Note on the Odd Perfect Numbers

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Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.

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1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n:

$$\sum_{d \mid n} d$$

where $d \mid n$ means the integer d divides n, $d \nmid n$ means the integer d does not divide n and $d^k \parallel n$ means $d^k \mid n$ and $d^{k+1} \nmid n$. Define $f(n)$ and $G(n)$ to be $\frac{\sigma(n)}{n}$ and $\frac{f(n)}{\log \log n}$ respectively, such that log is the natural logarithm. We know these properties from these functions:

Proposition 1.1. *[1]. Let $\prod_{i=1}^{r} q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \cdots < q_r$ with natural numbers as exponents $a_1, \ldots, a_r$. Then,*

$$f(n) = \left(\prod_{i=1}^{r} \frac{q_i}{q_i - 1}\right) \times \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

Proposition 1.2. *For every prime power q^a, we have that $f(q^a) = \frac{q^{a+1}-1}{q^a \times (q-1)}$ [2]. If $m, n \geq 2$ are natural numbers, then $f(m \times n) \leq f(m) \times f(n)$ [2]. Moreover, if p is a prime number, and a, b two positive integers, then [2]:*

$$f(p^{a+b}) - f(p^a) \times f(p^b) = -\frac{(p^a - 1) \times (p^b - 1)}{p^{a+b-1} \times (p-1)^2}.$$

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Say $\mathsf{Robins}(n)$ holds provided

$$G(n) < e^{\gamma}$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The importance of this property is:

Proposition 1.3. $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann Hypothesis is true [3].*

In mathematics, $\Psi = n \times \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function. Say $\mathsf{Dedekind}(q_n)$ holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n)$$

where q_n is the nth prime number, $\zeta(x)$ is the Riemann zeta function and $\zeta(2) = \prod_{i=1}^{\infty} \frac{q_i^2}{q_i^2 - 1} = \frac{\pi^2}{6}$. The importance of this inequality is:

Proposition 1.4. $\mathsf{Dedekind}(q_n)$ *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann Hypothesis is true [4].*

Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$ is called an Hardy-Ramanujan integer [5]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 1.5. *If n is superabundant, then n is an Hardy-Ramanujan integer [6]. Let n be a superabundant number, then* $p \parallel n$ *where p is the largest prime factor of n [6]. For large enough superabundant number n, we have that* $q^{a_q} < 2^{a_2}$ *for* $q > 11$ *where* $q^{a_q} \parallel n$ *and* $2^{a_2} \parallel n$ *[6]. For large enough superabundant number n, we obtain that* $\log n < (1 + \frac{0.5}{\log p}) \times p$ *where p is the largest prime factor of n [7]. Let n be a superabundant number, then* $f(n) > (1 - \varepsilon(p)) \times \prod_{q|n} \frac{q}{q-1}$ *where* $\varepsilon(p) = 1 - \frac{1}{\log p} \times (1 + \frac{1.5}{\log p})$ *and p is the largest prime factor of n [7].*

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [7].

Proposition 1.6. *[7]. For* $x \geq 89909$*:*

$$\theta(x) > (1 - \frac{0.068}{\log(x)}) \times x.$$

In addition, we will use this property:

Proposition 1.7. *[4]. For* $n \geq 2$*:*

$$\prod_{q > q_n} \frac{q^2}{q^2 - 1} \leq e^{\frac{2}{q_n}}.$$

In number theory, a perfect number is a positive integer n such that $f(n) = 2$. Euclid proved that every even perfect number is of the form $2^{s-1} \times (2^s - 1)$ whenever $2^s - 1$ is prime. It is unknown whether any odd perfect numbers exist, though various results have been obtained:

Proposition 1.8. *Any odd perfect number N must satisfy the following conditions:* $N > 10^{1500}$ *and the largest prime factor of N is greater than* 10^8 *[8], [9].*

Using these results, we finally prove that there is not any odd perfect number at all.

2. Results

Theorem 2.1. *Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.*

Proof. Suppose that N is the smallest odd perfect number, then we will show its existence implies that the Riemann Hypothesis is false. There is always a large enough superabundant number n such that n is a multiple of N. We would have

$$f(n) \leq f(N) \times f(\frac{n}{N})$$

according to the Proposition 1.2. That is the same as

$$f(n) \leq 2 \times f(\frac{n}{N})$$

since $f(N) = 2$, because N is a perfect number. Hence,

$$\begin{aligned} \frac{f(n)}{2} &= \frac{(2 - \frac{1}{2^{a_2}}) \times f(\frac{n}{2^{a_2}})}{2} \\ &= f(\frac{n}{2^{a_2}}) \times \frac{(2 - \frac{1}{2^{a_2}})}{2} \\ &= f(\frac{n}{2^{a_2}}) \times \frac{2^{a_2+1} - 1}{2^{a_2+1}} \end{aligned}$$

when $2^{a_2} \parallel n$ due to the Proposition 1.2. In this way, we have

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq \frac{2^{a_2+1}}{2^{a_2+1} - 1}.$$

However, we know that $p < 2^{a_2}$ because of $p > 10^8 > 11$ and the Propositions 1.5 and 1.8, where p is the largest prime factor of n. Consequently,

$$\frac{2^{a_2+1}}{2^{a_2+1} - 1} \leq \frac{2 \times p}{2 \times p - 1}$$

since $\frac{x}{x-1}$ decreases when $x \geq 2$ increases. In addition, we know that

$$\frac{2 \times p}{2 \times p - 1} \leq f(p)$$

where we know that $f(p) = \frac{p+1}{p}$ from the Proposition 1.2. Certainly,

$$\begin{aligned} 2 \times p^2 &\leq (p+1) \times (2 \times p - 1) \\ &= 2 \times p^2 + 2 \times p - p - 1 \\ &= 2 \times p^2 + p - 1 \end{aligned}$$

where this inequality is satisfied for every prime number p. So,

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq f(p)$$

where we know that $p \parallel n$ from the Proposition 1.5. Under the assumption of the Riemann Hypothesis, we have that

$$\begin{aligned} e^{\gamma} &> G(n) \\ &= \frac{f(\frac{n}{p}) \times f(p)}{\log \log n} \\ &\geq \frac{f(\frac{n}{p}) \times f(\frac{n}{2^{a_2}})}{f(\frac{n}{N}) \times \log \log n} \end{aligned}$$

since $f(\ldots)$ is multiplicative and as a consequence of Proposition 1.3. This is equivalent to

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} < \frac{e^{\gamma}}{f(\frac{n}{2^{a_2}})} \times \log \log n.$$

Under the assumption of the Riemann Hypothesis:

$$\prod_{q \leq p} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(p)$$

due to the Proposition 1.4. From the Propositions 1.1 and 1.5, we know that

$$f(\frac{n}{2^{a_2}}) = \left(\prod_{i=2}^{k} \frac{q_i}{q_i - 1}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

where $q_k = p$ and $q_1 = 2$. We know that

$$\frac{q_i}{q_i - 1} = \frac{q_i + 1}{q_i} \times \frac{q_i^2}{q_i^2 - 1}.$$

Using the previous inequality, we obtain that

$$\begin{aligned} e^{\gamma} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \log \theta(p) &< \zeta(2) \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= f(\frac{n}{2^{a_2}}) \times \frac{3}{2} \times \prod_{q > p} \frac{q^2}{q^2 - 1} \\ &\leq f(\frac{n}{2^{a_2}}) \times \frac{3}{2} \times e^{\frac{2}{p}} \end{aligned}$$

according to the Proposition 1.7. Taking into account that $p > 10^8 > 3$ and n is superabundant:

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log \theta(p)} > \frac{e^{\gamma}}{f(\frac{n}{2^{a_2}})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

We use the previous inequality to show that

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < \frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log \theta(p)} \times \log \log n.$$

For large enough superabundant number n and $p > 10^8$, then

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log \theta(p)} \times \log \log n \leq \frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log \left((1 - \frac{0.068}{\log 10^8}) \times 10^8\right)} \times \log \left((1 + \frac{0.5}{\log 10^8}) \times 10^8\right)$$

because of the Propositions 1.6 and 1.5. We obtain that

$$\frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log \left((1 - \frac{0.068}{\log 10^8}) \times 10^8\right)} \times \log \left((1 + \frac{0.5}{\log 10^8}) \times 10^8\right) < 1.51.$$

Thus,

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.51.$$

For every prime p_i that divides N such that $p_i^{a_i} \parallel N$ and $p_i^{a_i+b_i} \parallel n$ for a_i, b_i two natural numbers, we have that

$$f(p_i^{a_i+b_i}) - f(p_i^{a_i}) \times f(p_i^{b_i}) = -\frac{(p_i^{a_i} - 1) \times (p_i^{b_i} - 1)}{p_i^{a_i+b_i-1} \times (p_i - 1)^2}$$

in the Proposition 1.2. This is equal to

$$\frac{f(p_i^{a_i+b_i})}{f(p_i^{b_i})} = f(p_i^{a_i}) - \frac{(p_i^{a_i} - 1) \times (p_i^{b_i} - 1)}{f(p_i^{b_i}) \times p_i^{a_i+b_i-1} \times (p_i - 1)^2}.$$

Hence,

$$\begin{aligned}
\frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) &= \prod_{i} \left(\frac{f(p_i^{a_i+b_i})}{f(p_i^{b_i})}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\
&= \prod_{i} \left(f(p_i^{a_i}) - \frac{(p_i^{a_i} - 1) \times (p_i^{b_i} - 1)}{f(p_i^{b_i}) \times p_i^{a_i+b_i-1} \times (p_i - 1)^2}\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\
&\approx \prod_{i} \left(f(p_i^{a_i})\right) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\
&= f(N) \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\
&= 2 \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\
&> 2 \times \left(1 - \frac{1}{\log p} \times (1 + \frac{1.5}{\log p})\right) \\
&> 2 \times \left(1 - \frac{1}{\log 10^8} \times (1 + \frac{1.5}{\log 10^8})\right) \\
&> 1.88 \\
&> 1.51
\end{aligned}$$

using the Proposition 1.5 since we know that the expression

$$\frac{(p_i^{a_i} - 1) \times (p_i^{b_i} - 1)}{f(p_i^{b_i}) \times p_i^{a_i+b_i-1} \times (p_i - 1)^2}$$

tends to 0 as b tends to infinity for every odd prime p. Certainly, the fraction $\frac{f(\frac{n}{p})}{f(\frac{n}{N})}$ gets closer to 2 as long as we take n bigger and bigger. However,

$$1.51 < \frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^{k} \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.51$$

is a contradiction. By contraposition, the number N does not exist under the assumption of the Riemann Hypothesis. In this way, we prove there is not any odd perfect number at all. Indeed, according to the Proposition 1.8, the smallest counterexample N must comply that $N > 10^{1500}$ and therefore, we will always be capable of obtaining a large enough superabundant number n that is multiple of N. □

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