An Inconsistency

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Abstract. This paper proves an inconsistency in ZFC. We show that under two assumptions – a strengthened form of the strong Goldbach conjecture and its negation – a specific set is equal on the one hand and different on the other.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1, let $\mathbb{N}_n$ denote the natural numbers starting from n > 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3. Furthermore, we denote the exclusive OR by "$\underline{\vee}$".

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

Theorem. *ZFC is contradictory, i.e. the statement FALSE can be derived*.

Proof. We define the set $S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N}; p, q \in \mathbb{P}_3, p < q; m = (p + q) / 2 \}$.

SSGB is equivalent to saying that every integer $x \geq 4$ is the arithmetic mean of two different odd primes and so it is equivalent to saying that all integers $x \geq 4$ appear as m in a middle component mk of S_g. So, by the definitions we have

$$\text{SSGB} \iff \forall x \in \mathbb{N}_4 \quad \exists (pk, mk, qk) \in S_g \quad x = m.$$

$$\neg\text{SSGB} \iff \exists x \in \mathbb{N}_4 \quad \forall (pk, mk, qk) \in S_g \quad x \neq m.$$

The set S_g has the following two properties.

First, the whole range of $\mathbb{N}_3$ can be expressed by the triple components of S_g ("*covering*"), because every integer $x \geq 3$ can be written as some pk with k = 1 when x is prime, as some pk with $k \neq 1$ when x is composite and not a power of 2, or as $(3 + 5)k / 2$ when x is a power of 2; $p \in \mathbb{P}_3$, $k \in \mathbb{N}$. So we have

(C) $\forall x \in \mathbb{N}_3 \quad \exists (pk, mk, qk) \in S_g \quad x = pk \;\vee\; x = mk = 4k.$

Second, due to the definition of the set S_g, all pairs (p, q) of distinct odd primes are used ("*maximality*"). So we have

(M) $\forall p, q \in \mathbb{P}_3, p < q \quad \forall k \in \mathbb{N} \quad (pk, mk, qk) \in S_g$, where $m = (p + q) / 2$.

There are two possibilities for S_g, exactly one of which must occur: Either there is an $n \in \mathbb{N}_4$ in addition to all the numbers m defined in S_g or there is not. The latter is equivalent to SSGB and the former is equivalent to ¬SSGB.

The following proof is independent of the choice of n if there is more than one. For example, the minimal such n works. The basic idea is this:

Since, due to (C), every n given by ¬SSGB as well as every multiple nk, $k \in \mathbb{N}$, equals a component of some S_g triple that exists by definition, S_g in the case n exists (¬SSGB) is equal to S_g in the case n does not exist (SSGB). This leads to a contradiction because in the case SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas in the case ¬SSGB they don't.

The above properties (C) and (M) rule out the two possibilities that an n different from all m exists because n is different from all S_g triple components pk, mk, qk or because n is the arithmetic mean of a pair of primes not used in S_g. That is, we have the logical structure ((C) $\wedge$ (M)) => (F), where (F) is the statement FALSE which we will now derive.

We split S_g into two complementary subsets: For any $y \in \mathbb{N}_3$, $S_g = S_g+(y) \cup S_g-(y)$, where

$S_g+(y) := \{ (pk, mk, qk) \in S_g \mid \exists\, k' \in \mathbb{N} \quad pk = yk' \vee mk = yk' \vee qk = yk' \}$ and

$S_g-(y) := \{ (pk, mk, qk) \in S_g \mid \forall\, k' \in \mathbb{N} \quad pk \neq yk' \wedge mk \neq yk' \wedge qk \neq yk' \}$.

Let $n \in \mathbb{N}_4$ be given by ¬SSGB as above. Then, we have

(*) ¬SSGB => $S_g = S_g+(n) \cup S_g-(n)$.

More precisely, under the assumption ¬SSGB with the associated n the set S_g can be written as the disjoint union of the following triples.

(i) S_g triples of the form (pk = nk', mk, qk) with k = k' in case n is prime, due to (C)

(ii) S_g triples of the form (pk = nk', mk, qk) with k ≠ k' in case n is composite and not a power of 2, due to (C)

(iii) S_g triples of the form (3k, 4k = nk', 5k) in case n is a power of 2, due to (C)

(iv) all remaining S_g triples of the form (pk = nk', mk, qk), (pk, mk = nk', qk) or (pk, mk, qk = nk')

and

(v) S_g triples of the form (pk ≠ nk', mk ≠ nk', qk ≠ nk'), i.e. those S_g triples where none of the nk' equals a component.

So, S_g+(n) is the union of the triples of the above types (i) to (iv) and S_g-(n) is the union of the triples of type (v).

Now, we define

$S_1 := \{ (pk, mk, qk) \in S_g \mid \neg SSGB \text{ holds} \}$

$S_2 := \{ (pk, mk, qk) \in S_g \mid SSGB \text{ holds} \}$.

Since S_g is non-empty, we have

(1) $\neg$SSGB <=> $S_g = S_1$

(2) SSGB <=> $S_g = S_2$.

So, by (*) and (1) we obtain

(3) $\neg$SSGB => $S_1 = S_g = S_g+(n) \cup S_g-(n)$.

Since $S_g+(n) \cup S_g-(n)$ is independent of n, we can write

(3') $\forall y \in \mathbb{N}_3$ $\neg$SSGB => $S_1 = S_g = S_g+(y) \cup S_g-(y)$.

Under the assumption SSGB there is no n as above. Therefore, under this assumption, we can choose an arbitrary $y \in \mathbb{N}_3$ such that $S_g = S_g+(y) \cup S_g-(y)$. So, using (2), we obtain

(4) $\forall y \in \mathbb{N}_3$ SSGB => $S_2 = S_g = S_g+(y) \cup S_g-(y)$.

So, by (3') and (4) we have

(5) $\forall y \in \mathbb{N}_3$

(($\neg$SSGB => $S_1 = S_g = S_g+(y) \cup S_g-(y)$)

$\wedge$

(SSGB => $S_2 = S_g = S_g+(y) \cup S_g-(y)$)).

We will make use of the following trivial principle.

If two sets of (possibly infinitely many) x-tuples are equal, then the sets of their corresponding i-th components are equal; $1 \le i \le x$.

To this end, for each $k \ge 1$ we define

$M(k) := \{ mk \mid (pk, mk, qk) \in S_g \}$

$M_1(k) := \{ mk \mid (pk, mk, qk) \in S_1 \}$

$M_2(k) := \{ mk \mid (pk, mk, qk) \in S_2 \}$.

Then, applying the principle above to the middle component of the triples (pk, mk, qk), (5) implies

$\forall k \in \mathbb{N} \;\; \forall y \in \mathbb{N}_3$

$((\neg SSGB \;\Rightarrow\; M_1(k) = M(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \})$

$\wedge$

$(\quad SSGB \;\Rightarrow\; M_2(k) = M(k) = \{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \}))$.

Since by definition $S_g+(y) \cup S_g-(y)$ equals S_g for every $y \in \mathbb{N}_3$ regardless of whether or not SSGB holds and since for every $k \in \mathbb{N}$ and every $y \in \mathbb{N}_3$

$\{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \} = k\mathbb{N}_4$

$\veebar$

$\{ mk \mid (pk, mk, qk) \in S_g+(y) \cup S_g-(y) \} \ne k\mathbb{N}_4$,

we obtain

$\forall k \in \mathbb{N} \;\; (\neg SSGB \Rightarrow M_1(k) = M(k) = k\mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2(k) = M(k) = k\mathbb{N}_4)$

$\veebar$

$\forall k \in \mathbb{N} \;\; (\neg SSGB \Rightarrow M_1(k) = M(k) \ne k\mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2(k) = M(k) \ne k\mathbb{N}_4)$.

For k = 1 we set M := M(1), M_1 := M_1(1) and M_2 := M_2(1), and we obtain

$(\neg SSGB \Rightarrow M_1 = M = \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 = M = \mathbb{N}_4)$

$\veebar$

$(\neg SSGB \Rightarrow M_1 = M \neq \mathbb{N}_4 \quad \wedge \quad SSGB \Rightarrow M_2 = M \neq \mathbb{N}_4)$.

Since M is non-empty, we have

(1') $\neg$SSGB <=> M = M_1

(2') SSGB <=> M = M_2.

Since under the assumption SSGB the numbers m defined in S_g take all integer values $x \geq 4$ whereas under $\neg$SSGB they don't, we have

(6) SSGB <=> M = $\mathbb{N}_4$.

Because of SSGB => M_1 = { } and $\neg$SSGB => M_2 = { } and because M is non-empty in any case, the implications above are in fact equivalences. Then, using (1'), (2') and (6), we obtain

$(M_1 = M \neq \mathbb{N}_4 \Leftrightarrow M_1 = M = \mathbb{N}_4 \quad \wedge \quad M_2 = M = \mathbb{N}_4 \Leftrightarrow M_2 = M = \mathbb{N}_4)$

$\veebar$

$(M_1 = M \neq \mathbb{N}_4 \Leftrightarrow M_1 = M \neq \mathbb{N}_4 \quad \wedge \quad M_2 = M = \mathbb{N}_4 \Leftrightarrow M_2 = M \neq \mathbb{N}_4)$.

So, we get

(FALSE $\wedge$ TRUE)

$\veebar$

(TRUE $\wedge$ FALSE).

This yields FALSE $\underline{\vee}$ FALSE, which is equivalent to FALSE.

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