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# Homomorphisms and direct sum of uniserial modules

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**Abstract:** A right module $M$ over an associative ring $R$ with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules. Here we characterize the finitely generated submodule $N$ of a $QTAG$-module $M$ such that all homomorphisms or monomorphisms of the finitely generated submodule $N$ into the $QTAG$-module $M$, or all endomorphisms of the finitely generated submodule $N$, extends to an endomorphism of the $QTAG$-module $M$.

**Key words:** QTAG-modules, $Ulm$ sequences, valuation.

## 1. Introduction and Basic Setup

The 1950s, 1960s and 1970s were a period of tremendous progress in the study of abelian groups. Many people interested in module theory have worked on generalizing the theory of abelian groups. In fact, the theory of modules is highly motivated by abelian groups. It is obvious that virtually anything in the theory of abelian groups were generalized for modules over dedekind rings, prime rings, noetherian/artinian rings and hnp-rings etc.

In the 1970s, a particularly interesting approach to this search for generalizations was developed. Over an arbitrary associative ring with unity, a class of modules was defined by Singh [14] using two conditions relating to uniserial modules.

($I$) Every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules.

($II$) Given any two uniserial submodules $U$ and $V$ of a homomorphic image of $M$, for any submodule $W$ of $U$, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

It was shown that the theory of these modules very closely paralleled the theory of torsion abelian groups; for this reason they were referred to as $TAG$-modules. Later on, it was shown that, for almost all applications, one of these conditions was not needed; ignoring this nearly superfluous condition, the slightly more general concept of a $QTAG$-module was initiated by the same author in [15]. These definitions allowed for the translation of almost all of the theory of torsion abelian groups into parallel results about $QTAG$-modules. Facchini and Salce [3] also have considered the problem of detecting finite direct sums of uniserial modules. Bican and Torrecillas [1] worked on torsion free modules but the structures of torsion free $QTAG$-modules and torsion

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$QTAG$-modules are entirely different. The present paper is a natural extension of work already done in this field and certainly contributes to the overall knowledge of the structure of $QTAG$-modules.

Some basic definitions used in this paper have also been used in the works of one of the co-author and these are presented as quotations and referred appropriately here.

"All the rings $R$ considered here are associative with unity $(1 \neq 0)$, and the modules $M$ are unital right modules satisfying $(I)$ and $(II)$. A module $M$ over a ring $R$ is called uniserial if it has a unique decomposition series of finite length. A module $M$ is called uniform if intersection of any two of its non-zero submodules is non-zero. An element $x$ in $M$ is called uniform if $xR$ is a non-zero uniform (hence uniserial) module. For any module $M$ with a unique decomposition series, $d(M)$ denotes its decomposition length. For any uniform element $x$ of $M$, its exponent $e(x)$ is defined to be equal to the decomposition length $d(xR)$. For any $0 \neq x \in M$, $H_M(x)$ (the height of $x$ in $M$) is defined by $H_M(x) = \sup\{d(yR/xR) : y \in M,\ x \in yR \text{ and } y \text{ uniform}\}$. For $k \geq 0$, $H_k(M)$ is defined as $H_k(M) = \{x \in M |\ H_M(x) \geq k\}$. The module $M$ is $h$-divisible [11] if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$, where $M^1$ is the submodule of $M$ generated by uniform elements of $M$ of infinite height. The module $M$ is $h$-reduced if it does not contain any $h$-divisible submodule. In other words, it is free from the elements of infinite height. The module $M$ is called bounded [14], if there exists an integer $k$ such that $H_M(x) \leq k$, for all uniform elements $x \in M$."

"A submodule $N$ of $M$ is $h$-pure [10] in $M$ if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule $B$ of $M$ is called a basic submodule [11] of $M$, if $B$ is an $h$-pure submodule of $M$, $B$ is a direct sum of uniserial modules and $M/B$ is $h$-divisible. Mimicking [12], for any uniform element $x \in M$, there exist uniform elements $x_1, x_2, \ldots$ such that $xR \supseteq x_1R \supseteq x_2R \supseteq \ldots$ and $d(x_iR/x_{i+1}R) = 1$. For any $0 \neq x \in M$, the $Ulm$-sequence of $x$ in $M$ is defined as $U_M(x) = (H_M(x),\ H_M(x_1),\ H_M(x_2), \ldots)$. These sequences are partially ordered because $U_M(x) \leq U_M(y)$ if $H_M(x_i) \leq H_M(y_i)$ for every $i$."

It is worthwhile noticing that several results which hold for $TAG$-modules are also valid for $QTAG$-modules [13]. Many results, stated in the present paper, are clearly generalizations of [2]. For the better understanding of the mentioned topic here one must go through the papers [3, 8, 9]. Our notations and terminology are standard and follow essentially those from [4, 5]. As usual, for any uniform element $x \in M$, $N_{xR}$ denotes the submodule of $M$.

## 2. Main Results

We start with the following.

**Definition 2.1.** Let $e(M)$ denote an exponent of the $QTAG$-module $M$ such that for any $k \geq 0$, $H_k(M) = 0$, and $H_M(x) \leq k$ for all uniform elements $x$ of $M$. If $M$ is not bounded then $e(M) = \infty$.

Now, we are ready to formulate the following lemma.

**Lemma 2.1.** *Let $N_{xR}$ be the submodule of the $QTAG$-module $M$ such that the extension of all monomorphisms of $N_{xR} \to M$ is an endomorphism of $M$.*

*(i) If $M$ has unbounded basic submodule then $N_{xR} \cap M^1 = \{0\}$.*

*(ii) If $M = S \oplus T$ where $S$ is bounded and $T$ is $h$-divisible then either*

(*a*) $e(x) \leq e(S)$ *and* $N_{xR} \cap T = 0$ *or*
(*b*) $e(x) > e(S)$ *and* $x = s + t$ *where* $s \in S, t \in T$ *such that* $e(S) = e(s)$ *and* $e(t) > e(S)$*.*

*Proof.* (*i*) If $M$ has unbounded basic submodule and $x, y \in M$ then $N_{yR}$ is a direct sum uniserial module such that $e(N_{yR}) \geq e(x)$. Thus, it is a routine matter to see that there exists a monomorphism $N_{xR} \to N_{yR}$ which is the extension of an endomorphism of $M$. Consequently, $N_{xR} \cap M^1 = \{0\}$ and we are done.
(*ii*) Assume that $M = S \oplus T$ with $e(S) = n$ and $T$ is $h$-divisible.
(*a*) If $e(x) \leq n$ then we are done.
(*b*) If $e(x) > n$, and let $x = s + t$ such that $s \in S$ and $t \in T$. Then $e(t) > n \geq e(s)$. Let $s_1 \in S$ be any element such that $e(s_1) = n$. Then $e(s_1 + t) = e(t) = e(x)$, and there exists an endomorphism $\phi : M \to M$ such that $\phi(x) = s_1 + t$. Since $H_M(s_1) = 0$, we observe that $H_M(x) = 0$, whence $e(s) = n$. □

We continue with other statement, namely

**Lemma 2.2.** *If $M$ and $x$ are as in Lemma 2.1(ii)(b) then $N_{xR}$ is the submodule of $M$ such that the extension of all homomorphisms of $N_{xR} \to M$ is an endomorphism of $M$.*

*Proof.* Since $e(s) = e(S)$, $N_{sR}$ is a direct summand of $S$. Let $\phi : N_{xR} \to M$ be a homomorphism such that $\phi(x) = y + t_1$, where $y \in S$ and $t_1 \in T$. Since $N_{sR}$ is a summand of $S$ and $e(s) \geq e(y)$, then there exists a map $\phi_1 : S \to S$ defined by $\phi_1(s) = y$ extends to a homomorphism $\varphi_1 : S \to M$. Since $T$ is injective and $e(t) = e(x) \geq e(t_1)$, then there exists a map $\phi_2 : T \to T$ defined by $\phi_2(t) = t_1$ extends to a homomorphism $\varphi_2 : T \to M$. Hence $\psi = \varphi_1 + \varphi_2$ is an extension of an endomorphism $\phi$ from $M$ to $M$. □

Now we investigate the criteria for uniserial modules in which the extension of all homomorphisms of $N_{xR} \to M$ is an endomorphism of $M$. We consider therefore uniserial modules $N_{xR}$ containing no elements of infinite height in $M$.

**Definition 2.2.** [2]. For any uniform element $x$ of $M$ and $n \in \mathbb{N}$, an *Ulm* sequence of length $n$ is a strictly increasing infinite sequence $U_M(x) = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \infty, \ldots)$, with each $\alpha_i$ an ordinal, under the conventions that each ordinal $\alpha_i < \infty$, $\infty < \infty$ and the constant sequence $(\infty)$ is the unique *Ulm* sequence of length zero.

**Remark 2.1.** *[2]. The set of Ulm sequences is well-ordered pointwise with maximum $(\infty)$, no minimum but infimum $\mathbb{N} = (0, 1, \ldots, n, n+1, \ldots)$. This means in particular that if $x, y \in M$ and $U_M(x) \leq U_M(y)$, where $U_M(x)$ has length $n$ and $U_M(y)$ has length $m$, then $n \geq m$.*

**Definition 2.3.** [2]. For any uniform element $x \in M$, an *Ulm* sequence $U_M(x)$ is called finite if all its non-infinity entries are finite. In particular, $(\infty)$ is a finite *Ulm* sequence.

An *Ulm* sequence $U_M(x)$ has a gap before $k$ if $\alpha_k > \alpha_{k-1} + 1$. The gap before $n$, where $n$ is the length of $U_M(x)$, is called the trivial gap.

Regarding Lemma 2.1, the following immediately follows.

**Corollary 2.1.** *Let $M$ be a QTAG-module with $x \in M$, and $N_{xR}$ is the submodule of $M$ such that the extension of all monomorphisms of $N_{xR} \to M$ is an endomorphism of $M$, then $U_M(x)$ is finite.*

*Proof.* Let $x \in M$ such that $e(x) = n$. Then the *Ulm* sequence of $x$ in $M$ of length $n$ is given by $U_M(x) = (H_M(x), H_M(x_1) \ldots, H_M(x_{n-1}), \infty \ldots)$.
$\Rightarrow U_M(x)$ is finite if and only if $N_{xR} \cap M^1 = \{0\}$.
$\Rightarrow$ for every $k$, $H_M(x') = \infty$ where $d(xR/x'R) = k$ if and only if $H_k(x) \in M'$, the $h$-divisible part of $M$.
$\Rightarrow U_M(x) = (0, 1, \ldots, n-1, \infty \ldots)$ if and only if $N_{xR}$ is a summand of $M$ such that $e(N_{xR}) = n$, and for $x, y \in M$, $U_M(x+y) \geq \min\{U_M(x), U_M(y)\}$.

Lastly, if $H_M(x) = 0$ and $U_M(x)$ has the first non-trivial gap before $k$, then $M$ has a direct summand of exponent $k$. The proof is over. □

Now we are able to characterize uniserial modules with no elements of infinite height.

**Theorem 2.1.** *Let $M$ be a QTAG-module and $x \in M$ be an element of exponent $n$ such that $N_{xR} \cap M^1 = \{0\}$. The following are equivalent.*
*(i) $N_{xR}$ is the submodule of $M$ such that the extension of all homomorphisms of $N_{xR} \to M$ is an endomorphism of $M$;*
*(ii) $N_{xR}$ is the submodule of $M$ such that the extension of all monomorphisms of $N_{xR} \to M$ is an endomorphism of $M$;*
*(iii) $U_M(x)$ has at most one non-trivial gap and if a gap occurs before the index $m \geq 0$ and $H_M(x') = m + \ell$, $d(xR/x'R) = m$ then $M$ has no uniserial summand of exponents between $m+1$ and $n + \ell - 1$.*

*Proof.* $(i) \Rightarrow (ii)$. This is obvious.
$(ii) \Rightarrow (iii)$. Let $x \in M$ and $N_{xR}$ is the submodule of $M$ such that the extension of all monomorphisms of $N_{xR} \to M$ is an endomorphism of $M$. Now the *Ulm* sequence $U_M(x)$ has a trivial gaps, therefore the submodule $N_{xR}$ is a direct summand of $M$. Henceforth, we get that $U_M(x)$ has at least one non-trivial gap.

Let us consider $U_M(x) = (\alpha_0, \ldots, \alpha_{n-1}, \infty, \ldots)$ has at least two non-trivial gaps. By hypothesis, all heights $\alpha_i$ are integers, we write down $X = N_{x_1R} \oplus \cdots \oplus N_{x_rR}$ is a direct summand of $M$. Thereby, because by assumption a strictly increasing chain of positive integers $0 < m_1 < m_2 < \cdots < m_r$ such that
$(i)$ $r \geq 3$,
$(ii)$ $e(x_1) < e(x_2) < \cdots < e(x_r) = m_r + n$, and
$(iii)$ $x = H_{m_1}(x_1) + H_{m_2}(x_2) + \cdots + H_{m_r}(x_r)$.

Let $y \in M$ such that $e(y) = H_{m_1}(x_1) + H_{m_2-1}(x_2) + H_{m_3}(x_3) + \cdots + H_{m_r}(x_r) = n$. But

$$H_M(y') = e(x_1) - m_1 + m_2 - 1 < e(x_1) - m_1 + m_2 = H_M(x'),$$

where $d(xR/x'R) = d(yR/y'R) = e(x_1) - m_1$. Hence $U_M(y) \ngeq U_M(x)$, which is a contradiction. Therefore $U_M(x)$ has exactly one non-trivial gap. Choose $m$ be the index such that $U_M(x)$ has a gap before $m$. Then $H_M(x') = m + \ell$ where $d(xR/x'R) = m$ and $\ell > 0$.

Suppose that $N_{aR}$ is a direct summand of $M$ such that $e(N_{aR}) = n \leq k \leq n + \ell - 1$, for some $a \in M$. If $b = H_{k-n}(a)$ then $H_m(b) \neq 0$ since $n > m$. Moreover,

$$H_M(b') = k - n + m \leq n + \ell - 1 - n + m = m + \ell - 1 < H_M(x'),$$

where $d(xR/x'R) = d(bR/b'R) = m$. Therefore $U_M(b) \ngeq U_M(x)$, but $e(b) = e(x)$, a contradiction.

Now suppose that $N_{aR}$ is a direct summand of $M$ such that $e(N_{aR}) = m+1 \leq k \leq n-1$, for some $a \in M$. We observe that $b = x+a$ such that $e(b) = n$. But $H_M(x') > m = H_M(a')$, where $d(xR/x'R) = d(aR/a'R) = m$, hence $H_M(b') = H_M(x'+a') = m < H_M(x')$, where $d(xR/x'R) = d(aR/a'R) = d(bR/b'R) = m$. We therefore obtain that $U_M(b) \ngeq U_M(x)$, a contradiction.

$(iii) \Rightarrow (i)$. Let $x$ be as in $(iii)$. Now the *Ulm* sequence $U_M(x)$ has a trivial gaps then $N_{xR}$ is a direct summand of $M$.

Suppose that $U_M(x)$ has a gap before the index $m$, and let $y$ be a uniform element of $M$ such that $e(y) = k \leq e(x)$. We claim that $U_M(x) \leq U_M(y)$.

Let us assume that $U_M(y) = (\beta_0, \dots, \beta_{k-1}, \infty, \dots)$. We have two cases to consider.

**Case(i):** $\beta_{k-1}$ is finite. In order to demonstrate $U_M(x) \leq U_M(y)$, we have to prove that $H_M(x') \leq H_M(y')$ where $d(xR/x'R) = d(yR/y'R) = m$, since $U_M(x)$ has only one gap and this occurs before $m$.

Suppose that $H_M(x') > H_M(y')$ where $d(xR/x'R) = d(yR/y'R) = m$. If $n_1, \dots, n_r$ are the positive indexes before the gaps occur and we write $\beta_{n_i} = n_i + m_{i+1}$ and $m_1 = \beta_0$ then we get the uniserial direct summands with $d(xR) = n_i + m_i$, where $i = 1, \dots, r$.

If $m = 0$ then we have no uniserial direct summands with $d(xR) = 1, \dots, n+\ell-1$. Then every element $y$ with $e(y) \leq n$ must have height $\geq \ell$.

If $m > 0$, $n_j \leq m$ be the largest index $n_i \leq m$. Then $H_M(y') = m+m_{j+1} < m+\ell$ such that $d(yR/y'R) = m$ and $M$ has a direct summand of exponent $n_{j+1}+m_{j+1}$. Since $m < n_{j+1} \leq n$, then $M$ has a uniserial direct summand with $d(xR) = k$ and $m < k < n+\ell$, which is a contradiction.

**Case(ii):** $\beta_{m-1}$ is infinite. Let $c = H_M(x')$ such that $d(xR/x'R) = n-1$. If $B = \bigoplus_{i>0} B_i$ is a basic submodule of $M$. Let $M = B_1 \oplus \cdots \oplus B_c \oplus H_c(M)$ be the direct decomposition and set $y = y_1 + \cdots + y_c + y_1$ where $y_i \in B_i$ for all $i = 1, \dots, c$ and $y_1 \in H_c(M)$. Thus $U_M(x) \leq U_M(y_1)$. But $y - y_1$ satisfies Case (i) and $e(y-y_1) < e(x)$. Therefore $U_M(x) \leq \min\{U_M(y-y_1), U_M(y_1)\}$, and it follows that $U_M(x) \leq U_M(y)$. □

Recall from [6] that if $M$ is a $QTAG$-module and $N \subseteq M$, the valuation of $N$ induced by height in $M$ is defined by $v(x) = H_M(x)$, the height of $x$ in $M$, for all $x \in N$ and $N = P \oplus Q$ is a valuated direct sum if $v(p+q) = \min\{v(p), v(q)\}$ for all $p \in P$ and $q \in Q$.

**Lemma 2.3.** *Let $M$ be a $QTAG$-module, and let $P, Q$ be the submodules of $M$ such that the extension of all homomorphisms of $P$ and $Q$ into $M$ is an endomorphism of $M$ with $P \cap Q = 0$. If $P \oplus Q$ is the submodule of $M$ such that the extension of all endomorphisms of $N$ is an endomorphism of $M$, then $P \oplus Q$ is a valuated direct sum.*

*Proof.* Assume that $P \oplus Q$ is not valuated direct sum. Then there exists a pair $(p,q) \in P \oplus Q$ such that $H_M(p,q) > \min\{H_M(p), H_M(q)\}$. For example, say $H_M(p,q) > H_M(p)$. Let $\phi \in \mathrm{End}(P \oplus Q)$ be the natural projection onto $P$. Then $H_M(\phi(p,q)) = H_M(p) < H_M(p,q)$ and hence $\phi$ cannot be extended to an endomorphism of $M$, which is a contradiction. We are done. □

**Lemma 2.4.** *Let $L$ be an $h$-pure submodule of the $QTAG$-module $M$ such that $L$ is a direct sum of uniserial modules and $M/L$ is $h$-divisible. Let $\phi : N \to M$ be a homomorphism for some finitely generated submodule $N$ of $L$. Then the following are equivalent.*

*(i)* $\phi$ *can be extended to an endomorphism of* $M$ *;*

*(ii)* $\phi$ *can be extended to a homomorphism* $\phi_1 : L \to M$ *;*

*(iii)* $\phi$ *can be extended to a homomorphism* $\phi_1 : L \to M$ *such that* $\phi_1(L)$ *is bounded.*

*Proof.* $(i) \Rightarrow (ii)$ is obvious.

$(ii) \Rightarrow (iii)$. Let us first extend that $\phi$ to a homomorphism $\phi_2 : L \to M$. Since $N$ is finitely generated and $L$ is a direct sum of uniserial modules, there is a finite direct summand $S$ of $L$ such that $N \subseteq S$. For every complement $T$ of $S$ in $L$, $S \oplus T = L$, we have $\phi_1 : L \to M$ such that $\phi_1(a+b) = \phi_2(a)$ for all $a \in S$ and $b \in T$.

$(iii) \Rightarrow (i)$. Let $\phi_1$ be as in $(iii)$. Then the extension of $\phi_1$ is an endomorphism of $M$ such that $\phi_1(L)$ is bounded by $t$ for some $t \geq 0$. By the $h$-divisiblity of $M/L$, we have $M = L + H_t(M)$. Thus, for every $a \in M$ there are $b \in L$ and $c \in M$ such that $a = b + H_t(M)$. Since $L$ is $h$-pure in $M$, there exists a map $\phi_2 : M \to M$ such that $\phi_2(a) = \phi_1(b)$. Henceforth, it follows that $\phi_2$ is an endomorphism of $M$ which extends $\phi$. □

We are now able to characterize finitely generated submodules of $M$.

**Theorem 2.2.** *Let* $M$ *be a QTAG-module and* $N = \bigoplus_{i=1}^{n} N_i$ *a finitely generated submodule such that* $N \cap H_\omega(M) = 0$ *and each* $N_i = N_{x_iR}$ *is a uniserial module of exponent* $e_i$*. The following are equivalent.*

*(i) All the endomorphisms of* $N$ *can be extended to endomorphisms of* $M$*.*

*(ii)(a) If* $e_j \leq e_i$ *then* $U_M(x_i) \leq U_M(x_j) \leq U_M(x'_i)$ *such that* $d(x_iR/x'_iR) = e_i - e_j$*;*

*(ii)(b)* $N = \bigoplus_{i=1}^{n} N_i$ *is a valuated direct sum of uniserial modules.*

*Proof.* $(i) \Rightarrow (ii)$. As for the part $(ii)(a)$, let $i$, $j$ be two indices such that $e_j \leq e_i$. Then there are homomorphisms $\phi_1 : N_i \to N_j$ such that $\phi_1(x_i) = x_j$ and $\phi_2 : N_j \to N_i$ such that $\phi_2(x_j) = x'_i$ where $d(x_iR/x'_iR) = e_i - e_j$. Since these homomorphisms are the extensions of the endomorphisms of $N$, then so is endomorphisms of $M$. Moreover, $U_M(x_i) \leq U_M(x_j) \leq U_M(x'_i)$ such that $d(x_iR/x'_iR) = e_i - e_j$. Thus, by what we have just seen above, in view of the fact that endomorphisms do not decrease heights.

The part $(ii)(b)$ follows directly from Lemma 2.3.

$(ii) \Rightarrow (i)$. Let $\phi_1 : N_j \to N$ be a homomorphism, where $j \in \{1, \ldots, n\}$. Since $U_M(kx) = U_M(x)$ for all integers $k$, it is enough to prove that $U_M(x'_j) \leq U_M(H_k(\phi_1(x_j)))$ such that $d(x_jR/x'_jR) = t$ and $0 \leq t < e_j$. For every uniform element $y \in M$, we have $U_M(y')$ such that $d(yR/y'R) = t$ for some $t \geq 0$. This can be obtained by deleting the first $t$ components of $U_M(y)$, it is enough to prove $U_M(x_j) \leq U_M(\phi_1(x_j))$.

Let $\phi_1(x_j) = \sum_{i=1}^{n} k_i x_i$. Note that if $e_j < e_i$ then $e_i - e_j$ divides $k_i$. Then

$$\phi_1(x_j) = (\sum_{e_j < e_i} n_i x'_i) + (\sum_{e_i \leq e_j} k_i x_i),$$

where $d(x_iR/x'_iR) = e_i - e_j$, and hence

$$U_M(\phi_1(x_j)) = \min\{U_M(n_i x'_i)) : e_j < e_i\} \cup \{U_M(k_i x_i) : e_i \leq e_j\} \geq U_M(x_j),$$

such that $d(x_iR/x'_iR) = e_i - e_j$ and the proof is complete. □

The following lemma is of some interest.

**Lemma 2.5.** *Let $N$ be a submodule of the $QTAG$-module $M$ such that the extension of all monomorphisms of $N \to M$ is an endomorphisms of $M$. If $V$ is a uniserial direct summand of $N$ then all monomorphisms of $V$ into $M$ can be extended to endomorphisms of $M$.*

*Proof.* Let $W$ be a complement of $V$ in $N$, so $N = V \oplus W$, and $\phi : V \to M$ be a monomorphism.

If $\phi(V) \cap W = 0$, then the homomorphism $\varphi : V \oplus W \to M$ such that $\varphi(a, b) = \phi(a) + b$ is a monomorphism, hence it can be extended to an endomorphism $\psi : M \to M$. It easy to see that $\psi$ also extends $\phi$.

If $\phi(V) \cap W \neq 0$, we first observe that $Soc(\phi(V)) \subseteq W$ since $\phi(V)$ is a uniserial module. Let $\phi_1 : V \to M$ be the homomorphism defined by $\phi_1(a) = \phi(a) - a$. Suppose that $\phi_1$ is not monomorphism. Then there exists a non-zero element $a \in V$ such that $\phi_1(a) = 0$. Then $\phi(a) = a \in V$, hence $\phi(V) \cap V \neq 0$. It follows that $Soc(\phi(V)) \subseteq V$, and this contradicts $V \cap W = 0$. Hence $\phi_1$ is a monomorphism.

Now suppose that $\phi_1(V) \cap W \neq 0$. Then there exists $a \in V$ such that $0 \neq \phi(a) - a \in W$. If $e(a) = m$ then $H_{m-1}(\phi(a) - a) \in Soc(W)$. But $H_{m-1}(\phi(a)) \in Soc(\phi(W)) \subseteq Soc(W)$, hence $H_{m-1}(a) \in Soc(W)$, a contradiction. Then $\phi_1(V) \cap W = 0$ and there exists an endomorphism $\varphi : M \to M$ which extends $\phi$. Then for every $a \in V$ we have $\phi(a) = \varphi(a) + a$, hence $\varphi + I_M$ extends $\phi$. □

Let $M$ be a $QTAG$-module and $N \subseteq M$. If $U$ is an $Ulm$ sequence, we denote

$$N_U = \{x \in N : U_M(x) \geq U\}, \ N^U = \{x \in N : U_M(x) > U\}$$

and

$$N(U) = \frac{N_U + H_1(N)}{N^U + H_1(N)}$$

Recall from [7] that a $*$-basis for $M$ is constructed in the following way.

Let $M$ be a $QTAG$-module. For each ordinal $\sigma$, let $B_\sigma$ be a set of representatives of the nonzero cosets of $H_\sigma(M)$ mod $H_{\sigma+1}(M)$; in other words, $B_\sigma$ contains exactly one element from each of the nonzero cosets of $H_{\sigma+1}(M)$ in $H_\sigma(M)$. If each element $x$ in $M$ can be expressed as

$$x = b_1 + b_2 + \cdots + b_n \tag{1}$$

where $b_i \in B_{\sigma(i)}$ with $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$, then $B = \bigcup B_\sigma$ is called a $*$-basis of $M$. The expression (1) is called a representation of $x$ with respect to the $*$-basis $B$.

With the help of above discussion, we are able to infer the following.

**Lemma 2.6.** *Let $M$ be a $QTAG$-module and $N = \bigoplus_{i=1}^{n} N_i$ a finitely generated submodule such that all $N_i = N_{x_iR}$ are uniserial modules such that*
*(i) $e(x_1) \leq e(x_2) \leq \cdots \leq e(x_n)$,*
*(ii) for all $k \in \{1, \ldots, n\}$ and for all $y \in \bigoplus_{i=1}^{k} N_i$ we have $U_M(x_k) \leq U_M(x)$, and*
*(iii) if $e(x_i) = e(x_j)$ then $U_M(x_i) = U_M(x_j)$.*
*Then the direct sum $\bigoplus_{i=1}^{n} N_i$ is a valuated direct sum of uniserial modules.*

*Proof.* The proof is by induction. For $n = 1$ the property is obvious. Suppose that $(ii)$ is valid for all $k < n$. Then $\bigoplus_{i=1}^{n-1} N_i$ is a valuated direct sum of uniserial modules. Let $t$ be the minimal index such that $e(N_n) = e(N_t)$. We observe that the sequence $U_M(x_i)$, $i = 1, \ldots, n$ is a decreasing sequence such that $U_M(x_i) = U_M(x_j)$ if and only if $e(N_i) = e(N_j)$. Moreover, it follows by $(ii)$ that $U_M(x_n) \leq U_M(z)$ for all $z \in N$. Now, for every $U$ we fix a basis in $N(U)$, and we choose one representative whose $Ulm$ sequence is $U$ for each element in this basis; the union of all these representatives is a $*$-basis for $N$. Moreover, in this hypothesis every $*$-basis is linearly independent and it generates $N$. Therefore, it is enough to prove that $\{x_i : i = 1, \ldots, n-1\}$ is a $*$-basis for $N$.

Since $L = \bigoplus_{i=1}^{n-1} N_i$ is a direct sum of uniserial valuated modules, it follows that the set $\{x_i : i = 1, \ldots, n-1\}$ is a $*$-basis for $L$. Let $U'$ be the $Ulm$ sequence of $x_n$. If $U < U'$ then $N_U = N^U = N$, so $N(U) = 0$. If $U$ and $U'$ are not comparable then $N_U = N^U$ since $U'$ is minimal as $Ulm$ sequence of an element of $N$, so $N(U) = 0$. It is easy to see that $N_{U'} = N$, and $N^{U'} = (\oplus_{i<t} N_i) \oplus (\oplus_{i=t}^{n} H_1(N_i))$, so $x_t, \ldots, x_n$ represent a basis in $N(U')$.

If $U' < U$ then

$$N_U + H_1(N) = (\bigoplus_{U_M(x_i) \geq U} N_i) + H_1(N) = L_U \oplus H_1(N_n)$$

and

$$N'_U + H_1(N) = (\bigoplus_{U_M(x_i) \geq U} N_i) + H_1(N) = L_{U'} \oplus H_1(N_n)$$

Therefore every set in $L = \bigoplus_{i=1}^{n-1} N_i$ which represents a basis in $L(U)$ is also a representative set for a basis in $N(U)$. It follows that $\{x_1, \ldots, x_n\}$ is a $*$-basis, and the proof is complete. □

We end this paper by the following lemma.

**Lemma 2.7.** *Let $M$ be a QTAG-module and $a, b \in M$. If $U_M(a + b) = U_M(a)$ then $U_M(a) \leq U_M(b)$.*

*Proof.* Suppose that $U_M(a) \nleq U_M(b)$. Then there exists a positive integer $t$ such that $H_M(b') < H_M(a')$ where $d(aR/a'R) = d(bR/b'R) = t$. It follows that $H_M(c') = H_M(b') \neq H(a')$ where $d(aR/a'R) = d(bR/b'R) = d(cR/c'R) = t$ and $c = a + b$. This contradicts our hypothesis, and we are done. □


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