The Seybold Report

ISSN 1533-9211

NOTE ON DIFFERENT TYPES OF PRIMARY IDEALS IN NEAR-RINGS

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Highlights

• This paper focuses on ideal theory of near-rings.

• Classical algebraic substructures of near-rings are introduced in this study.

• Highly useful results are obtained about the characterizations of near-rings.

DOI: 10.5281/zenodo.6553311

ABSTRACT

We introduce the notions of 0-(1-2)-primary and almost primary ideals which are the generalizations of 0-(1-2)-prime ideals in near-rings. Moreover, some characterizations are also obtained and are demonstrated with suitable examples.

Key words: Near-rings, 0-primary ideals, 1-primary ideals 2- primary ideals, almost primary ideals

1 Introduction and Preliminaries

An algebraic system N with two binary operation "$+$" and "$\cdot$" is said to be a (right) near-ring, if it is a group (not necessarily abelian) under addition, semigroup under multiplication and N satisfies (right) distributive law i.e., for any $x, y, z \in N$; $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ [7]. Similarly, a left near-ring can be defined by replacing the right distributive law with left distributive law. We call a (left) near-ring N is a zero-symmetric, if $0 \cdot n = 0$ for all $n \in N$. Similarly, an element x of (left) near-ring N is distributive, if $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in N$, and if its all the elements of N satisfies right distributive property we say that N is distributive near-ring. The near-ring N is called a distributively generated (d.g), if it contains a multiplicative sub-semigroup of distributive elements which generates additive group $(N, +)$. Every distributively generated near-rings are zero-symmetric near-rings. We refer [7] for the fundamental concepts and notions for near-rings. Let N and N' be two near-rings and $_N\Gamma$ and $_N\Gamma'$ be two N-groups, then $\phi: N \to N'$ satisfying $\phi(n_1 + n_2) = \phi\ (n_1) + \phi(n_2)$; $\phi(n_1 n_2) =$

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$\phi(n_1)\ \phi(n_2)$; and a map $h\colon {}_N\Gamma \to {}_N\Gamma'$ satisfying $h(\gamma+\delta) = h(\gamma) + h(\delta)$; $h(n\gamma) = nh\,(\gamma)$ are said to be near-ring homomorphism and N-homomorphism, respectively. We call a subset I of a near-ring N is an ideal if: (i) $(I, +)$ is a normal subgroup of a $(N, +)$, (ii) For each $n \in N$, $i \in I$, $ni \in I$ i.e., $NI \subseteq I$, and (iii) $(n_1 + i)n_2 - n_1 n_2 \in I$ for each $n_1, n_2 \in N$ and $i \in I$. But A. Frohlich [4] showed that for $d.g$- near-rings the third condition is equivalent to $in \in I$ i.e., $IN \subseteq I$. A proper ideal P of a near-ring N is said to be a prime ideal if for ideals A and B of N, $AB \subseteq P \Rightarrow A \subseteq P$ or $B \subseteq P$. Different types of prime ideals have been introduced in the literature (see [5], [2]&[8]). Almost prime ideals in near-rings have been endorsed by B. Elavarasan (see [3]). A proper ideal P of a near-ring N is said to be an almost prime if for any ideals A and B of N such that $AB \subseteq P$ and $AB \nsubseteq P^2$, we have $A \subseteq P$ or $B \subseteq P$ [3, page 47]. The author established few relationships between almost prime and prime ideals as well [3]. Notions of 0-(1-2)-prime ideals have been introduced in ([2], [5] & [8]). Following [5], an ideal P of near-ring is said to be a 0-prime ideal, if for any two ideals $I_1, I_2 \subseteq N$ such that $I_1 I_2 \subseteq P$ implies $I_1 \subseteq P$ or $I_2 \subseteq P$ [5]. Subsequently, Ramakotaiah and Rao [8] introduced the concepts of 0-prime, 1-prime and 2- prime ideals of a near-rings. Furthermore, G. Birkenmeier et al. [2] discussed the connections between prime ideals and type one prime ideals in near-rings. Following [2], an ideal I is said to be a type-zero or simply a prime ideal if A and B are ideals of N, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. Further to this, an ideal I of a near-ring is of type-1(or 1-prime) if $x, y \in N$, $xNy \in I$ then $x \in I$ or $y \in I$. Similarly, an ideal P of a near-ring N is called 2-prime if for any two subgroup K_1, K_2 of $(N, +)$ such that $K_1 K_2 \subseteq P$ implies that $K_1 \subseteq P$ or $K_2 \subseteq P$. It is well-known that 2-prime $\Rightarrow$ 1-prime $\Rightarrow$ 0-prime, but the converse doesn't exist in any of the implication. Recently, P-ideals and their P-properties in near rings have been introduced in [1]. On the other hand,few concepts of nearrings have been shifted towards seminearrings in [6].

In this note, we introduce the notions of 0-(1-2)-primary ideals and almost primary ideals in a near-rings. We investigate that 0-prime ideal is always 0-primary but converse is not true. We also establish that 2-primary $\Rightarrow$ 1-primary $\Rightarrow$ 0-primary ideals but the converse does not hold true in any of implication. Furthermore, several characterizations are obtained and supported by suitable examples.

2 Primary ideals in near-rings

In this section, we introduce and discuss different types of primary ideals of near-rings. We also investigate some relationships among them.

Definition 1. A proper ideal P of N is called 0-primary if A, B are any two ideal of N such that $AB \subseteq P$ implies that $A \subseteq P$ or $B^n \subseteq P$ for some $n \in \mathbb{Z}^+$.

Example 1 Suppose that $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a right near-ring with addition and multiplication defined in the tables set 1.

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***Tables set* 1**

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Here $P = \{0, 2\}$, $I_1 = \{0, 1, 2, 3\}$ *and* $I_2 = \{0, 1\}$ *are ideals of* N. *Also* $I_1 I_2 = \{0\} \subseteq P$ *implies* $I_2^2 \subseteq P \Rightarrow P$ *is a* 0*-primary ideal of near-ring* N, *however* P *is not a* 0*-prime ideal.*

Proposition 1 Let I be an ideal of a zero-symmetric near-ring N. Then I is a 0-primary ideal if and only if every zero-divisor in N/I is a nilpotent.

Proof ⇒) Let I be a 0-primary ideal of a near-ring N and consider N/I is a non-trivial. Let $n + I \in N/I$ be a zero-divisor and $n_1 \in N/I$. Consider $n_1 n + I = (n_1 + I)(n + I) = 0 + I \Rightarrow n_1 n \in I$, $n_1 \notin I \Rightarrow n^k \in I$ for some $k \in \mathbb{Z}^+$. Hence $(n + I)^k = n^k + I = 0 + I \Rightarrow n + I$ is nilpotent. ⇐) Suppose N/I is non-trivial and every nonzero zero-divisor in N/I is nilpotent. Since $I \neq N$, let $n_1, n_2 \in N$ such that $n_1 . n_2 \in I$, then either $n_1 \in I$ or $n_1 \notin I$, suppose $n_1 \notin I$ then consider $(n_2 + I)(n_1 + I) = n_2 . n_1 + I = 0 + I \Rightarrow n_2 . n_1 = 0$, so $n_2 + I$ is a zero-divisor and by assumption $(n_2 + I)^k = n_2^k + I = 0 + I \Rightarrow n_2^k \in I$, hence I is a primary (0-primary) ideal.

Example 2 Let $N = \{0, a, b, c\}$ be a zero-symmetric near-ring under the addition and multiplication defined in the tables set 2.

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Tables set 2

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

Clearly, $P = \{0, a\}$ *is* 0*-primary ideal and the quotient* $N/P = \{0 + P, b + P\}$ *along with operations given in tables set* 3.

Tables set 3

+	$0 + P$	$b + P$
$0 + P$	$0 + P$	$b + P$
$b + P$	$b + P$	$0 + P$

$\cdot$	$0 + P$	$b + P$
$0 + P$	$0 + P$	$0 + P$
$b + P$	$0 + P$	$0 + P$

Here the zero divisors of N/P *are* $0 + P$ *and* $b + P$*, which are nilpotents.* Intersection of any two 0-primary ideals of a near-ring need not be a 0-primary ideal, we provide an example.

Example 3 Suppose $N = \{0, a, b, c\}$ be a commutative near-ring with addition and multiplication defined in the tables set 4.

Tables set 4

+	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	0	0
c	0	a	0	a

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Let us consider 0*-primary ideals* $P_1 = \{0, a\}$ *and* $P_2 = \{0, b\}$ *of a near-ring* N*. But* $P_1 \cap P_2 = \{0\}$ *is not a* 0*-primary ideal of* N*.*

Proposition 2 Every 0-prime ideal in a near-ring N is a 0-primary ideal of N.

Proof Let N be a near-ring and P be a 0-prime ideal then for all $x, y \in N$, $xy \in P \Rightarrow x \in P$ or $y \in P$, while considering $n = 1$ the result follows.

Remark 1 Every maximal ideal in near-ring is 0-prime and hence a 0-primary ideal $\Rightarrow$ a maximal ideal is a 0-primary.

Definition 2. An ideal I of a near-ring N is a semi-primary ideal if for any ideal J of N, $J^2 \subseteq I$ implies that $J \subseteq I$.

It is well known that in any near-ring the intersection of any prime ideals is a semi-prime ideal. We also know that a semi-prime ideal of a near-ring N is the intersection of minimal prime ideals of I in N such that the ideal I can be written as the intersection of all prime ideals containing I. However, the intersection of two primary ideals need not be a semi-primary ideal for instance see in example3 i.e., $I = \{0\}$ is the intersection of primary ideals $\{0, a\}$ and $\{0, b\}$ but is not a semi-primary i.e., $P_2^2 \subseteq \{0\} = I$, but $P_2 \nsubseteq I$.

Definition 3 An ideal P of near-ring N is said to be 1-primary ideal if for any right ideals A, B of N, $AB \subseteq P \Rightarrow A \subseteq P$ or $B^n \subseteq P$ where $n \in \mathbb{Z}^+$.

Example 4 Let $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a right near-ring with addition and multiplication defined in the tables set 5.

***Tables set* 5**

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	7	6	5	0	3	2	1
5	5	4	7	6	1	0	3	2
6	6	5	4	7	2	1	0	3
7	7	6	5	4	3	2	1	0

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	0	1	1	0
2	0	2	0	2	0	2	2	0
3	0	3	0	3	0	3	3	0
4	4	4	4	4	4	4	4	4
5	4	5	4	5	4	5	5	4
6	4	6	4	6	4	6	6	4
7	4	7	4	7	4	7	7	4

Let $A = \{0, 1, 2, 3\}$ *and* $B = \{0, 2\}$ *be the two right ideals of* N. *Let* $P = \{0, 4\}$ *be an ideal of* N*then the product* $AB = \{0\} \subseteq P$ *implies* $B^2 \subseteq P$. *Hence* P *is a* 1*-primary ideal of near-ring* N.

Definition 4. An ideal P of near-ring N is called 2-primary ideal if A, B are any two N-subgroups such that $AB \subseteq P$ implies that $A \subseteq P$ or $B^n \subseteq P$ for some $n \in \mathbb{Z}^+$.

Proposition 3 Let N be a near-ring. Then the following statements are equivalent.

(i) P is a 2-primary ideal of N.

(ii) If A is an N-subgroup and B is an ideal of N then $AB \subseteq P$ implies $A \subseteq P$ or $B^k \subseteq P$ where $n \in \mathbb{Z}^+$.

Proof. $(i) \Rightarrow (ii)$ If P is 2-primary ideal and B is an N-subgroup then (ii) is straightaway.

$(ii) \Rightarrow (i)$ Let A and B be two N-subgroups of N such that $AB \subseteq P$. Let $A \nsubseteq P$ and assume $B^k \subseteq (P: A) = \{n \in N: An \subseteq P\} = S$. Since S is an ideal of N, we have if $r \in S$ and $n, n_1 \in N$ then for all $a \in A$, $a(-n + r + n) = -an + ar + an \in P$, as P is an ideal thus $a[(n + r)\, n_1 - nn_1] = (an + ar)n_1 - ann_1 \in P$ which implies $Anr \subseteq Ar \subseteq P$. Hence $AS \subseteq P$ but we have assumed that $A \nsubseteq P$ which implies $S \subseteq P$ so $B^k \subseteq S \subseteq P$.

Proposition 4 Let P be a 2-primary ideal and $A_1, \ldots, A_k$ are N-subgroups. Then $A_1A_2 \ldots A_k \subseteq P$ implies $A_i^n \subseteq p$ for some $i \in \{1, \ldots, k\}$ and $n \in \mathbb{Z}^+$.

Proof. Let $A_1A_2 \ldots A_k \subseteq P$ and $A_1 \nsubseteq P$ such that $(A_2, \ldots, A_k)^n \subseteq (P: A_1)$. Thus $A_1.(P: A_1) \subseteq P$ which implies $(P: A_1) \subseteq P$ given that P is 2-primary ideal. By using proposition3 (ii), we get $(A_2, \ldots,$

$A_k)^n \subseteq P$. Similarly, we can repeat procedure for $A_2 \nsubseteq P$ and eventually $A_i^n \subseteq P$ for some $i \in \{1, .., k\}$.

Definition 5. An ideal P of a near-ring N is said to be 3-primary ideal if for $a, b \in N$ such that $aNb \subseteq P \Rightarrow a \subseteq P$ or $b^n \subseteq P$ for $n \in \mathbb{Z}^+$.

Example 5 Let $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a (right) near-ring under the addition and multiplication defined in tables set 6.

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Tables set 6

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	3	0	1	7	6	4	5
2	2	3	0	1	5	4	7	6
3	3	0	1	2	6	7	5	4
4	4	7	5	6	2	0	1	3
5	5	6	4	7	0	2	3	1
6	6	4	7	5	1	3	0	2
7	7	5	6	4	3	1	2	0

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	2	2	0	0
3	0	3	2	1	5	4	6	7
4	0	4	2	5	4	5	6	7
5	0	6	2	4	5	4	6	7
6	0	6	0	6	0	0	0	0
7	0	7	0	7	2	2	0	0

Let $P = \{0, 7\}$ *is a left ideal of* N *which is* 3*-primary ideal.*

Definition 6. A proper ideal P of near-ring is called (completely) c-primary ideal if for $a, b \in N$ suth that $ab \in P$ implies $a \in P$ or $b^n \in P$ for $n \in \mathbb{Z}^+$.

Example 6 Let $N = \{0, 1, 2, 3, 4, 5\}$ whose addition and multiplication are defined in the tables set 7.

Tables set 7

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	5	1	0	5	1
2	0	4	2	0	4	2
3	0	3	3	0	3	3
4	0	2	4	0	2	4
5	0	1	5	0	1	5

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Clearly, $P = \{0, 2, 4\}$ *is a c-primary ideal of* N *as* $1.3 = 0 \in P$ *implies* $3^2 = 0 \in P$ *and* $5.3 = 0 \Longrightarrow 3^2 = 0 \in P$.

Now we try to find the relationships among different types of primary ideals. Refer to example4, it is easy to verify that $P = \{0, 4\}$ is a 0-primary ideal. Thus, every 1-primary ideal $\Rightarrow$ 0-primary ideal. From example4, we have observed that an ideal $P = \{0, 4\}$ is 1-primary ideal but it is not 1-prime ideal. Similarly, from example5, we see that $P = \{0, 7\}$ is 3-primary ideal but it is not 3-prime ideal as $3N7 = \{0\} \subseteq P$ but 3 or 7 doesn't belong to P. Similarly, in example6, $P = \{0, 2, 4\}$ is a c-primary ideal of N which is not a c-prime ideal, however it is easy to verify that an ideal P is simultaneously 3-primary, 2-primary, 1-primary and 0-primary ideal . Hence we concluded that

$$c - \text{primaryideal} \Rightarrow 3 - \text{primary} \Rightarrow 2 - \text{primary} \Rightarrow 1 - \text{primary} \Rightarrow 0 - \text{primary}.$$

But the converse doesn't hold true in the above implication. After discussing different types of primary ideals in a near-ring now we introduce 0-(1-2)-primary near-ring.

Definition 7. A near-ring N is said to be a 0-(1-2)-primary near-ring if $\{0\}$ is 0-(1-2)-primary ideal of N.

We can say that a near-ring N is said to be a 0-primary (primary) near-ring, if for any two ideals A, B of N, $AB \subseteq \{0\}$ implies $A \subseteq \{0\}$ or $B^n \subseteq \{0\}$. In a similar manner, we can define 1-primary and 2-primary near-rings.

Example 7 Consider the left near-ring $N = \{0, 1, 2, 3\}$ defined in tables set 8.

***Tables set* 8**

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

·	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	1	1

Let $I = \{0, 1\}$ *and* $J = \{0, 2\}$ *be the two right ideals of* N *such that* $IJ = \{0\}$ *where* $J^2 = \{0\}$. Hence N is a 1-primary near-ring.

Proposition 5 Each 0-prime near-ring is a 0-primary near-ring.

Proof. Immediate.

Example 8 Every integral near-rings are prime near-rings and hence primary (0-primary) near-rings.

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We have introduced different primary ideals now we discuss the prime radical of these ideal. Following [7, definition 2.93], if I be an ideal of a near-ring N then the intersection of all prime ideals containing I is said to be the prime radical and is denoted by $\wp(I)$ i.e., $\wp(I) = \bigcap_{P \supseteq I} P$, where P is the prime. Hence, if $n \in \wp(I) \Rightarrow \exists\ k \in \mathbb{N}: n^k \in I$. In other words, an ideal I of a near-ring N is a semiprime ideal in N iff $\wp(I) = I$. Likewise rings, we will see that if I is the 0-(1-2)-primary ideal of a near-ring then its prime radical is the corresponding 0-(1-2)-prime ideal.

Example 9 Refer to example1, $\{0, 2\}$ is 0-primary ideal and $\sqrt{\{0,2\}} = \{0, 1, 2, 3\}$ which is 0-prime ideal of N.

It is easy to verify that if I is a 0-(1-2)-primary ideal then its prime radical is a 0-(1-2)-prime which we have already seen in example9. On the other hand, the converse doesn't hold true i.e., if the prime radical of an ideal I is 0-(1-2)-prime then it is not necessary that I is a 0-(1-2)-primary ideal.

Proposition 6 Let I be the both primary and semiprime ideal of a near-ring N. Then I is a prime ideal.

Proof. Immediate.

It is well known that an ideal I of a nearring N is said to be a completely prime (or c-prime) if $a, b \in N$, $ab \in I$ implies $a \in I$ or $b \in I$.

Definition 8. Let Q be a c-primary (completely primary) ideal of a nearring N such that $\sqrt{Q} = P$, where P is a c-prime ideal of N. Then we call Q a cP-primary ideal.

Definition 9. Let Q be a cP-primary ideal of a near-ring N. For $x \in N - Q$, we have $(Q: x) = \{a \in N: ax \in Q\}$.

Proposition 7 Let Q be a cP-primary ideal of a near-ring N and let $n \in N$. Then we have the following.

(i) If $n \in Q$, then $(Q: n) = N$.

(ii) If $n \notin Q$, then $(Q: n)$ is cP-primary ideal and $\sqrt{(Q: n)} = P$.

(iii) If $n \notin P$, then $(Q: n) = Q$.

Proof. Proof is omitted because it is similar to that of rings.

Remark 2 Let Q be a cP-primary ideal of a near-ring N such that$\sqrt{(Q: n)}$ is c-prime and $\sqrt{Q_i} = P_i$, then it must be contained in the set $\sqrt{(Q: n)}$ where $n \in N$.

We illustrate proposition7 and remark2 in the below example.

Example 10 Refer to example1, we have $Q = \{0, 2\}$ is 0-primary ideal. Then the only possible

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0-prime ideal of N containing Q is the ideal $P_1 = \{0, 1, 2, 3\}$ and hence a prime radical of Q implies Q is a P-primary. On the other hand, let $3 \in N$ and consider $(Q{:}\,3) = \{n \in N{:}\; 3n \in Q\} = \{0, 1, 2, 3\}$, which is clearly a 0-prime ideal of N. Hence $(Q{:}\,3)$ is an associated 0-prime ideal of a 0-primary ideal Q. Thus, every associated prime ideal must be contained in $\sqrt{(Q{:}\,x)}$.

3 Almost primary ideal in near-ring

In this section, we introduce and discuss some generalizations of primary ideals of near-rings. We initiate with the following definition.

Definition 10. A proper ideal P of a near-ring N is said to be an almost primary ideal if for ideals I and J of N, $IJ \subseteq P - P^2$, we have $I \subseteq P$ or $J^n \subseteq P$ for some $n \in \mathbb{Z}^+$.

Theorem 1 Let P be the proper ideal of a near-ring N. Then the followings are equivalent.

(1) P be an almost primary ideal of N.

(2) For any ideals I and J of N, $(IJ) \subseteq P$ such that $(IJ) \nsubseteq P^2 \Rightarrow I \nsubseteq P^2$ or $J^n \subseteq P$.

(3) For any $i, j \in N$, $i \notin P$ and $j^n \notin P$ for some $n \in \mathbb{Z}^+ \Rightarrow (i)(j) \subseteq P^2$ or $(i)(j) \nsubseteq P$.

Proof. (1) $\Leftrightarrow$ (2) is trivial.

(1) $\Leftrightarrow$ (3) Let P be an almost primary. Let $(i)(j) \subseteq P - P^2$ then $(i) \subseteq P$ or $(j)^n \subseteq P$ implies $i \in P$ or $j^n \in P$.

(2) $\Leftrightarrow$ (3) is immediate.

Example 11 Let $N = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a right near-ring whose tables are given below.

Tables set 9

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	0	2	2	2	0	0
3	0	3	2	1	5	4	6	7
4	0	4	2	5	4	5	6	7
5	0	6	2	4	5	4	6	7
6	0	6	0	6	0	0	0	0
7	0	7	0	7	2	2	0	0

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	7	6	4	5
2	2	3	0	1	5	4	7	6
3	3	0	1	2	6	7	5	4
4	4	7	5	6	2	0	1	3
5	5	6	4	7	0	2	3	1
6	6	4	7	5	1	3	0	2
7	7	5	6	4	3	1	2	0

An ideal $I = \{0, 2\}$ is an almost primary ideal. However, I is not a prime ideal and also I is neither a weakly prime nor an almost prime ideal. On the other hand, $J = \{0, 6\}$ is a weakly prime and almost prime but not an almost primary ideal in N.

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ISSN 1533-9211

Remark 3 Every weakly prime ideal is not an almost primary ideal.

Remark 4 Every 0-primary ideals and idempotent ideals in a near-ring are almost primary ideals. But the converse doesn't hold true in all cases.

Example 12 *An ideal* $P = \{0, 2\}$ *in example*1 is an idempotent ideal and also an almost primary ideal. Now we provide an example of an almost primary ideal which is not a prime ideal in a near-ring.

Example 13 *In example1 we have a prime ideal* $P = \{0, 1, 2, 3\}$ *of a near-ring* N. *Let* $S = \{4, 5\}$ *be a multiplicative closed set in* N. *We note that* $P \cap S = \emptyset$ *and also* $P_S \cap N = \{0, 1, 2, 3\} = P$ *and* $P_S \cap N = \{x \in N: xs \in P, for\ x \in S\}$. *On the other hand, if we take an almost primary ideal* $P = \{0, 2\}$ *of* N *and*$S = \{4, 5\}$ *be a multiplicative set of* N. *Then* $P = \{0, 2\}$ *is an almost primary ideal of N but not a prime ideal.*

Remark 5 If P is an almost primary ideal of N and S is a multiplicative set of N with $P \cap S = \phi$ then P_S is an almost primary ideal in N_S.

Proposition 8 Let P be a nonzero almost primary ideal of N and $(P^2 : P) \subseteq P$ then P is primary ideal.

Proof. Let P be an almost primary ideal and $(P^2 : P) \subseteq P$. Suppose that P is not a primary ideal of N, then there exist $x \notin P$ and $y^n \notin P$ such that $< x >< y^n > \subseteq P$. If $< x >< y^n > \nsubseteq P^2$ we are done and hence $< x >< y^n > \subseteq P - P^2$. Consider $< x > (< y^n > + P) \subseteq P$. If $< x > (< y > + P) \nsubseteq P^2$, then we have $x \in P$ or $y^n \in P$, a contradiction. Otherwise $< x > (< y^n > + P) \subseteq P^2$then $< x > P \subseteq P^2$ implies $x \in (P^2 : P) \subseteq P$.

Theorem 2 Let N_1 and N_2 be any two near-rings with identity and P be a proper ideal of N_1. Then P is an almost primary if and only if $(P \times N_2)$ is an almost primary ideal of $N_1 \times N_2$.

Proof. Suppose that P be an almost primary ideal of N_1 and let $(I_1 \times J_1)$ and $(I_2 \times J_2)$ be the ideals of $N_1 \times N_2$ such that $(I_1 \times J_1)(I_2 \times J_2)^n \subseteq (P \times N_2)$ and $(I_1 \times J_1)\ (I_2 \times J_2)^n \nsubseteq (P \times N_2)^2$. Then $(I_1 I_2^n \times J_1 J_2^n) \subseteq (P \times N_2)$ and $(I_1 I_2^n \times J_1 J_2^n) \nsubseteq (P^2 \times N_2)$, so $I_1 I_2^n \subseteq P$ and $I_1 I_2^n \nsubseteq P^2$ which implies $I_1 \subseteq P$ or $I_2^n \subseteq P$. Conversely, suppose that $(P \times N_2)$ is an almost primary ideal of $N_1 \times N_2$ and let A and B be ideals of N_1 such that $AB \subseteq P$ and $AB \nsubseteq P^2$. Then $(A \times N_2)(B \times N_2) \subseteq (P \times N_2)$ and $(A \times N_2)(B \times N_2) \nsubseteq (P \times N_2)^2$. By assumption, we have $(A \times N_2) \subseteq (P \times N_2)$ or $(B \times N_2) \subseteq (P \times N_2)$. So $A \subseteq P$ or $B^n \subseteq P$.

Proposition 9 If P be an almost primary ideal of a near-ring N such that an ideal $I \subseteq P$, then $\frac{P}{I}$ is an almost primary ideal of $\frac{N}{I}$.

Proof. Let $(a + I)(b + I) \in \frac{P}{I} - (\frac{P}{I})^2$ and $(a + I) \notin \frac{P}{I}$. Then $ab \in P$, $ab + I \notin (\frac{P}{I})^2$ and $a \notin P$. At the present, if $ab \in P^2$ then for some $n \in \mathbb{Z}^+$, we have $ab = \sum_{i=1}^{n} a_i, b_i$, where $a_i b_i \in P$ for all i, and we have $ab + I = \sum_{i=1}^{n} a_i b_i + I = \sum_{i=1}^{n} (a_i + I)(b_i + I) \in \frac{P}{I}\frac{P}{I} = (\frac{P}{I})^2$, a contradiction arises, so $ab \notin P^2$ and hence $b^m \in P$, for some $m \in \mathbb{Z}^+$, it implies $(b + I)^m \in$

$\frac{P}{I}$. Thus $\frac{P}{I}$ is an almost primary ideal of $\frac{R}{I}$.

Example 14 In example11, $P = \{0, 2\}$ is an almost primary ideal. The ideal which are subsets of P are only $\{0\}$ and P itself. According to proposition9 $\frac{P}{P} = \{0 + P\}$ and $\frac{N}{P} = \{0 + P, 1 + P, 4 + P, 6 + P\}$. For near-ring $\frac{N}{P}$,the addition and multiplication tables are given below.

Tables set 10

$+$	$0+P$	$1+P$	$4+P$	$6+P$
$0+P$	$0+P$	$1+P$	$4+P$	$6+P$
$1+P$	$1+P$	$2+P$	$6+P$	$4+P$
$4+P$	$4+P$	$6+P$	$0+P$	$1+P$
$6+P$	$6+P$	$4+P$	$1+P$	$0+P$

$\cdot$	$0+P$	$1+P$	$4+P$	$6+P$
$0+P$	$0+P$	$0+P$	$0+P$	$0+P$
$1+P$	$0+P$	$1+P$	$4+P$	$6+P$
$4+P$	$0+P$	$4+P$	$4+P$	$6+P$
$6+P$	$0+P$	$6+P$	$0+P$	$0+P$

Since $(6 + P)\ (4 + P) = 0 + P \in \frac{P}{P}$, *which implies* $(6 + P)^2 = 0 + P \in \frac{P}{P}$, *hence* $\frac{P}{P}$ *is a primary ideal in* $\frac{N}{P} \Rightarrow \frac{P}{P}$ *is an almost primary ideal of* $\frac{N}{P}$.

References

1. A. O. Atagün, H. Kamac, İ. Takekin & A. Sezgin, P-properties in near-rings, J. Math. Fund. Sci., **51**(2), 162-167 (2019).
2. G. Birkenmeier, H. Heatherly & E. Lee, Prime ideals in near-rings, Results in Mathematics, **24**, 27-48 (1993).
3. B. Elavarasan, Generalizations of prime Ideals in near-rings, Int. J. Open Problems Compt. Math., **4** (4), 47-53 (Dec., 2011).
4. A. Frohlich, Distributively generated near-rings, Proc. London Math. Soc., **3** (8), 76-108 (1958).
5. W. L. M. Holcombe, Primitive near-rings, Dissertation University of Leeds, (1970).
6. K. Koppula, B. S. Kedukodi & S. P. Kuncham, On perfect ideals of seminearrings, Beitr Algebra Geom (2020). https://doi.org/10.1007/s13366-020-00535-2
7. G. Pilz, Near-rings, North-Holland Publishing Co., Amsterdam, second edition, 1983.
8. D. Ramakotaiah & G. K. Ra