

# Note on the Odd Perfect Numbers

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## Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.

*Keywords:* Riemann Hypothesis, Prime numbers, Odd perfect numbers, Superabundant numbers, Sum-of-divisors function

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## 1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part  $\frac{1}{2}$ . As usual  $\sigma(n)$  is the sum-of-divisors function of  $n$ :

$$\sum_{d|n} d$$

where  $d | n$  means the integer  $d$  divides  $n$ ,  $d \nmid n$  means the integer  $d$  does not divide  $n$  and  $d^k \parallel n$  means  $d^k | n$  and  $d^{k+1} \nmid n$ . Define  $f(n)$  and  $G(n)$  to be  $\frac{\sigma(n)}{n}$  and  $\frac{f(n)}{\log \log n}$  respectively, such that  $\log$  is the natural logarithm. We know these properties from these functions:

**Proposition 1.1.** [1]. Let  $\prod_{i=1}^r q_i^{a_i}$  be the representation of  $n$  as a product of primes  $q_1 < \dots < q_r$  with natural numbers as exponents  $a_1, \dots, a_r$ . Then,

$$f(n) = \left( \prod_{i=1}^r \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^r \left( 1 - \frac{1}{q_i^{a_i+1}} \right).$$

**Proposition 1.2.** For every prime power  $q^a$ , we have that  $f(q^a) = \frac{q^{a+1}-1}{q^a \times (q-1)}$  [2]. If  $m, n \geq 2$  are natural numbers, then  $f(m \times n) \leq f(m) \times f(n)$  [2]. Moreover, if  $p$  is a prime number, and  $a, b$  two positive integers, then [2]:

$$f(p^{a+b}) - f(p^a) \times f(p^b) = -\frac{(p^a - 1) \times (p^b - 1)}{p^{a+b-1} \times (p - 1)^2}.$$

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Say Robins( $n$ ) holds provided

$$G(n) < e^\gamma$$

where the constant  $\gamma \approx 0.57721$  is the Euler-Mascheroni constant. The importance of this property is:

**Proposition 1.3.** Robins( $n$ ) holds for all natural numbers  $n > 5040$  if and only if the Riemann Hypothesis is true [3].

The Chebyshev function  $\theta(x)$  is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers  $p$  that are less than or equal to  $x$  [4]. We state the following property about this function:

**Proposition 1.4.** [4]. For  $x \geq 89909$ :

$$\theta(x) > \left(1 - \frac{0.068}{\log(x)}\right) \times x.$$

In mathematics,  $\Psi = n \times \prod_{q|n} \left(1 + \frac{1}{q}\right)$  is called the Dedekind  $\Psi$  function. Say Dedekind( $q_n$ ) holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^\gamma}{\zeta(2)} \times \log \theta(q_n)$$

where  $q_n$  is the  $n$ th prime number,  $\zeta(x)$  is the Riemann zeta function and  $\zeta(2) = \prod_{i=1}^{\infty} \frac{q_i^2}{q_i^2 - 1} = \frac{\pi^2}{6}$ . The importance of this inequality is:

**Proposition 1.5.** Dedekind( $q_n$ ) holds for all prime numbers  $q_n > 3$  if and only if the Riemann Hypothesis is true [5].

Let  $q_1 = 2, q_2 = 3, \dots, q_k$  denote the first  $k$  consecutive primes, then an integer of the form  $\prod_{i=1}^k q_i^{a_i}$  with  $a_1 \geq a_2 \geq \dots \geq a_k \geq 0$  is called an Hardy-Ramanujan integer [6]. A natural number  $n$  is called superabundant precisely when, for all natural numbers  $m < n$

$$f(m) < f(n).$$

**Proposition 1.6.** If  $n$  is superabundant, then  $n$  is an Hardy-Ramanujan integer [7]. Let  $n$  be a superabundant number, then  $p \parallel n$  where  $p$  is the largest prime factor of  $n$  [7]. For large enough superabundant number  $n$ , we have that  $q^{a_q} < 2^{a_2}$  for  $q > 11$  where  $q^{a_q} \parallel n$  and  $2^{a_2} \parallel n$  [7]. For large enough superabundant number  $n$ , we obtain that  $\log n < \left(1 + \frac{0.5}{\log p}\right) \times p$  where  $p$  is the largest prime factor of  $n$  [4]. Let  $n$  be a superabundant number, then  $f(n) > (1 - \varepsilon(p)) \times \prod_{q|n} \frac{q}{q-1}$  where  $\varepsilon(p) = \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)$  and  $p$  is the largest prime factor of  $n$  [4].

Define  $R(N_k)$  to be  $\frac{f(N_k)}{\log \log N_k}$  where  $N_k = \prod_{i=1}^k q_i$  is the primorial number of order  $k$ .

**Proposition 1.7.** [5], [8]:

$$\lim_{k \rightarrow +\infty} \frac{R(N_k)}{2} = \frac{e^\gamma}{\zeta(2)}.$$

**Proposition 1.8.** [9]. Under the assumption of the Riemann Hypothesis, the inequality  $R(N_k) > R(N_{k+1})$  is violated for infinitely many  $k$ 's.

In addition, we will use this property:

**Proposition 1.9.** [5], [6]. For  $n \geq 2$ :

$$\prod_{q > q_n} \frac{q^2}{q^2 - 1} \leq e^{\frac{2}{q_n}}.$$

In number theory, a perfect number is a positive integer  $n$  such that  $f(n) = 2$ . Euclid proved that every even perfect number is of the form  $2^{s-1} \times (2^s - 1)$  whenever  $2^s - 1$  is prime. It is unknown whether any odd perfect numbers exist, though various results have been obtained:

**Proposition 1.10.** Any odd perfect number  $N$  must satisfy the following conditions:  $N > 10^{1500}$  and the largest prime factor of  $N$  is greater than  $10^8$  [10], [11].

Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.

## 2. Main Insight

**Lemma 2.1.** Under the assumption of the Riemann Hypothesis, we prove that

$$\frac{\pi^2}{6.4} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$$

is satisfied for infinitely many prime numbers  $q_n$ .

*Proof.* We know that:

$$\lim_{k \rightarrow +\infty} R(N_k) = \lim_{k \rightarrow +\infty} R(N_k) = \frac{e^\gamma}{\frac{\pi^2}{6}}$$

according to the Proposition 1.7 and the properties of limit [8]. For any positive real number  $\varepsilon$ , there exists a natural number  $m$  such that

$$R(N_k) > \frac{e^\gamma}{\frac{\pi^2}{6}} - \varepsilon$$

for all  $k > m$ , because of the definition of limit inferior [8]. If we multiply the previous inequality by  $\frac{6.4}{6}$ , then we obtain that

$$\frac{6.4}{6} \times R(N_k) > \frac{e^\gamma}{\frac{\pi^2}{6.4}} - \varepsilon'$$

no matter how large or small could be the value of  $\varepsilon'$ . However, there are infinitely many prime numbers  $q_k$  such that there always exists some prime  $q_n$ :

$$R(N_n) \geq \frac{6.4}{6} \times R(N_k) + \varepsilon'$$

under the assumption of the Riemann Hypothesis due to the Proposition 1.8. Certainly, there could be many primes  $q_n$  such that

$$R(N_n) \gg \frac{6.4}{6} \times R(N_k)$$

for every large enough and fixed prime number  $q_k$  where  $q_n > q_k$  and

$$6 \times \prod_{q_k < q \leq q_n} \left(1 + \frac{1}{q}\right) \gg 6.4 \times \left(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k \times \log \log N_k}\right).$$

Here, the symbol  $\gg$  means “much greater than”. Indeed,

$$\begin{aligned} \frac{\log \log N_n}{\log \log N_k} &= \frac{\log \left(\log N_k + \log \frac{N_n}{N_k}\right)}{\log \log N_k} \\ &= \frac{\log \left(\log N_k \times \left(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k}\right)\right)}{\log \log N_k} \\ &= \frac{\log \log N_k + \log \left(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k}\right)}{\log \log N_k} \\ &= 1 + \frac{\log \left(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k}\right)}{\log \log N_k} \\ &\leq 1 + \frac{\log \frac{N_n}{N_k}}{\log N_k \times \log \log N_k} \end{aligned}$$

since when  $x > -1$ , then  $x \geq \log(1 + x)$  [12]. Therefore, the proof is done.  $\square$

### 3. Main Theorem

**Theorem 3.1.** *Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.*

*Proof.* Suppose that  $N$  is the smallest odd perfect number, then we will show its existence implies that the Riemann Hypothesis is false. There is always a large enough superabundant number  $n$  such that  $n$  is a multiple of  $N$ . We would have

$$f(n) \leq f(N) \times f\left(\frac{n}{N}\right)$$

according to the Proposition 1.2. That is the same as

$$f(n) \leq 2 \times f\left(\frac{n}{N}\right)$$

since  $f(N) = 2$ , because  $N$  is a perfect number. Hence,

$$\begin{aligned}\frac{f(n)}{2} &= \frac{(2 - \frac{1}{2^{a_2}}) \times f(\frac{n}{2^{a_2}})}{2} \\ &= f(\frac{n}{2^{a_2}}) \times \frac{(2 - \frac{1}{2^{a_2}})}{2} \\ &= f(\frac{n}{2^{a_2}}) \times \frac{2^{a_2+1} - 1}{2^{a_2+1}}\end{aligned}$$

when  $2^{a_2} \parallel n$  due to the Proposition 1.2. In this way, we have

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq \frac{2^{a_2+1}}{2^{a_2+1} - 1}.$$

However, we know that  $p < 2^{a_2}$  because of  $p > 10^8 > 11$  and the Propositions 1.6 and 1.10, where  $p$  is the largest prime factor of  $n$ . Consequently,

$$\frac{2^{a_2+1}}{2^{a_2+1} - 1} \leq \frac{2 \times p}{2 \times p - 1}$$

since  $\frac{x}{x-1}$  decreases when  $x \geq 2$  increases. In addition, we know that

$$\frac{2 \times p}{2 \times p - 1} \leq f(p)$$

where we know that  $f(p) = \frac{p+1}{p}$  from the Proposition 1.2. Certainly,

$$\begin{aligned}2 \times p^2 &\leq (p+1) \times (2 \times p - 1) \\ &= 2 \times p^2 + 2 \times p - p - 1 \\ &= 2 \times p^2 + p - 1\end{aligned}$$

where this inequality is satisfied for every prime number  $p$ . So,

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq f(p)$$

where we know that  $p \parallel n$  from the Proposition 1.6. Under the assumption of the Riemann Hypothesis, we have that

$$\begin{aligned}e^\gamma &> G(n) \\ &= \frac{f(\frac{n}{p}) \times f(p)}{\log \log n} \\ &\geq \frac{f(\frac{n}{p}) \times f(\frac{n}{2^{a_2}})}{f(\frac{n}{N}) \times \log \log n}\end{aligned}$$

since  $f(\dots)$  is multiplicative and as a consequence of the Proposition 1.3. This is equivalent to

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} < \frac{e^\gamma}{f(\frac{n}{2^{a_2}})} \times \log \log n.$$

Under the assumption of the Riemann Hypothesis and the Lemma 2.1, we deduce that:

$$\frac{\pi^2}{6.4} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(p)$$

which is the same as

$$\frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log((\theta(p))^{0.8}).$$

From the Propositions 1.1 and 1.6, we know that

$$f\left(\frac{n}{2^{a_2}}\right) = \left(\prod_{i=2}^k \frac{q_i}{q_i - 1}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

where  $q_k = p$  and  $q_1 = 2$ . We know that

$$\frac{q_i}{q_i - 1} = \frac{q_i + 1}{q_i} \times \frac{q_i^2}{q_i^2 - 1}.$$

Using the previous inequality and the Lemma 2.1, we obtain that

$$\begin{aligned} e^\gamma \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \log((\theta(p))^{0.8}) &< \frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= f\left(\frac{n}{2^{a_2}}\right) \times \frac{3}{2} \times \prod_{q > p} \frac{q^2}{q^2 - 1} \\ &\leq f\left(\frac{n}{2^{a_2}}\right) \times \frac{3}{2} \times e^{\frac{2}{p}} \end{aligned}$$

according to the Proposition 1.9. Taking into account that  $p > 10^8 > 3$  and  $n$  is superabundant:

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} > \frac{e^\gamma}{f\left(\frac{n}{2^{a_2}}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

We use the previous inequality to show that

$$\frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < \frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log \log n.$$

For large enough superabundant number  $n$  and  $p > 10^8$ , then

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log \log n \leq \frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(\left(1 - \frac{0.068}{\log 10^8}\right) \times 10^8\right)^{0.8}} \times \log\left(\left(1 + \frac{0.5}{\log 10^8}\right) \times 10^8\right)$$

because of the Propositions 1.4 and 1.6. We obtain that

$$\frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(\left(1 - \frac{0.068}{\log 10^8}\right) \times 10^8\right)^{0.8}} \times \log\left(\left(1 + \frac{0.5}{\log 10^8}\right) \times 10^8\right) < 1.87811.$$

Thus,

$$\frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.87811.$$

For every prime  $p_j$  that divides  $N$  such that  $p_j^{a_j} \parallel N$  and  $p_j^{a_j+b_j} \parallel n$  for  $a_j, b_j$  two natural numbers, we have that

$$f(p_j^{a_j+b_j}) - f(p_j^{a_j}) \times f(p_j^{b_j}) = -\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

in the Proposition 1.2. This is equal to

$$\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})} = f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}.$$

Hence,

$$\begin{aligned} \frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) &= \prod_j \left(\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= \prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &> 1.999 \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &> 1.999 \times \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) \times \frac{1}{\left(1 - \frac{1}{2^{a_2+1}}\right)} \\ &> 1.999 \times \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) \\ &> 1.999 \times \left(1 - \frac{1}{\log 10^8} \times \left(1 + \frac{1.5}{\log 10^8}\right)\right) \\ &> 1.88 \\ &> 1.87811 \end{aligned}$$

using the Propositions 1.6 and 1.1 since we know that the expression

$$\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

tends to 0 as  $b_j$  tends to infinity for every odd prime  $p_j$  where

$$\begin{aligned} \prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) &\approx \prod_j (f(p_j^{a_j})) \\ &= f(N) \\ &= 2. \end{aligned}$$

Certainly, the fraction  $\frac{f(\frac{n}{p})}{f(\frac{n}{N})}$  gets closer to 2 as long as we take  $n$  bigger and bigger. In addition, we note that

$$\begin{aligned} \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) &< \prod_{i=1}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \left(1 - \frac{1}{2^{a_2+1}}\right) \end{aligned}$$

after taking into account the Proposition 1.6. However,

$$1.87811 < \frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.87811$$

is a contradiction. By contraposition, the number  $N$  does not exist under the assumption of the Riemann Hypothesis. The smallest counterexample  $N$  must comply that  $N > 10^{1500}$  and therefore, we will always be capable of obtaining a large enough superabundant number  $n$  that is multiple of  $N$ . Note that, this proof fails for even perfect numbers.  $\square$

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## References

- [1] A. Hertlein, Robin's Inequality for New Families of Integers, *Integers* 18, (2018).
- [2] R. Vojak, On numbers satisfying Robin's inequality, properties of the next counterexample and improved specific bounds, arXiv preprint arXiv:2005.09307, (2020).
- [3] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, *J. Math. pures appl* 63 (2) (1984) 187–213.
- [4] S. Nazardonyavi, S. Yakubovich, Superabundant numbers, their subsequences and the Riemann hypothesis, arXiv preprint arXiv:1211.2147, (2012).
- [5] P. Solé, M. Planat, Extreme values of the Dedekind  $\psi$  function, *Journal of Combinatorics and Number Theory* 3 (1) (2011) 33–38.
- [6] Y. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin's criterion for the Riemann hypothesis, *Journal de Théorie des Nombres de Bordeaux* 19 (2) (2007) 357–372. doi:doi:10.5802/jtnb.591.
- [7] L. Alaoglu, P. Erdős, On highly composite and similar numbers, *Transactions of the American Mathematical Society* 56 (3) (1944) 448–469. doi:doi:10.2307/1990319.
- [8] G. H. Hardy, E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press, 1979.
- [9] Y. Choie, M. Planat, P. Solé, On Nicolas criterion for the Riemann hypothesis, arXiv preprint arXiv:1012.3613, (2010).
- [10] P. Ochem, M. Rao, Odd perfect numbers are greater than  $10^{1500}$ , *Mathematics of Computation* 81 (279) (2012) 1869–1877. doi:doi:10.1090/S0025-5718-2012-02563-4.
- [11] T. Goto, Y. Ohno, Odd perfect numbers have a prime factor exceeding  $10^8$ , *Mathematics of Computation* 77 (263) (2008) 1859–1868.
- [12] L. Kozma, Useful Inequalities, [http://www.lkozma.net/inequalities\\_cheat\\_sheet/ineq.pdf](http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf), accessed 4 June 2022 (2022).