcal{X} \subseteq Y$$

$$(ii) \ \forall x \in \mathcal{X}, \exists (a, b) : x \in (a, b) \subseteq \mathcal{X}$$

6.  $X, Y :=$  sets

7.  $(a, b) :=$  open interval in  $\mathbb{R}$

8.  $\text{open} := \text{open in } \mathbb{R} := \text{open set in } \mathbb{R}$

## 9. Proof

10.  $A, B =_{\ell} \text{open in } \mathbb{R}$

11.  $x \in_{\ell} A \cap B$  arbitrary

12.  $x \in A, B$

13. (5),  $A \equiv \text{open} \vdash \exists (a, b) : x \in (a, b) \subseteq A \subseteq \mathbb{R}$

14.  $B \equiv \text{open} \vdash \exists (c, d) : x \in (c, d) \subseteq B$

15.  $C =_{\ell} (a, b) \cap (c, d) \neq \emptyset$

16.  $x \in (a, b), (c, d) \vdash x \in C$

17. Lemma:  $(a, b) \cap (c, d) \equiv \emptyset \vee$  (open interval in  $\mathbb{R}$ ).
18.  $(15, 16, 17) \vdash C \equiv$  open interval in  $\mathbb{R}$
19. Lemma:  $A \cap B \subseteq A$ .
20. Lemma:  $\cap$  is commutative.
21.  $x \in (a, b) \subseteq A, x \in (c, d) \subseteq B, C = (a, b) \cap (c, d), A \cap B \subseteq A \vdash C \subseteq A, B$
22. Lemma:  $C \subseteq A, B \rightarrow C \subseteq A \cap B$ .
23.  $(21, 22) \vdash C \subseteq A \cap B$
24.  $x$  arbitrary,  $(5, 18, 23) \vdash A \cap B \equiv$  open in  $\mathbb{R}$  □

## Intersection of Closed Sets

### 1. Theorem

$$A \cap B \equiv \text{closed in } \mathbb{R}$$

2.  $A, B :=$  closed sets in  $\mathbb{R}$

### 3. Definitions

4.

$$(\mathbb{R} \setminus X \equiv \text{open in } \mathbb{R}) \rightarrow (X \equiv \text{closed in } \mathbb{R})$$

5.  $X \subseteq \mathbb{R}$

6.  $\mathbb{R} \setminus X :=$  complement of  $X$  in  $\mathbb{R}$

### 7. Proof

8.  $A, B =_{\ell}$  closed in  $\mathbb{R}$

9.  $(4, 8) \vdash \mathbb{R} \setminus A, \mathbb{R} \setminus B \equiv$  open in  $\mathbb{R}$

10. Lemma

$$(X := \text{set of open subsets of } \mathbb{R}) \rightarrow \bigcup X \equiv \text{open in } \mathbb{R}$$

11.  $(9,10) \vdash (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B) \equiv \text{open in } \mathbb{R}$

12.  $(4,11) \vdash \mathbb{R} \setminus [(\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)] \equiv \text{closed in } \mathbb{R}$

13.

$$\begin{aligned} (x \in A \cap B) &\leftrightarrow (x \in A \wedge x \in B) \leftrightarrow (x \notin \mathbb{R} \setminus A \wedge x \notin \mathbb{R} \setminus B) \leftrightarrow \\ &\leftrightarrow x \notin (\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B) \leftrightarrow x \in \mathbb{R} \setminus [(\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)] \end{aligned}$$

14.  $A \cup B = \mathbb{R} \setminus [(\mathbb{R} \setminus A) \cup (\mathbb{R} \setminus B)]$

15.  $(12,14) \vdash A \cap B \equiv \text{closed in } \mathbb{R}$

□

## Closed in $\mathbb{R}$ , Accumulation Points

1. Theorem

$$(C \equiv \text{closed in } \mathbb{R}) \leftrightarrow (x \equiv \text{accumulation point of } C \rightarrow x \in C)$$

2.  $C \subseteq \mathbb{R}$

3. Definitions

4.

$$\begin{aligned} x \equiv \text{accumulation point of } S &\leftrightarrow \\ \forall a, b \in \mathbb{R} (a < x < b \rightarrow \exists y \in S (a < y < b \wedge y \neq x)) &\equiv \\ \equiv \text{every open interval containing } x &\text{ contains at least one point of } S \\ &\text{different from } x \end{aligned}$$

5.  $S \subseteq \mathbb{R}; \quad x \in \mathbb{R}$



6.  $(\mathbb{R} \setminus X \equiv \text{open in } \mathbb{R}) \rightarrow (X \equiv \text{closed in } \mathbb{R})$
7.  $X \subseteq \mathbb{R}$
8.  $\mathbb{R} \setminus X := \text{complement of } X \text{ in } \mathbb{R}$
9. Proof
10.  $(\rightarrow)$
11. Assume:  $C$  closed in  $\mathbb{R}$ .
12.  $\mathbb{R} \setminus C \equiv \text{open in } \mathbb{R}$
13.  $x \notin C$  arbitrary
14.  $x \in \mathbb{R} \setminus C$
15.  $(12) \vdash \exists(a, b) : x \in (a, b) \subseteq \mathbb{R} \setminus C$
16. Suppose  $y \in (a, b)$  arbitrary.
17.  $(16), (a, b) \subseteq \mathbb{R} \setminus C \vdash y \notin C$
18.  $(4, 17) \vdash x \not\equiv \text{accumulation point of } C$
19.  $(13, 18) \vdash x \notin C \rightarrow x \not\equiv \text{accumulation point of } C$
20. contrapositive of (19):  $(x \equiv \text{accumulation point of } C) \rightarrow (x \in C)$  ■
21.  $(\leftarrow)$
22. Assume:  $(x \equiv \text{accumulation point of } C) \rightarrow (x \in C)$ .
23.  $\emptyset \equiv \text{open in } \mathbb{R}$
24. Assume:  $\mathbb{R} \setminus C \neq \emptyset$ .
25. contrapositive of (22):  $(x \notin C) \rightarrow (x \not\equiv \text{accumulation point of } C)$

$$26. x \in {}_\ell \mathbb{R} \setminus C \vdash x \notin C$$

$$27. (22,25,26) \vdash x \neq \text{accumulation point of } C$$

$$28. (4,27) \vdash \exists(a,b) : x \in (a,b), (a,b) \cap C = \emptyset$$

$$29. (a,b) \subseteq \mathbb{R} \setminus C$$

$$30. x \text{ arbitrary, } \mathbb{R} \setminus C \equiv \text{open in } \mathbb{R}, C \equiv \text{closed in } \mathbb{R}$$

□

## Closure, Accumulation Point, Closed in $\mathbb{R}$

### Theorems

1.

$$S \subseteq \overline{S}$$

2.

$$(C \equiv \text{closed in } \mathbb{R}, S \subseteq C) \rightarrow (\overline{S} \subseteq C)$$

3.

$$\overline{S} = S \cup \{x \in \mathbb{R} \mid x \equiv \text{accumulation point of } S\}$$

4.

$$(S \equiv \text{closed in } \mathbb{R}) \leftrightarrow (S = \overline{S})$$

5.

$$(x \in \overline{S}) \leftrightarrow \forall X (x \in X \rightarrow \exists s \in X)$$

$$6. s \in S \subseteq \mathbb{R}$$

$$7. \overline{S} := \text{closure of } S \text{ in } \mathbb{R}$$

$$8. X := \text{open interval}$$

$$9. \underline{\text{Definitions}}$$

10.

$$\overline{S} = \bigcap \{C \mid S \subseteq C, C \equiv \text{closed in } \mathbb{R}\}$$

11.  $\overline{S} := \text{closure of } S \text{ in } \mathbb{R}$

12.

$$\begin{aligned} x \equiv \text{accumulation point of } S &\leftrightarrow \\ \forall a, b \in \mathbb{R} (a < x < b \rightarrow \exists y \in S (a < y < b \wedge y \neq x)) &\equiv \\ \equiv \text{every open interval containing } x \text{ contains at least one point of } S & \\ &\text{different from } x \end{aligned}$$

13.  $S \subseteq \mathbb{R}; \quad x \in \mathbb{R}$

14. Proof of (1),

$$S \subseteq \overline{S}$$

15.  $S \subseteq \mathbb{R}$

16.  $x \in_\ell S$  arbitrary

17.  $C =_\ell$  closed in  $\mathbb{R}$

18. Lemma

$$(C \equiv \text{closed in } \mathbb{R}) \leftrightarrow (x \equiv \text{accumulation point of } C \rightarrow x \in C)$$

19.  $C \subseteq \mathbb{R}$

20.  $(16) \vdash S \subseteq C \rightarrow x \in C$

21.  $x \in C \vdash x \in \bigcap \{C \mid S \subseteq C, C \equiv \text{closed in } \mathbb{R}\} = \overline{S}$

22.  $x \in \overline{S}$

23.  $(16, 22) \vdash \forall x (x \in S \rightarrow x \in \overline{S}) \vdash S \subseteq \overline{S}$

□

24. Proof of (2),

$$(C \equiv \text{closed in } \mathbb{R}, S \subseteq C) \rightarrow (\overline{S} \subseteq C)$$

25.  $C =_\ell \text{closed in } \mathbb{R}, S \subseteq_\ell C$

26.  $x \in_\ell \overline{S}$  arbitrary

27.  $(10,26) \vdash x \in C$

28.  $(26,27) \vdash \forall x(x \in \overline{S} \rightarrow x \in C)$

29.  $\overline{S} \subseteq C$  ■

30. Proof of (3),

$$\overline{S} = S \cup \{x \in \mathbb{R} \mid x \equiv \text{accumulation point of } S\}$$

31.  $C =_\ell S \cup \{x \in \mathbb{R} \mid x \equiv \text{accumulation point of } S\}$

32.  $(10) \vdash \overline{S} \equiv \text{closed in } \mathbb{R}$

33.  $? \vdash (C \equiv \text{closed in } \mathbb{R})$

34.  $x =_\ell \text{accumulation point of } C$

35.  $x \in_\ell (a, b)$

36.  $y \in_\ell C, y \in_\ell (a, b), y \neq_\ell x$

37. case 1:  $x < y$

38.  $y \in C \vdash (y \in S) \vee (y \equiv \text{accumulation point of } S)$

39.  $y \in S \vdash \exists z \in S : z \in (x, b)$

40.  $y \equiv \text{accumulation point of } S \vdash \exists z \in S : z \in (x, b)$

41. case 2:  $x > y \vdash \exists z \in S : z \in (a, x)$

42. (40,41),  $z \neq x \vdash x := \text{accumulation point of } S$
43. (31,34,42)  $\vdash x \in C$
44.  $x$  arbitrary  $\vdash \forall x \in \mathbb{R} : (x \equiv \text{accumulation point of } C) \rightarrow x \in C$
45. (18,44)  $\vdash C \equiv \text{closed in } \mathbb{R}$  ■
46. ?  $\vdash \overline{S} \subseteq C$
47. (31)  $\vdash S \subseteq C$
48. (2,45,47)  $\vdash \overline{S} \subseteq C$  ■
49. ?  $\vdash C \subseteq \overline{S}$
50. suppose  $x \notin \overline{S}$
51. (1,50)  $\vdash x \notin S$
52.  $\overline{S} \equiv \text{closed in } \mathbb{R}$
53.  $\mathbb{R} \setminus \overline{S} \equiv \text{open in } \mathbb{R}$
54.  $x \notin \overline{S} \vdash x \in \mathbb{R} \setminus \overline{S}$
55. (53,54)  $\vdash \exists (a, b) : x \in (a, b) \subseteq \mathbb{R} \setminus \overline{S}$
56. (1)  $\vdash S \subseteq \overline{S} \vdash \mathbb{R} \setminus \overline{S} \subseteq \mathbb{R} \setminus S$
57.  $\subseteq$  is transitive
58. (55,56,57)  $\vdash (a, b) \subseteq \mathbb{R} \setminus S$
59. (12,58)  $\vdash x \not\equiv \text{accumulation point of } S$
60. (51,59)  $\vdash x \notin C$
61.  $x \notin \overline{S}$  arbitrary, (60)  $\vdash \forall x (x \in \overline{S} \rightarrow x \notin C)$
62. contrapositive of (61)  $\vdash \forall x (x \in C \rightarrow x \in \overline{S})$

63.  $(62) \vdash C \subseteq \overline{S}$  ■

64.  $(48,63) \vdash \overline{S} = C$  □

65. Proof of (4),  
 $(S \equiv \text{closed in } \mathbb{R}) \leftrightarrow (S = \overline{S})$

66.  $(\rightarrow)$

67. assume  $S$  closed in  $\mathbb{R}$

68.  $(1) \equiv (S \subseteq \overline{S})$

69.  $(2) \equiv ((S \equiv \text{closed in } \mathbb{R}, S \subseteq S) \rightarrow (\overline{S} \subseteq S))$

70.  $(1,2,67) \vdash \overline{S} \subseteq S$

71.  $(1,70) \vdash S = \overline{S}$  ■

72.  $(\leftarrow)$

73. assume  $S = \overline{S}$

74.  $\overline{S} \equiv \text{closed in } \mathbb{R}$

75.  $(73,74) \vdash S \equiv \text{closed in } \mathbb{R}$  ■

76.  $(71,75) \vdash (4)$

77. Proof of (5),  
 $(x \in \overline{S}) \leftrightarrow \forall X (x \in X \rightarrow \exists s \in X)$

78.  $s \in S \subseteq \mathbb{R}$

79.  $\overline{S} := \text{closure of } S \text{ in } \mathbb{R}$

80.  $X := \text{open interval}$

81.  $(\rightarrow)$

82.  $x \in_\ell \overline{S}$
83.  $x \in_\ell (a, b)$
84.  $(3) \equiv (\overline{S} = S \cup \{x \in \mathbb{R} \mid x \equiv \text{accumulation point of } S\})$
85.  $(3, 82) \vdash (x \in S) \vee (x \equiv \text{accumulation point of } S)$
86. case 1:  $x \in S$
87.  $x \in (a, b)$
88.  $x \in S, (a, b) \vdash (x \in \overline{S}) \rightarrow \forall X(x \in X \wedge \exists s \in X); \quad X \equiv \text{open interval} \quad \blacksquare$
89. case 2:  $x \equiv \text{accumulation point of } S$
90.  $(12, 89) \vdash \forall a, b \in \mathbb{R}(x \in (a, b) \rightarrow \exists y \in S(y \in (a, b), y \neq x)) \quad \blacksquare$
91.  $(\leftarrow)$
92. assume:  $\forall X(x \in X \rightarrow \exists s \in X); \quad s \in S \subseteq \mathbb{R}; \quad X := \text{open interval in } \mathbb{R}$
93.  $(1) \equiv (S \subseteq \overline{S})$
94.  $? \vdash x \in \overline{S}$
95. case 1:  $x \in S$
96.  $(1) \vdash x \in \overline{S} \quad \blacksquare$
97. case 2:  $x \notin S$
98.  $(92, 97) \vdash \forall X(x \in X \rightarrow \exists s \in S, s \neq x)$
99.  $(12, 98) \vdash x \equiv \text{accumulation point of } S$
100.  $(3) \equiv (\overline{S} = S \cup \{x \in \mathbb{R} \mid x \equiv \text{accumulation point of } S\})$
101.  $(3, 99) \vdash x \in \overline{S} \quad \blacksquare$
102.  $(92, 96, 101) \vdash (5) \quad \square$

# Open Invitation

*Review, add content, and co-author this white paper [4, 5].*

*Join the **Open Mathematics Collaboration**.*

*Send your contribution to `mplobo@uft.edu.br`.*

# Open Science

The **latex file** for this *white paper* together with other *supplementary files* are available in [6, 7].

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The author agrees with [5].



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