Note on the Odd Perfect Numbers

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Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.

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1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n:

$$\sum_{d \mid n} d$$

where $d \mid n$ means the integer d divides n, $d \nmid n$ means the integer d does not divide n and $d^k \parallel n$ means $d^k \mid n$ and $d^{k+1} \nmid n$. Define $f(n)$ and $G(n)$ to be $\frac{\sigma(n)}{n}$ and $\frac{f(n)}{\log \log n}$ respectively, such that log is the natural logarithm. We know these properties from these functions:

Proposition 1.1. *[1]. Let $\prod_{i=1}^{r} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_r$ with natural numbers as exponents $a_1, \ldots, a_r$. Then,*

$$f(n) = \left(\prod_{i=1}^{r} \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^{a_i+1}} \right).$$

Proposition 1.2. *For every prime power q^a, we have that $f(q^a) = \frac{q^{a+1}-1}{q^a \times (q-1)}$ [2]. If $m, n \geq 2$ are natural numbers, then $f(m \times n) \leq f(m) \times f(n)$ [2]. Moreover, if p is a prime number, and a, b two positive integers, then [2]:*

$$f(p^{a+b}) - f(p^a) \times f(p^b) = -\frac{(p^a - 1) \times (p^b - 1)}{p^{a+b-1} \times (p-1)^2}.$$

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Say $\mathsf{Robins}(n)$ holds provided

$$G(n) < e^{\gamma}$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The importance of this property is:

Proposition 1.3. $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann Hypothesis is true [3].*

The Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [4]. We state the following property about this function:

Proposition 1.4. *[4]. For* $x \geq 89909$*:*

$$\theta(x) > (1 - \frac{0.068}{\log(x)}) \times x.$$

In mathematics, $\Psi = n \times \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function. Say $\mathsf{Dedekind}(q_n)$ holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n)$$

where q_n is the nth prime number, $\zeta(x)$ is the Riemann zeta function and $\zeta(2) = \prod_{i=1}^{\infty} \frac{q_i^2}{q_i^2 - 1} = \frac{\pi^2}{6}$. The importance of this inequality is:

Proposition 1.5. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann Hypothesis is true [5].*

Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ is called an Hardy-Ramanujan integer [6]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 1.6. *If n is superabundant, then n is an Hardy-Ramanujan integer [7]. Let n be a superabundant number, then* $p \parallel n$ *where p is the largest prime factor of n [7]. For large enough superabundant number n, we have that* $q^{a_q} < 2^{a_2}$ *for* $q > 11$ *where* $q^{a_q} \parallel n$ *and* $2^{a_2} \parallel n$ *[7]. For large enough superabundant number n, we obtain that* $\log n < (1 + \frac{0.5}{\log p}) \times p$ *where p is the largest prime factor of n [4]. Let n be a superabundant number, then* $f(n) > (1 - \varepsilon(p)) \times \prod_{q \mid n} \frac{q}{q-1}$ *where* $\varepsilon(p) = \frac{1}{\log p} \times (1 + \frac{1.5}{\log p})$ *and p is the largest prime factor of n [4].*

Define $R(N_k)$ to be $\frac{f(N_k)}{\log \log N_k}$ where $N_k = \prod_{i=1}^{k} q_i$ is the primorial number of order k.

Proposition 1.7. *[5], [8]:*

$$\lim_{k \to +\infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

Proposition 1.8. *[9]. Under the assumption of the Riemann Hypothesis, the inequality $R(N_k) > R(N_{k+1})$ is violated for infinitely many k's.*

In addition, we will use this property:

Proposition 1.9. *[5], [6]. For $n \geq 2$:*

$$\prod_{q > q_n} \frac{q^2}{q^2 - 1} \leq e^{\frac{2}{q_n}}.$$

In number theory, a perfect number is a positive integer n such that $f(n) = 2$. Euclid proved that every even perfect number is of the form $2^{s-1} \times (2^s - 1)$ whenever $2^s - 1$ is prime. It is unknown whether any odd perfect numbers exist, though various results have been obtained:

Proposition 1.10. *Any odd perfect number N must satisfy the following conditions: $N > 10^{1500}$ and the largest prime factor of N is greater than 10^8 [10], [11].*

Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.

2. Main Insight

Lemma 2.1. *Under the assumption of the Riemann Hypothesis, we prove that*

$$\frac{\pi^2}{6.4} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log \theta(q_n)$$

is satisfied for infinitely many prime numbers q_n.

Proof. We know that:

$$\lim_{k \to +\infty} R(N_k) = \underline{\lim}_{k \to +\infty} R(N_k) = \frac{e^{\gamma}}{\frac{\pi^2}{6}}$$

according to the Proposition 1.7 and the properties of limit [8]. For any positive real number ε, there exists a natural number m such that

$$R(N_k) > \frac{e^{\gamma}}{\frac{\pi^2}{6}} - \varepsilon$$

for all $k > m$, because of the definition of limit inferior [8]. If we multiply the previous inequality by $\frac{6.4}{6}$, then we obtain that

$$\frac{6.4}{6} \times R(N_k) > \frac{e^{\gamma}}{\frac{\pi^2}{6.4}} - \varepsilon'$$

no matter how large or small could be the value of ε'. However, there are infinitely many prime numbers q_k such that there always exists some prime q_n:

$$R(N_n) \geq \frac{6.4}{6} \times R(N_k) + \varepsilon'$$

under the assumption of the Riemann Hypothesis due to the Proposition 1.8. Certainly, there could be some primes q_n such that

$$R(N_n) > \frac{6.4}{6} \times R(N_k)$$

for every large enough and fixed prime number q_k where $q_n > q_k$ and

$$6 \times \prod_{q_k < q \leq q_n} \left(1 + \frac{1}{q}\right) > 6.4 \times \left(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k \times \log\log N_k}\right).$$

Indeed,

$$\begin{aligned}
\frac{\log\log N_n}{\log\log N_k} &= \frac{\log\left(\log N_k + \log \frac{N_n}{N_k}\right)}{\log\log N_k} \\
&= \frac{\log\left(\log N_k \times (1 + \frac{\log \frac{N_n}{N_k}}{\log N_k})\right)}{\log\log N_k} \\
&= \frac{\log\log N_k + \log(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k})}{\log\log N_k} \\
&= 1 + \frac{\log(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k})}{\log\log N_k} \\
&\leq 1 + \frac{\log \frac{N_n}{N_k}}{\log N_k \times \log\log N_k}
\end{aligned}$$

since when $x > -1$, then $x \geq \log(1 + x)$ [12]. Therefore, the proof is done. □

3. Main Theorem

Theorem 3.1. *Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.*

Proof. Suppose that N is the smallest odd perfect number, then we will show its existence implies that the Riemann Hypothesis is false. There is always a large enough superabundant number n such that n is a multiple of N. We would have

$$f(n) \leq f(N) \times f(\frac{n}{N})$$

according to the Proposition 1.2. That is the same as

$$f(n) \leq 2 \times f(\frac{n}{N})$$

since $f(N) = 2$, because N is a perfect number. Hence,

$$\frac{f(n)}{2} = \frac{(2 - \frac{1}{2^{a_2}}) \times f(\frac{n}{2^{a_2}})}{2}$$
$$= f(\frac{n}{2^{a_2}}) \times \frac{(2 - \frac{1}{2^{a_2}})}{2}$$
$$= f(\frac{n}{2^{a_2}}) \times \frac{2^{a_2+1} - 1}{2^{a_2+1}}$$

when $2^{a_2} \parallel n$ due to the Proposition 1.2. In this way, we have

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq \frac{2^{a_2+1}}{2^{a_2+1} - 1}.$$

However, we know that $p < 2^{a_2}$ because of $p > 10^8 > 11$ and the Propositions 1.6 and 1.10, where p is the largest prime factor of n. Consequently,

$$\frac{2^{a_2+1}}{2^{a_2+1} - 1} \leq \frac{2 \times p}{2 \times p - 1}$$

since $\frac{x}{x-1}$ decreases when $x \geq 2$ increases. In addition, we know that

$$\frac{2 \times p}{2 \times p - 1} \leq f(p)$$

where we know that $f(p) = \frac{p+1}{p}$ from the Proposition 1.2. Certainly,

$$2 \times p^2 \leq (p + 1) \times (2 \times p - 1)$$
$$= 2 \times p^2 + 2 \times p - p - 1$$
$$= 2 \times p^2 + p - 1$$

where this inequality is satisfied for every prime number p. So,

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq f(p)$$

where we know that $p \parallel n$ from the Proposition 1.6. Under the assumption of the Riemann Hypothesis, we have that

$$e^{\gamma} > G(n)$$
$$= \frac{f(\frac{n}{p}) \times f(p)}{\log \log n}$$
$$\geq \frac{f(\frac{n}{p}) \times f(\frac{n}{2^{a_2}})}{f(\frac{n}{N}) \times \log \log n}$$

since $f(\ldots)$ is multiplicative and as a consequence of the Proposition 1.3. This is equivalent to

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} < \frac{e^{\gamma}}{f(\frac{n}{2^{a_2}})} \times \log \log n.$$

Under the assumption of the Riemann Hypothesis and the Lemma 2.1, we deduce that:

$$\frac{\pi^2}{6.4} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log \theta(p)$$

which is the same as

$$\frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^{\gamma} \times \log((\theta(p))^{0.8}).$$

From the Propositions 1.1 and 1.6, we know that

$$f(\frac{n}{2^{a_2}}) = \left(\prod_{i=2}^{k} \frac{q_i}{q_i - 1}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

where $q_k = p$ and $q_1 = 2$. We know that

$$\frac{q_i}{q_i - 1} = \frac{q_i + 1}{q_i} \times \frac{q_i^2}{q_i^2 - 1}.$$

Using the previous inequality and the Lemma 2.1, we obtain that

$$\begin{aligned} e^{\gamma} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \log((\theta(p))^{0.8}) &< \frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= f(\frac{n}{2^{a_2}}) \times \frac{3}{2} \times \prod_{q > p} \frac{q^2}{q^2 - 1} \\ &\leq f(\frac{n}{2^{a_2}}) \times \frac{3}{2} \times e^{\frac{2}{p}} \end{aligned}$$

according to the Proposition 1.9. Taking into account that $p > 10^8 > 3$ and n is superabundant:

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} > \frac{e^{\gamma}}{f(\frac{n}{2^{a_2}})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

We use the previous inequality to show that

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < \frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log \log n.$$

For large enough superabundant number n and $p > 10^8$, then

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log \log n \leq \frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(((1 - \frac{0.068}{\log 10^8}) \times 10^8)^{0.8}\right)} \times \log\left((1 + \frac{0.5}{\log 10^8}) \times 10^8\right)$$

because of the Propositions 1.4 and 1.6. We obtain that

$$\frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(((1 - \frac{0.068}{\log 10^8}) \times 10^8)^{0.8}\right)} \times \log\left((1 + \frac{0.5}{\log 10^8}) \times 10^8\right) < 1.87811.$$

Thus,

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.87811.$$

For every prime p_j that divides N such that $p_j^{a_j} \parallel N$ and $p_j^{a_j+b_j} \parallel n$ for a_j, b_j two natural numbers, we have that

$$f(p_j^{a_j+b_j}) - f(p_j^{a_j}) \times f(p_j^{b_j}) = -\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

in the Proposition 1.2. This is equal to

$$\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})} = f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}.$$

Hence,

$$\begin{aligned}
\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) &= \prod_j \left(\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\
&= \prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\
&> 1.999 \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\
&> 1.999 \times \left(1 - \frac{1}{\log p} \times (1 + \frac{1.5}{\log p})\right) \times \frac{1}{(1 - \frac{1}{2^{a_2+1}})} \\
&> 1.999 \times \left(1 - \frac{1}{\log p} \times (1 + \frac{1.5}{\log p})\right) \\
&> 1.999 \times \left(1 - \frac{1}{\log 10^8} \times (1 + \frac{1.5}{\log 10^8})\right) \\
&> 1.88 \\
&> 1.87811
\end{aligned}$$

using the Propositions 1.6 and 1.1 since we know that the expression

$$\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

tends to 0 as b_j tends to infinity for every odd prime p_j where

$$\begin{aligned}
\prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) &\approx \prod_j \left(f(p_j^{a_j})\right) \\
&= f(N) \\
&= 2.
\end{aligned}$$

Certainly, the fraction $\frac{f(\frac{n}{p})}{f(\frac{n}{N})}$ gets closer to 2 as long as we take n bigger and bigger. In addition, we note that

$$\left(1 - \frac{1}{\log p} \times (1 + \frac{1.5}{\log p})\right) < \prod_{i=1}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$
$$= \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times (1 - \frac{1}{2^{a_2+1}})$$

after taking into account the Proposition 1.6. However,

$$1.87811 < \frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.87811$$

is a contradiction. By contraposition, the number N does not exist under the assumption of the Riemann Hypothesis. The smallest counterexample N must comply that $N > 10^{1500}$ and therefore, we will always be capable of obtaining a large enough superabundant number n that is multiple of N. Note that, this proof fails for even perfect numbers. □

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