

JOHN NAPIER AND HENRY BRIGGS AT THE THRESHOLD OF ANALYSIS

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ABSTRACT. The logarithms of John Napier (1614) and Henry Briggs (1617) were landmarks in the history of mathematics, and their work is well known to math historians, but not so much to mathematicians who learn calculus from modern texts. This is odd because so many of the basic ideas of analysis are directly related to arithmetic techniques and concepts they employed or anticipated without calculus, and they can all be explained in algebraic terms with limiting values that are evident from calculus.

Napier called his logarithms wonderful, and they are. Napier's arithmetic model of continuous motion set up speed as a difference quotient, and its limit is the instantaneous speed he postulated. In effect, he used a Riemann sum for estimating his logarithm, integrating the area under a rectangular hyperbola, proving accuracy and preceding Gregory St. Vincent by some thirty years.

Briggs, perhaps with Napier, found an algorithm for natural logarithms (but he did not recognize it that way) that is powerful and practical for manual calculations. Incorporated in his *golden rule*, it provided his first method for calculating base-10 logarithms.

Briggs method can be inverted easily to define the number e and calculate it to seventeen significant figures with no more effort than he used to calculate logarithms. Not until 1691 was e identified by Leibniz and calculated to eight digits of accuracy. The formal connection between Briggs' method and the generating function for the Bernoulli numbers yields a recursion formula for natural logarithms.

This study explores the underlying mathematics of Napier and Briggs in a manner that underscores some of the modern concepts inherent in their work.

(Codes written in a Basic-like language are provided to demonstrate the calculations using Casio's online Keisan Calculator.)

ACKNOWLEDGEMENTS

I am indebted to several authors for information about early logarithms and their use by navigators and astronomenrs, and of course they are cited in the text. Denis Roegel and Ian Bruce explain the arithmetic methods used by John Napier and Henry Briggs for calculating their logarithms, and they provide tables that readers may use to compare with calculations generated by codes in the text. Roegel's *LOCOMAT* — *The Loria Collection of Mathematical Tables* and Bruce's *17th Century Maths* are online and can be accessed directly from the locations where they are cited.

Siegmund Probst discussed this project with me while we were writing “The Leibniz catenary and approximation of e ” (*Hist. Math*, 2019) and made important suggestions along the way. Niccolò Guicciardini wrote to us that Briggs, Huygens and Newton had computed $\log_{10} e$ without recognizing e . Astonishing!

Numerical testing was indispensable for this project, all done on Casio's online Keisan Calculator. Most of the codes are presented in the text so that they can be copied and executed while reading the text.

Adrian Rice was ready as always with helpful suggestions.

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I am responsible for remaining errors.

CONTENTS

Acknowledgements	2
List of Figures	4
List of Tables	4
Preface of April 28, 2022	5
Preface of March 31, 2022	5
1. Introduction	7
2. Logarithms defined kinematically	10
2.1. Simple example of a kinematic logarithm	12
2.2. A generic logarithm	12
2.3. Napier-style Logarithms: Stages of Construction	14
3. Napier's Logarithm — Foundations for numerics	19
4. Numerics — Napier's Tables	24
4.1. Recurring themes	24
4.2. Tables 1, 2 and 3	27
4.3. Short Table	31
4.4. Using the Canonical Table	36
5. Another Perspective — Riemann Sums	37
6. Transition from Napier to Briggs	40
6.1. Root-extraction for increasing accuracy	41
7. Typical Examples Based on Briggs' Golden Rule	42
8. Briggs' discovery of his golden multiplier	44
9. Briggs' golden rule and the Bernoulli numbers	46
9.1. Bernoulli numbers and the exponential function	48
9.2. The recursion algorithm for natural logarithms	49
10. Conclusion	54
Ontology	59
Vita	61
References	62
Index	66

LIST OF FIGURES

1	Napier's model logarithm	10
2	Convergence of recursion only if $d \leq D = 3.921553^+$	52
3	Comparing initial behavior of recursion for $d \leq D$ and $d > D$	53
4	Continuing example of divergence for $d = 5 > D$	53

LIST OF TABLES

1	Example of Interpolating proportionate intervals	13
2	Brigg's progressive square-roots of 10	45
3	Convergence to $\ln(10) = 1/\log(e)$	46

PREFACE OF APRIL 28, 2022

Hyperlinks were incorporated into the text. These links are most convenient when the pdf viewer has a function for back stepping to the location where the link was invoked. Not all pdf viewers offer this function. For example, the Adobe Acrobat Reader DC on Mac Monterey does not, but you may find bid-directional links for the table of contents in a sidebar, or if not use *Preview* with the “Back” option of “Go” on the top bar.

Convergence properties for the Bernoulli series of Equation 46 were stated incorrectly in Section 9.1 with no impact on calculations in the text. The error was corrected and implicated text was clarified. A few minor corrections were made as well as changes to improve the discussion and flow of text around Figures 2, 3 and 4.

PREFACE OF MARCH 31, 2022

This paper began as an exercise to answer questions about why an arithmetic technique of Briggs leads directly to natural logarithms and the Bernoulli numbers. It has resulted in an extended study of the direct connections between modern calculus and the arithmetic methods of Napier and Briggs. The paper draws heavily from Roegel’s *Napier’s ideal construction of the logarithms*, but the perspective on calculus here is different [36].

Because the major themes are treated fairly independently, with cross references, the Table of Contents (and Index) can be used for selective reading. Napier defined his logarithm by postulating time spent by a kinematic point traversing intervals, so the first 4 Sections illustrate and develop that idea. The difference equation and an early stage of differential calculus appear in Section 2. Section 3 develops theory used in Numerics. Section 4, Numerics, introduces the construction tables Napier created to reduce the massive computations required for a practical table; the numerics are complicated and are not necessary for understanding the theory, but they show the intimate connection between theory and applications.¹ Napier’s close encounter with integration is discussed in Section 5.

Further sections discuss an algorithm of Briggs with a final section connecting it to a recursion formula for calculating natural logarithms using Bernoulli numbers, as if to say they and logarithms are from the

¹“Although the logarithms are precisely defined by Napier’s procedure, their computation is not obvious. This is particularly interesting, as Napier managed to define a numerical concept, and to separate this concept from its actual computation,” Roegel [36, p. 6].

same cradle. Short computer codes that can be executed online appear throughout the paper to support or illustrate the text.

1. INTRODUCTION

Napier's logarithms have been discussed so much by historians and mathematicians for centuries that I wonder, does anything more need to be said?² I have two reasons: (1) I have not seen any treatment that simply converts the methods of Napier and Briggs into algebraic forms with limiting values that can stand as prototypes for calculus, and (2) Napier called his logarithms wonderful, and I think a younger generation of mathematicians would find it wonderful to see how he brought them to life using the arithmetic of decimal numbers.³

So I begin! I infer formulations based on the numerical methods of Napier and Briggs, but they did not think in this way. Although I rely on authors who explain the original methods in detail, primarily Denis Roegel and Ian Bruce, my purpose has been different: it is to examine the underlying mathematics lying close to the surface for eyes trained in modern algebraic formulations and calculus [36, 8].⁴ This can help us understand how deeply Napier and Briggs ventured into matters that lay at the foundation of analysis.

Archimedes defined the archetypal logarithm by pairing integer exponents with a geometric progression to represent large numbers and simplify arithmetic [3]. But not until Napier and his successor Briggs was the idea fully realized with practical widely published tables of logarithms.⁵

Napier's Table of Logarithms was published in 1614 as *Mirifici Logarithmorum Canonis Descriptio* (The Description of the Wonderful Canon of Logarithms), and his prior explanation of his logarithms was

²See the bibliography in Roegel [36, Pp.40–57].

³Ian Bruce writes: “This book [Napier's *Mirifici Logarithmorum Canonis Constructio*] is a ‘must read’ for serious students of mathematics, young or old” [10]. Gauss, quoted by former student Moritz Cantor some fifty years later: “Sie ahnen nicht, wieviel Poesie in der Berechnung einer Logarithmentafel enthalten ist.” “You have no idea how much poetry there is in the construction of a table of logarithms” [14, p. 254]. Gauss had a lifelong interest in logarithms, acquired a large collection of tables making notes on their qualities, and he created his own table of “Gaussian logarithms” for simplifying usage. See Karin Reich's “Logarithmentafeln — Gauß' ‘tägliches Arbeitsgeräth’ ” in *Carl Gauß in Göttingen* [32, Pp. 73–89].

⁴Others I have consulted are Naux [30], Goldstine [21], Havil [22], Burn [12] and Bower [5].

⁵For antecedents to Napier, see Naux for an encyclopedic history of logarithms and consequential development of analysis in its wake: Volume I reviews history up to the invention of differential and integral calculus; Volume II continues with developments through the use of calculus [30, See Chronologies, p. 11, Tome I]. Truemper features the contemporaneous development of Bürgi's logarithm [44, Ch. 3].

published posthumously in 1617 as *Mirifici Logarithmorum Canonis Constructio* (The Construction of the Wonderful Canon of Logarithms) [37, 29, 28]. Briggs, professor of geometry at Gresham College in London, visited Napier at Edinburgh in 1615 and 1616, where they planned a better kind of logarithms (base-10) [27]. Briggs' first table appeared in 1617 [35].

Napier defined his logarithm kinematically, as a correlation between the distance a point would move along a line at a constant speed with another kinematic point moving along a parallel half-ray at an instantaneous rate equal to its distance from the end; he coordinated the two by having both begin at the same initial velocity. Both points were illustrated moving rightward. The analytical determination of this logarithm requires calculus not available to Napier, but Napier devised a discretized method that he used arithmetically for approximation, and his calculations did indeed approximate the calculus solution.

Napier and Briggs anticipated many foundational ideas in analysis. In 1638, Galileo characterized motion at constant speeds and speeds of constant acceleration some 24 years after Napier's work had already characterized hyperbolic motion as an instantaneous speed corresponding to the limit of a difference quotient, a much more sophisticated concept [19, 20]. This topic is discussed in Section 2.3. Napier's discretization effectively integrates the area under a rectangular hyperbola by Riemann sums, preceding Gregory St. Vincent by more than thirty years [48, 12, 40]. And Riemann presented his integral in 1854, more than 240 years after Napier had published his logarithmic tables [34, Sec. 4]. This topic is discussed in Theory Part 2.

Briggs demonstrated in his *golden rule* that at the root of Napier's logarithm lies a powerful computational tool recognized by us as:⁶

$$\ln x \approx \frac{(x^\epsilon - 1)}{\epsilon}, \quad (\epsilon \text{ small}). \quad (1)$$

This formulation, with ϵ a variable and $x = e$, presents a formal relationship to the generating function for the Bernoulli numbers. It will be discussed in Section 9. Its inverse, the exponential function, can be formulated as,

$$e^x \approx \left(1 + \frac{x}{n}\right)^n, \quad (n \text{ large}) \quad (2)$$

⁶Equivalents of the two following formulas were published in Euler's 1748 calculus text *Introductio In Analysisn Infinitorim*, translated with original Latin text and comments by Bruce [17, Vol 1, Ch. 7, p192]. The Latin with a German translation appear in the Euler Archive [16, Summary].

Briggs used progressive square roots ($\epsilon = 1/2^n$) in Equation 1, a refinement of Napier's idea that a base-10 logarithm for the number, say 5, could be found by progressive combinations of square roots beginning with 5.⁷

Equations 1 and 2 are expressions given later by Euler in his *Introductio in Analysin Infinitorum Vol. 1* to characterize the "hyperbolic" or "natural" logarithm and exponential function, but we will see how they arise by progressive root-taking from a method of Napier for estimating logarithms.⁸ In Section 6 Equation 2 is used to calculate e to 17 significant figures with no more difficulty than Briggs would have used Equation 1 to calculate his *golden multiplier*, $\log_{10} e$, with comparable accuracy, although e was not known until recognized by Leibniz.⁹

The expression on the right-hand side of Equation 2 and its expansion as a power series were known later to Jacob Bernoulli but only as a formula for compound interest; he estimated the value between 2.5 and 3.0 and did not record a connection with the natural logarithm.¹⁰

Briggs' golden rule may have been conceived during his collaboration with Napier, plausible because Equation 1 is implicit in Napier's work.¹¹ We can express the golden rule as the familiar identity, $\log_{10} x = (\log_{10} e)(\ln x)$, or in words: To compute the base-10 logarithm of a

⁷See [27, pp. 50–51, 61–63, 97–100] and [35, pp. 4–5].

⁸See *Transition from Napier to Briggs*, Section 6, below.

⁹Bruce writes, "Napier's tables gives e the value 2.71828288267020; the underlined places are in error. Thus, one presumes for the first time in the history of mathematics, a set of numbers had emerged from which one could calculate e ." Bruce's *Modern Generation* [29, Decriptio, Liber I, Item 9, p. 13].

¹⁰Rough and Probst discuss what may be the first record of an accurate estimate for e as the base of the natural logarithm, by Leibniz (ca. 1691); their footnote 7 provides references to the Bernoulli series and estimate [31]. At the time of our study, we received an email in May 13 of 2018 from Niccolò Guicciardini informing us that "In 1661 Huygens baptized the 'logarithmic' curve ($y = cb^x$) and calculated $\log_{10}(e)$ to 18 decimals [0.434294481903251804] (see *Oeuvres complètes*, vol. 14, p. 464: a number that appears also in Briggs, *Arithmetica logarithmica* (1624), p. 455..... It is interesting to find $\log_{10}(e) = 0.4342944819032518$ in Newton's *Mathematical Papers*, vol. 3, p. 232 (in the so-called *De methodis serierum et fluxionum*, early 1670s)." With a table of base 10 logarithms, they could have found an approximation for e , since to high accuracy $10^{0.434294481903251804} = 2.718281828459045087326$.

¹¹Napier himself originated the idea of base-10 logarithms, acknowledged by Briggs [21, pp. 11-13]. Briggs, carrying out the program agreed with Napier, invented powerful finite-difference methods for reducing the burdens of calculating the logarithms. We will see the centrality of square roots in the Briggs method in Section 7. "This must be regarded the first time the Binomial theorem was developed for a noninteger exponent" [21, p. 19].

number, multiply its base- e logarithm by Briggs' multiplier. A generalization of the golden rule, $\log_a c = (\log_a b)(\log_b c)$ is now common knowledge. It is a kind of compliment to Napier and Briggs that several examples in Courant and John's classic *Introduction to Calculus and Analysis* could have been presented as adaptations of their results, but they make no mention of priority [15]. Some examples of these are referenced in the Conclusion.

2. LOGARITHMS DEFINED KINEMATICALLY

Napier characterized his logarithm kinematically, in terms of instantaneous motion:

A precise definition of the logarithm was given by Napier as follows. He considered two lines (Figure 1), with two points moving from left to right at different speeds.

On the first line, the point $\Lambda(t)$ is moving with a constant speed equal to 10^7 units of distance per unit of time. The figure shows the positions of the point at time t .

On the second line (which is of length 10^7), the point $x(t)$ is moving geometrically, in the sense that the distances left to traverse at instants determined arithmetically at chosen times $T, 2T, 3T, \dots$, form a geometric sequence.

(Adapted from Roegel [36, p.4, Sec. 2.2])

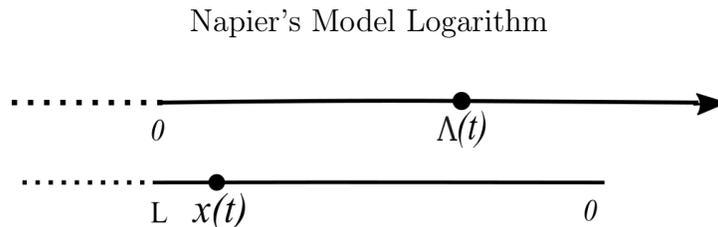


FIGURE 1. In this paper, $x(t) \in [L, 0)$ indicates that $x(t)$ is the positive distance of x from 0 ; t is time. The point x begins at $x(0) = L$ and travels toward 0 at a rate equal to $x(t)$ units of distance per unit of time. For Napier, $L = 10^7$. This notation differs from Roegel's representation of Napier, but the resulting logarithm is equivalent [36, p.4, Sec. 2.2].

The preceding definition differs from Roegel’s only slightly. I use different variable names. And the geometric point I have labelled $x(t)$ *decreases* from 10^7 to 0, whereas Roegel’s variable *increases* from 0 to 10^7 . This reversal simplifies the algebra. I represent the logarithm $\Lambda(t)$ increasing from 0, the same as Roegel.

Napier’s model of a continuously declining speed requires calculus to find the function connecting the number to its logarithm. But neither the word “function” nor calculus was available at the time of Napier.

Napier used arithmetic. He discretized the motion into segments partitioned by the numbers determined by a specified quasi-geometric progression (a progression that doesn’t start with 1) beginning with 10^7 , with the average speed over each segment equal to the distance of a chosen point (near the segment’s midpoint) from the origin. This method resulted in identical travel times over each segment, so that his logarithm at any number in the progression was proportional to the sum of the number of segments traveled from the start.

This is a straightforward concept. But it is impossibly laborious to complete a practical table of logarithms so simply, and so the story is complicated. To help clarify the exposition, I use special terms to emphasize the distinctions mentioned above.

Napier’s model point, whose speed declined continuously, I refer to as the *continuous* traveling point or *model* point; analytical characterization of that motion requires calculus, so I also use the term *calculus traveler*, the choice of term depending on which seems most helpful in context. Napier had to estimate the motion in discrete steps arithmetically, so a point that is characterized by discrete positions is called a *discrete traveler*. Napier made mistakes in his arithmetic that were corrected by Roegel in “Napier’s ideal construction of the logarithms” [36]. So, to refer to a point that moves in discrete steps corresponding to Roegel’s corrected logarithms, I use terms like *idealized traveler*. Finally, there is the *log point*, which moves at a constant speed over distances that depend on the time spent by the traveling point it is associated with.

These along with other basic terms used throughout are collected in Ontology, Section 10. I think a reader would find it useful to look at the ontology before going farther.

The following two subsections present the simplest examples of kinematic logarithms defined by a point moving at discrete speeds. The third subsection sets Napier’s logarithm in a general class of Napier-style logarithms, where basic theory is developed.

2.1. Simple example of a kinematic logarithm. The simplest illustration is built from the partition of the unit interval using the geometric progression $(1/2)^k$, $k = 0, 1, 2, \dots$, separating progressive halves of what remains:¹²

$$[1, 0.5), [0.5, 0.25), [0.25, 0.125), \dots$$

A discretized traveling point starts at $x = 1$ and advances rightward over each interval at a constant speed equal to the length of the interval; in one unit of time the discrete traveler has reached 0.5, in two units 0.25, etc. At that speed, the time taken by the traveler to reach the end of the k -th interval at $(1/2)^k$ is k . Matching these two numbers — the position reached and time taken — gives an Archimedean logarithm. Moreover, the speed of the traveling point over the n -th interval is equal to the distance of the interval from the point $x = 0$.

2.2. A generic logarithm. Instead of restricting the partition to the unit interval, let its length be $L > 0$, and instead of using a partition by powers of $1/2$, use powers of $R > 0$. Partition the interval into segments by using the quasi-geometric progression $x_k = LR^k$, $k = 0, 1, 2, \dots$. Number the intervals $I_k = [x_{k-1}, x_k)$, $k = 1, 2, \dots$. Suppose a discretized traveler traverses the k -th interval at the constant rate of $x_k = LR^k$ units of distance per unit of time, and that the rate is equal to the distance of the k -th interval from the origin.

At that rate, the length of time it takes the discretized traveler to move over the k th interval is $(LR^{k-1} - LR^k)/(LR^k) = 1/R - 1$, a result that depends only on R and is independent of k and of L . The time $T_R(k)$ to traverse the first k intervals is therefore,¹³

$$T_R(k) = k(1/R - 1). \tag{3}$$

The function defining T_k has the value 0 at $k = 0$. Although there is no interval numbered 0, we can pair $T_R(k)$ with $x_k = LR^k$ for all $k = 0, 1, 2, \dots$ to obtain a match-up between an arithmetic progression and a quasi-geometric progression. For the case $L = 1$, the result is an Archimedean logarithm — with the qualification that a constant multiple of k instead of simply k is used. For $L \neq 1$ it is a different kind of logarithm. With a slight modification of the above, and R and L suitably coordinated, it is Napier's logarithm.

¹²Notice that the unit interval is represented here as $[1, 0]$, reversing the standard notation $[0, 1]$. This practice of placing the length of the interval first and 0 second was mentioned above.

¹³This expression is basic. Similar expressions will arise repeatedly at the basis of the logarithms discussed in this paper.

A Refinement. To see where this is leading, for a fixed integer $n > 1$ we can refine the partition by inserting intermediate n -th roots into each of the original segments to define a refined quasi-geometric series of terms $x_k^{(n)} = LR^{k/n}$, $k = 0, 1, 2, \dots$, defining speed of travel over each of these segments to be, again, the distance of the interval from the origin. Higher roots could be interpolated for greater refinement. Table 1 shows an example, where cube roots are interpolated into Napier’s fundamental progression.

TABLE 1. Example: Interpolating cube roots into the series LR_1^n , $k = 0, 1, 2, \dots$ where $L = 10^7$, $R_1 = 0.9999999$. Initial terms of the refined series are shown in the right-most column; exponents of the interpolated cube roots are in the third column. The original series is bolded.

n	Exponent	Interpolated Exp.	
n	γ	γ	LR_1^γ
0	0		10000000.0000000000000000
1		1/3	9999999.6666666555555555
2		2/3	9999999.3333333222222222
3	1		9999999.0000000000000000
4		4/3	9999998.6666666888888889
5		5/3	9999998.33333338888889
6	2		9999998.00000010000000
7		7/3	9999997.66666682222222
7		8/3	9999997.33333355555555
9	3		9999997.00000029999999

This scheme retains the endpoints of the original partition among the new ones created by interpolation, and it retains the original speed at each of the original points, but it refines the speed at the intermediate points. The travel time over each refined subinterval is $1/R^{1/n} - 1$, so the time it will take to cover n refined intervals is,

$$T_R^{(n)} = n (1/R^{1/n} - 1). \tag{4}$$

Our derivation showed that for each n the result is a generic logarithm, and so will the limit be. L’Hopital’s Rule shows that,¹⁴

$$\lim_{n \rightarrow \infty} T_R^{(n)} = \ln(R).$$

¹⁴See the derivation for a similar case at Equation 24.

The use of root-extraction to refine a quasi-geometric series is applied again in Section 6.1 to improve the accuracy of Napier's model logarithm.¹⁵

2.3. Napier-style Logarithms: Stages of Construction. A *Napier-style logarithm* is based on a traveling point that travels at a rate *proportional* to its distance from the origin. Napier postulated a point traveling at a rate *exactly* equal to its distance from the origin, but his approximation resulted in a point that travels at a slightly different proportional rate, as we shall see.

A Napier-style logarithm is discretized and allows the choice of a point within each interval at a fixed proportion between the endpoints to determine the constant speed on the interval. In Roegel's exposition, Napier chose a point near the midpoint of the interval [36, Sec. 2.4]. The more general logarithm defined here will be adapted to Napier's logarithm in Section 3, applying the parameters chosen by Napier:

(Stage 1) **Napier's model speed.** A traveling point with coordinate $x(t)$, with $t = \text{time}$, traverses an interval $[L, 0]$ moving rightward from $x(0) = L$ toward the origin at a speed $\sigma^*(x)$ units of distance per unit of time equal to its distance from the origin:

$$\sigma^*(x(t)) = x(t), \quad t \geq 0. \quad (5)$$

This is an instantaneous speed decreasing rightward with x .

(Stage 2) **The Fundamental Progression and Partition.** Napier assumed correctly that a point traveling at the speed expressed by Equation 5 would traverse proportionate intervals in identical times. Suppose in general that we are given a partition on the segment $[L, 0]$ of proportionate intervals $I_k = [x_{k-1}, x_k]$, $n = 1, 2, \dots$ with the ratio of proportionality $R = x_k/x_{k-1}$.

$$x_k = LR^k, \quad k = 0, 1, 2, \dots \quad (6)$$

The intervals defined in this way are referred to as *Fundamental Intervals*, and the set of all the intervals is a *Fundamental Partition*. Napier used $L = 10^7$ and $R = 0.9999999$.

(Stage 3) **Constancy of travel time across Fundamental Intervals.** Starting with a set of fundamental intervals, it requires calculus to determine the mean speed of the model traveler over each interval. Calculus was not available to Napier, so he had

¹⁵The subtraction of 1 from the root in Equation 4 explains the systematic result obtained by Briggs in his tabular process explained in Section 8.

to guess the mean speed, and we will see how he did that. To generalize, we define a mean speed over each interval as

$$\bar{\sigma}_k = \alpha x_{k-1} + \beta x_k, \quad k = 1, 2, \dots, \quad (7)$$

with $\beta = 1 - \alpha$. The restriction $0 \leq \alpha \leq 1$ would ensure that $\bar{\sigma}_k \in [x_{k-1}, x_k]$, but that is not required.

Using the notation $\Delta_k = x_{k-1} - x_k$, the travel time over I_k may be expressed,

$$\Delta_t(k) = \frac{\Delta_k}{\bar{\sigma}_k} = \frac{1 - x_k/x_{k-1}}{\alpha + \beta x_k/x_{k-1}} = \frac{1 - R}{\alpha + \beta R}, \quad (8)$$

a quantity that is independent of k and L , proving that if a traveling point moves over proportionate intervals of ratio R , and the mean speed over each interval is a fixed proportion between its endpoints, then the travel times are identical.

Crucial for Napier, the converse is also true. Given the proportionate intervals defined by Equation 6, suppose travel time over each is a constant, say Δ_t . Then, for mean speed over the k -th interval, we have,

$$\begin{aligned} \bar{\sigma}_k &= \frac{\Delta_k}{\Delta_t} = x_{k-1} \left(\frac{1 - R}{\Delta_t} \right) \\ &= x_k \left(\frac{1 - R}{R\Delta_t} \right), \quad k = 1, 2, \dots \end{aligned} \quad (9)$$

whose proportional position \bar{x}_k within the interval I_k is,

$$\bar{x}_k = \frac{x_{k-1} - \bar{\sigma}_k}{\Delta_k} = \frac{x_{k-1} - \bar{\sigma}_k}{x_{k-1}(1 - R)} = \frac{1}{1 - R} - \frac{1}{\Delta_t}, \quad (10)$$

a constant for all k . Comparing this constant with the constant final term of Equation 8, it is apparent that constant values for α and $\beta = 1 - \alpha$ can be found, placing this case in the category previously considered. This validates a key property that Napier claimed for his *model logarithm* — proportionate intervals have identical traversal times — proving this property for Napier-style *discretized logarithms*: logarithms constructed on a discrete set of intervals determined by a fundamental partition.

The constancy of travel times over fundamental intervals is what makes it possible to use them to define a logarithm, because the travel time to traverse the first k intervals is just k times that constant, providing an arithmetic progression to pair with the quasi-geometric progression $x_k = LR^k$. In fact

any constant multiplied by k would yield a logarithmic pairing. But the choice of that constant will determine how well the resulting logarithm matches the model logarithm postulated by Napier.

As we will see in Equation 13 of the next Stage, Napier's choice that we label C for convenience, approximates well the amount of time Napier's postulated continuous traveler would travel over each of Napier's fundamental intervals, resulting in Ck as an accurate estimate of the time taken to traverse the first k intervals. Napier defined a log point that travels at a constant rate of L units of distance in 1 unit of time, the same as the initial speed of the model traveler. Then he defined his logarithm as $\lambda_N(x_k) = LCk$, which is just the distance the log point travels in time Ck . Napier's logarithm can be defined as illustrated in Roegel's Figure 1 for points defined by a fundamental partition, [36, p.4, Sec. 2.2].¹⁶ One of the special features of Napier's method is that geometric proportionality can be used to extend the definition to points not belonging to the fundamental partition.

(Stage 4) **Discrete Napier-style logarithms and the continuous case.** The foregoing provides the basis for a logarithm defined on the discrete set of points $x_k = LR^k$, $k = 0, 1, 2, \dots$:

$$\lambda(x_k) = k \cdot \Delta_t, \quad k = 0, 1, 2, \dots \quad (11)$$

This definition is broad. It pairs an arithmetic progression with a geometric progression in the case $L = 1$ and with a "quasi-geometric" progression in the case $L \neq 1$, called "quasi" because the progression doesn't start with 1. It covers the kinematic logarithms discussed in earlier sections, and it includes Napier's logarithm.

The differential equation. Reinterpreting Equation 8 as a functional relationship between Δ_x and Δ_t , it can be rewritten as a difference quotient that, taken to the limit, yields a differential equation characteristic of kinematic logarithms.

Let $x_{k-1} = x + \Delta_x$, $x_k = x$, then

$$\frac{\Delta_t}{\Delta_x} = \frac{1}{\alpha x + \alpha \Delta_x + \beta x} = \frac{1}{x + \alpha \Delta_x}$$

¹⁶Roegel's axes and labels are different than ones I use, but Roegel gives Napier's own graphic description of his logarithm. See Figure 1 to see the convention for this paper.

Taking the limit as $\Delta_x \rightarrow 0$,

$$\frac{dx}{dt} = x.$$

Using L as the upper boundary value, the solution determines the time a continuous traveler going at Napier's postulated speed takes to traverse the interval from x^* to L (x increases leftward):

$$T(x^*) = \int_{x^*}^L \frac{dx}{x} = \ln \frac{L}{x^*}. \quad (12)$$

The result is independent of α (and β). It is implicit that the *model traveler*, like the Napier-style discretized traveler, traverses intervals with a common proportion in identical times. This is true for proportionate intervals of arbitrary size, not just the fundamental intervals of the discrete model.

$T(x^*)$ can be multiplied by L , an effect achieved by Napier, to synchronize the starting speeds of the continuous traveler and the corresponding log traveler: L units of distance in 1 unit of time.¹⁷

The determination of C and definition of c . The preceding theory was general for Napier-style logarithms. In the remainder of this Stage we assume the values used by Napier: $L = 10^7$ and $R_1 = 1 - 1/L = 0.9999999$.

Napier used the midpoint of his first fundamental interval as an initial estimate for the mean speed: $\bar{\sigma}_1 = (L + LR_1)/2$. Equation 8, with $L = 10^7$, $R_1 = 1 - 1/L$ and $\alpha = \beta = 1/2$, gives as an estimate of time to travel over the first interval,

$$\begin{aligned} \Delta_t &= 2 \cdot \frac{1 - R_1}{1 + R_1} \\ &\approx 1.0000000500000025 \cdot 10^{-7}. \end{aligned} \quad (13)$$

Napier settled on the value $C = 1.00000005 \cdot 10^{-7}$ discussed at the end of (Stage 3).¹⁸

It is his approximation for the amount of time spent by his model traveler on a fundamental interval defined by the series LR_1 .

¹⁷See the quote from Roegel at the head of Section 2.

¹⁸See Roegel [36, p.9, Sec. 2.4.1]

Scaling by L yields the constant $c = LC$ we shall use throughout the remainder of this study,

$$c = 1.00000005, \quad (14)$$

fixing Napier's definition of his logarithm on the fundamental intervals as,

$$\lambda_N(x_k) = LCk = ck. \quad (15)$$

We will use a scaled version of the quotient in Equation 13 as a general estimator for interpolation; it will be referred to as the *midpoint estimator*:¹⁹

$$\text{MidpointEst}([X_1, X_2]) = L \cdot 2 \cdot \left(\frac{X_1 - X_2}{X_1 + X_2} \right). \quad (16)$$

The value for c fixed the definition of Napier's logarithm at points of the fundamental partition, and the Midpoint Estimator can be used to interpolate the logarithm to other points required for Napier's tables. We shall see the unfolding consequences. But let's pause to understand more fully how Napier's logarithm differs from the behavior of his postulated continuous model deduced by calculus in Equation 12.

What Napier's logarithm is. According to Equation 9, the implicit mean speed over the k th interval determined by $x_k = LR_1^k$ can be extended to a continuous function defining an instantaneous speed,

$$\sigma(x) = s \cdot x, \quad s = \left(\frac{1 - R_1}{cR_1} \right); \quad (17)$$

$$s \approx 1.0000000500000005.$$

To see how slight the difference is from the logarithm implied by Napier's model speed x , solve the differential equation $dx/dt = \sigma(x)$ over the interval $[L, x^*]$ and compare it with Equation 12:

$$t(x^*) = \int_{x^*}^L \frac{dx}{\sigma(x)} \approx 0.99999995 \cdot \ln(L/x^*). \quad (18)$$

The full characterization of Napier's logarithm is developed in Section 3.

¹⁹The midpoint estimator is accurate for $\lambda_N(X_2/X_1)$ if $R = X_2/X_1 \in [1, 0.9999999]$. Note that for $X_2/X_1 = R_1$, the rounded value is the midpoint estimator c .

(Stage 5) **Error in approximated travel time.** This is not a defining property but a consequence of the foregoing. A gross error bound for the approximated travel time follows from the fact that the extreme speeds on the interval (x_1, x_2) are $\zeta \cdot x_1$ and $\zeta \cdot x_2$, where $\zeta = 1$ for the calculus logarithm or $\zeta = s$ is nearly 1 as defined in Equation 17 for the continuous extension of Napier’s discretized logarithm. So the error in a guessed mean speed for traversal at the rate of $\sigma(x) = \zeta x$ cannot exceed,

$$t_{error}(x_1, x_2) \leq \left| (x_1 - x_2) \left(\frac{1}{\zeta x_2} - \frac{1}{\zeta x_1} \right) \right| = \frac{(x_2 - x_1)^2}{\zeta x_2 x_1}. \quad (19)$$

Like the quotient in Equation 8, it is homogeneous of degree 0, hence identical in value for proportionate intervals. Since $1 - 1/1.00000005 \approx 5.0 \cdot 10^{-8}$, the value chosen for ζ is insignificant, so we will simplify numerics by using $\zeta = 1$. This error in time is converted to a distance error (i.e., error in a logarithm) by multiplying by speed L .

For intervals on which the bound exceeded his desired accuracy, Napier used a proportionate interval with better accuracy; see Roegel on construction of the third table [36, Sec. 2.4.3].

3. NAPIER’S LOGARITHM — FOUNDATIONS FOR NUMERICS

The basic theory of Napier’s logarithm has been consolidated in the Stages of Construction of Section 2.3. It is only a matter of specifying parameters to define Napier’s logarithm: For Napier, $L = L_N = 10^7$, and $R = R_1 = 1 - 1/L_N = 0.9999999$. The fundamental progression for Napier was therefore $x_k = L_N R_1^k$, $k = 0, 1, 2, \dots$. All the intervals of the fundamental partition are proportionate, and so a point traveling at a rate proportional to its distance from the origin will require the identical traversal times for each interval. The midpoint estimator is used to estimate times over intervals not in the fundamental partition. Napier’s estimate for the time of traversal over a fundamental interval was derived in (Stage 4) and multiplied by speed L to produce his basic unit of the logarithm: $c = 1.00000005$. Therefore, repeating Equation 15, Napier’s logarithm was defined on the discrete set of points x_k of the fundamental progression as,

$$\lambda_N(x_k) \equiv c \cdot k, \quad c = 1.00000005, \quad k = 0, 1, \dots \quad (20)$$

Napier’s first problem is that the definition could not be applied directly for a complete table because it would have been impossible to

compute enough points manually. Napier's first table, which presents entries only for x_k and $\lambda_N(x_k)$ for $k = 0, 1, \dots, 100$, illustrates the problem. You can see them by executing the following code online using Casio's Keisan Calculator.²⁰ Napier's numbers and logarithms for this table are scaled by the factor L , rounded to the 8th decimal place. The code shows that Table 1 reaches only so far as the final entry at logarithm 100.0000050 corresponding to the number 9999900.0004950. At that rate it would take about 6,900,000 entries to reach only about half way down the interval $[L, 0)$: $x_{6900000} \approx 5015761$:

```
L=10^7; R1=0.9999999; c=1.00000005;
if( k >=0 and k <= 100) {
  Xk = round(L*R1^k,8);
  NapLogk = round(c*k,8);
  print( NapLogk, Xk ); }
```

To overcome this problem, Napier used three more progressions to construct tables with longer intervals, but he made some arithmetic errors. Roegel has recomputed Napier's Tables correctly based on these progressions, and they are accessible online in *Napier's ideal construction of the logarithms*; see Tables 1, 2 and 3 [36, Pp. 58–83]. The *logarithms* appear in a column to the right of the corresponding *numbers*: they are all scaled by $L = 10^7$, producing integer parts of up to seven decimal digits.

Table 2 shows numbers that advance by an initial stride 100 times longer than the stride of the Table 1 series, and the initial stride of Table 3 is 5,000 times longer than that of Table 1. Table 3 has elements of an even longer stride. These comparisons translate into inversely proportional numbers of computations required to fill the tables. The Short Table is used to extend the reach of Table 3.

Napier's second problem lies with Tables 2 and 3 and the Short Table. They don't match the defining fundamental progression exactly, and

²⁰ Computations for this paper are specified by codes written for use on Casio's free service, the online Keisan Calculator (KC) at <https://keisan.casio.com/calculator>. KC's language is similar to Basic, explained in the KC's *How to use* section. The codes can be pasted into the *expression field* of the KC and executed using the KC standard settings with two exceptions: (1) always select the *Textarea* editor instead of *Ace*, and (2) select *degrees* instead of *radians* for codes executing trigonometric functions. The typical accuracy used for this article is the KC standard 22 digits, but 38-digit accuracy was used where noted, and much higher is possible.

For more, see <https://keisan.casio.com/keisan/service.php> and FAQ at <https://keisan.casio.com/keisan/faq.php\#faq>.

their discrepancies need to be corrected to produce accuracy in the logarithms. These are problems of approximation using the midpoint estimator dealt with in Numerics, Section 4. Two analytical methods shown next will be helpful there as checks on accuracy, and they will appear in several codes. But they are not necessary for understanding Napier's own methods.

Analytics. The coefficient c transforms exponents into logarithms. Solving $x_k = L_N R_1^k$ for k :

$$k = \ln(x/L) / \ln(R_1), \quad (21)$$

from which, by Equation 20,

$$\lambda_N(x) = c \cdot \ln(x/L) / \ln(R_1) \quad (22)$$

for $x = x_k, k = 0, 1, 2, \dots$

Napier could not know this formulation. But his given logarithm of Table 1 is rigorously extended to a continuous function for all real numbers $x > 0$ by letting x be any positive real number.²¹ We refer to this extension as the *idealized Napier logarithm*. It is not the same as the model logarithm derived by calculus as Equation 12.

The following code takes as input any number X from the tables corrected by Roegel's *Napier's ideal construction of the logarithms* [36, Pp. 58–83] or Bruce's *Modern generation of Napier's logarithms* [11]. The corresponding logarithm can be compared with the values produced by Equation 22 and the calculus solution from Equation 12 scaled by L .

Reminder: Set the Keisan *Editor* to *Textarea*, not *ACE*.

```
L=10^7; R1=0.9999999; c=1.00000005;
IdealizedNapierLog = c* ln(X/L)/ln(R1);
CalculusLog = L*ln(L/X);
print ( IdealizedNapierLog, CalculusLog );
print ( IdealizedNapierLog - CalculusLog );
```

The extension of Napier's logarithm by algebra. One further matter of theoretical interest will conclude this section, demonstrating how

²¹ This is not to be confused with the ideal logarithm of Equation 18 implied by Napier's choice of c . This is a subtle point worth making because it is not clear exactly which logarithm Napier was striving to approximate: the logarithm determined by a point traveling at a speed equal to its distance from the origin, the logarithms of Equation 18 or the logarithm of Equation 22. It doesn't matter in practical terms because all three were within his error tolerance.

Equation 22 extends Napier's logarithm beyond the original points $x_k = LR_1^k$.

For any integer $n > 0$, we can refine Napier's logarithm by using $R_1^{1/n}$ as the base of a partition finer than the one produced by LR_1^k .²²

$$x_k^{(n)} = L_N R_1^{k/n}, k = 0, 1, 2, \dots \quad (23)$$

This inserts n points between every point of the partition based on R_1 , and the intervals they define are proportionate with ratio $R_1^{1/n}$. Any choice of a time Δ_t for traversing an interval will define a logarithm as the accumulated time for traversing the first k intervals: $\lambda(x_k) = \Delta_t \cdot k/n$. Using Napier's value we have denoted as c , the definition $\lambda_N(x_k^{(n)}) \equiv ck/n$ retains Napier's original values on all points $kn, k = 0, 1, 2, \dots$ and preserves the logarithms on all the original intervals.

Let $y > 0$, choose integers such that $k/n \rightarrow y$, then $L_N R_1^{k/n} \rightarrow L_N R_1^y$, and $\lambda_N(x_k^{(n)}) = ck/n \rightarrow cy$. The function extending Napier's logarithm is $\lambda_N(LR_1^y) = cy$. By construction, all finite extensions given by integers k and n agree exactly at all coincident points and at Napier's fundamental progression; for all other points, it extends by arithmetic Napier's logarithm to the analytic limit of Equation 22.

To see how this extension works numerically, pick an integer power m , and divide the first m fundamental intervals into n segments using Equation 23. For example, try $m = 4$ and $n = 2$ and check the four clusters against Roegel's Table 1 [36, Pp. 58–59].

```
L=10^7; R1=0.9999999; c=1.00000005;
for(j=0; j <= m*n; j=j+1) {
  Xjn = L*R1^(j/n);
  IdealizedNapLog = c*ln(Xjn/L)/ln(R1);
  CalculusIdealLog = L*ln(L/Xjn);
  print( 888888888888888888 );
  print( round(Xjn,7) , c*j/n );
  print( IdealizedNapLog, round(CalculusIdealLog,16) );}
```

The method used to divide the intervals $L[R_1^{k-1}, R_1^k]$, $k = 1, 2, 3, \dots$ into n proportionate slices was designed to merely *extend the arithmetic progression* defined by Equation 20. It does not *change* the traversal times over any combination of intervals previously evaluated.

²²This kind of refinement was introduced in Section 2.2. See Table 1.

The method just discussed differs from the method of Section 2.2, where logarithms were recomputed on smaller slices to change and *improve* the approximation of the model calculus logarithm. We can use that method to refine the intervals of LR_1 by using $LR_1^{1/n}$ as the defining ratio of the fundamental partition. By setting the constant speed over a fundamental interval $[x_{k-1}, x_k]$ as the distance from the origin by the coordinates of an intermediate point $\bar{x}_k = \alpha x_{k-1} + \beta x_k$, with $\beta = 1 - \alpha$, gives a travel time t_k over the k th interval of,

$$t_k = \frac{x_{k-1} - x_k}{\bar{x}_k} = \frac{1 - R_1^{1/n}}{\alpha + \beta R_1^{1/n}},$$

a result independent of k . So the travel time T_k to cover the first k intervals is, $T_k = kt_k$, and the time to reach the interval ending at LR_1^N is,

$$NT_n = Nn \frac{1 - R_1^{1/n}}{\alpha + \beta R_1^{1/n}}.$$

We want to find the limiting value of T_n . Since $\lim_{n \rightarrow \infty} (\alpha + \beta R_1^{1/n}) = 1$, L'Hopital's rule will verify that,

$$\lim_{n \rightarrow \infty} T_n = \lim_{n \rightarrow \infty} n \cdot (1 - R_1^{1/n}) = -\ln(R_1). \quad (24)$$

To see this, use $\epsilon = 1/n$ and rewrite the above as,

$$\lim_{n \rightarrow \infty} n \cdot (1 - R_1^{1/n}) = \lim_{\epsilon \rightarrow 0} \frac{1 - R_1^\epsilon}{\epsilon}.$$

Regarding ϵ as a continuous variable, the second member of the preceding equation has a numerator and denominator that are differentiable functions of ϵ , and so the indeterminate quotient will have the same limit as the quotient of the derivatives:²³

$$\lim_{\epsilon \rightarrow 0} \frac{1 - R_1^\epsilon}{\epsilon} = \lim_{\epsilon \rightarrow 0} \frac{-\ln(R_1) e^{\epsilon \ln(R_1)}}{1} = -\ln(R_1).$$

A similar case was presented for Equation 4. Although Napier used $\alpha = \beta = 0.5$, the limiting result is independent of α and $\beta = 1 - \alpha$. We saw above (Equation 13) that Napier used his midpoint approximation,

$$\lambda_N(R_1) = 2L \frac{1 - R_1}{1 + R_1} = 1.0000000500000025 \cdot 10^{-7},$$

²³Courant and John index the name as "L'Hospital's Rule." See the rule at [15, Indeterminate Expressions, A.I.3]. It can be found on the internet.

and here we see again (as a consequence of multiplying Equation 12 by L) that the ultimate refinement of his discretized model yields $\Lambda(R_1) = -L \ln(R_1) = 1.0000000500000033333334$. (See Equation 30.) So $\Lambda(LR_1^N) = -LN \ln(R_1)$.

4. NUMERICS — NAPIER’S TABLES

Napier’s arithmetic methods were intricate, and so are the details of this section. It is not needed for the theory of this paper, but here one can understand the ingenuity and labor required by Napier to create his Canonical Table. Short segments of computer code for the Keisan Calculator support and illustrate the text.²⁴ The codes are based on algebraic representations inferred from Roegel’s exposition of Napier’s methods set forth as the “Stages of Construction” of Section 2.3 above. To see how Napier himself performed his computations, see also Napier [27] and Havil [22].

Napier’s fundamental partition, marked off by the LR_1^k series, was too refined for practical computation. He used alternative progressions with larger subinterval sizes, and from these created four tables: three *construction tables* (Tables 1,2, and 3), the third of which spanned a little more than half of the interval $[L_N, 0]$; the fourth table (*Short Table*) enabled him to bring any number into the range of the third table. Roegel published the *Construction Tables* 1–3 and the *Short Table* in “Napier’s ideal construction of the logarithms” [36, Pp. 58–83] and [36, Pp. 20–21].

Napier used the tables to derive logarithms for every sine of the Canonical Table. For the Canonical Table, see Bruce’s *A modern generation of Napier’s logarithms* [11]. Also see Roegel’s Figures showing the first page of Napier’s table from the publication of 1620, with slight differences from Bruce’s first page [36, p. 25]. Napier’s full table appears in the English translation of *The Wonderful Canon of Logarithms* [26].²⁵ Specific references are given below for each table.

4.1. Recurring themes. Parameters, subtractive multiplication, and other matter that appears repeatedly or implicitly in codes for table calculations:

²⁴See Footnote 20.

²⁵With persistence, it is possible to find a complete and clear copy of the table in “The Wonderful Canon” in Google Books.

Parameters. We label the parameters Napier used to define his logarithm and tables:

$$\begin{aligned} L &= 10^7, \\ R_1 &= 0.9999999, R_2 = 0.99999, R_3 = 0,9995, R_4 = 0.99, \\ c &= 1.00000005. \end{aligned} \quad (25)$$

Subtractive multiplication. Napier used the decimal system to simplify multiplication. The representations $R_1 = 1 - 1/10^7$, $R_2 = 1 - 1/10^5$, $R_3 = 1 - 5/10^4$, $R_4 = 1 - 1/10^2$ make it easy to recognize a subtraction method for performing repetitive multiplication: for example, $R_1^{k+1} = R_1^k - R_1^k/10^7 = R_1^k - r_1 R_1^k$, where $r_i = 1 - R_i$.²⁶

The following code emulates subtractive multiplication to generate *numbers*, as distinguished from *logarithms*, for Tables 1 and 2, the first-column entries for Table 3, and the LR_4 series comprising the first row of Table 3.²⁷ Results can be compared with Roegel's [36, Pp. 58–83]. Inputs are *Tb* for Table number and *Row* for row number. See Footnote 20 for instructions for the Keisan Calculator.

```
L=10^7;
r1=1/10^7; r2=1/10^5; r3=5/10^4; r4=1/10^2;
if (Tb==1) {R = r1; nRows=100;}
elseif (Tb==2) {R= r2; nRows=50;}
elseif (Tb==3) {R= r3; nRows=20; }
elseif (Tb==4) {R= r4; nRows=68; }
else {exit;}
print( L );
num=1;
for (k=0; k < nRows; k=k+1) {
num=num - R*num;
print( L*num ); }
```

Decimal points in the tables. Napier represented decimal numbers with principal digits in the range $[L, 1]$, with the decimal point placed to the right of the unit's digit. He used digits to the right of the decimal point in computation and recorded logarithms to the seventh decimal place in Tables 1 and 2, but only to the first decimal place in Table 3. For his Canonical Table, the fractional parts are omitted and numbers are expressed in 6 to 7 digits and their logarithms in 1 to 8 digits.

²⁶Roegel uses r_1, r_2 , etc., for parameters that I denote by R_1, R_2, \dots

²⁷Table 3 comprises 69 columns (0–68). Napier simplifies the arithmetic for computing the number in a given column (*Col*) by multiplying the leading row term by R_4^{Col} using the subtraction technique.

Calculations to assess accuracy. Three kinds of calculations are performed for each table. Foremost are calculations that I suppose Napier could carry out arithmetically; executed here on a modern computational platform, they are meant to stand in for methods that could be done manually. Roegel explains Napier’s own methods for estimating the accuracy of his calculations [36, Sec. 2.4.2].

Two codes using modern analytical formulations are implemented to assess accuracy of the “arithmetic” calculations. They were expressed in Equations 21 and 22 and in Equation 12 with Napier’s synchronization factor L included in the *Analytics* paragraphs above along with code for assessing accuracy of Napier’s tables.

Error estimation by arithmetic. Equation 19 gives an error bound for an estimated time to traverse an interval $[x_1, x_2]$ where the speed of the ideal point equals $\sigma(x) = \zeta \cdot x$, an arbitrary proportion of its distance from the origin is given by,

$$T_{MaxErr}[x_1, x_2] = \frac{(\Delta x)^2}{\zeta x_1 x_2}$$

As mentioned in (Stage 5), a satisfactory simplification for Napier’s logarithm is given by the formula,

$$T_{MaxErr}[x_1, x_2] = \frac{(\Delta x)^2}{x_1 x_2}.$$

For example, in estimating $\lambda_N(LR_2)$ it is necessary to estimate the time taken to traverse the gap from $x_1 = LR_1^{100}$ to $x_2 = LR_2$ of length $\Delta x = x_1 - x_2$. Denoting the ratio $R = x_2/x_1 = R_2/R_1^{100}$, and multiplying by speed L , the foregoing formula provides the error estimate,

$$\begin{aligned} L \cdot T_{MaxErr}[1, R] &= L \cdot \frac{(1 - R)^2}{R} \\ &\approx Error_2 = 2.4502831 \cdot 10^{-14}. \end{aligned} \tag{26}$$

It can be used to validate the accuracy of an estimate for $\lambda_N(LR_2)$. This case is simple. Napier defined $\lambda_N(LR_1^{100}) = 100 \cdot c = 100.000005$, and all that is needed is to estimate the logarithm of the gap $[LR_1^{100}, LR_2]$ using the midpoint estimator:

$$MidEst = 2L \cdot \frac{LR_1^{100} - LR_2}{LR_1^{100} + LR_2} = 4.95003333 \cdot 10^{-4}$$

Adding the two will give an estimate of $\lambda_N(LR_2)$ accurate to within the bounds given by Equation 26:

$$\begin{aligned}\lambda_N(R_2) &\approx 0.0004950033 + 100.000005 \\ &= 100.0005000033 \pm 2.4502831 \cdot 10^{-14}.\end{aligned}\tag{27}$$

Roegel's value for $\lambda_N(R_2)$ is 100.0005000 [36, Sec. 10.2, p. 60]. The comparative analytics computed by the Keisan Calculator are: Napier's idealized logarithm = 100.0005000033, and the calculus logarithm = 100.0005000033.

Looking ahead. $\lambda_N(LR_2)/\lambda_N(R_1) \approx 100.000495$ implies that it takes about 100 computations of the LR_1^k series to get as far as one computation of the LR_2^k series; to reach the halfway mark with LR_2^k would require about 69000 computations — still impracticable for manual calculation. But in theory it would hold the error in the logarithm at that point to within an outside estimate of $\pm 69000 \cdot 2.4502831 \cdot 10^{-14} \approx \pm 1.69 \cdot 10^{-9}$. Napier will reduce the burden of computation even farther and ensure accuracy of the series LR_3^k and LR_4^k .

4.2. Tables 1, 2 and 3. See entry for each of Tables 1–3. The Short Table is treated in the following subsection:

Table 1 [36, Sec. 10.1, Pp. 58–9]:

Table 1 consists of the first 101 terms of the fundamental progression, $x_k = L_N R_1^k$, $k = 0, 1, \dots, 100$, paired with their logarithms. The magnitude of effort required to construct a complete table of logarithms based on the “ LR_1 ” progression becomes evident on examining the Table: x_{100} reaches only to 9999900.0004950. A stupifying 6,900,000 iterations of the ratio R_1 would reach only to

$$LR_1^{6900000} = 5015760.5176168,$$

not quite halfway down the interval $[L, 1]$. Getting to halfway will be Napier's first objective. The code below uses the Keisan Calculator's methods for estimating x_{100} and $x_{6900000}$.

Reminder: Set the Keisan *Editor* to *Textarea*, not *ACE*.

```
L=10^7; R1=0.9999999; print(L*R1^100, L*R1^6900000);
```

Napier defined his logarithm for $L_N R_1^k$ as $\lambda_N(x_k) = ck = 1.00000005 k$, for k a non-negative integer. The following code can be used to choose k and find the entry x_k and its logarithm $\lambda_N(x_k)$ for Table 1; although these logarithms are known exactly, the values are rounded to 7 decimal places to fit the table.

Reminder: Set the Keisan *Editor* to *Textarea*, not *ACE*.

```

L=10^7; R1=0.9999999; c=1.00000005;
Xk = round(L*R1^k, 7);
LogXk = round(c*k, 7);
print( Xk, LogXk );

```

Table 2 [36, Sec. 10.2, p. 60]:

The foregoing code shows that $x_{100} = 9999900.000495$ and $\lambda_N(x_{100}) = 100.0000050$. The value of x_{100} is so close to 9999900 that Napier adopted $R_2 = 0.99999$ as more convenient than R_1 for iteration, increasing advancement of the progression by a factor of 100. He computed $LR_2^k, k = 0, 1, \dots, 50$ and included those terms with their logarithms in Table 2. It would take 69,000 terms of LR_2^k to reach about halfway down the interval $[L, 1]$, still too large a number.

But there is another catch: there are gaps between terms of the original LR_1 series and the LR_2 series. Because Napier's logarithms are defined by the LR_1 series, he must assign logarithms for the LR_2 series consistent with those of LR_1 , a problem solved by Equation 27.

Table 3 [36, Sec. 10.3 Pp. 61–83]:

The final entry of Table 2 is $LR_2^{50} = 9995001.2248040230$, close to 9995000, leading to adoption of $R_3 = 0.9995$, a number even more convenient than R_2 for iteration. The stride length of the LR_3 series is 5000 times greater than that of the LR_1 series, which implies it would take about 1380 terms of the LR_3 series to reach nearly halfway down the interval: $LR_3^{1380} = 5014895.2580976$.

Napier deemed this within reach of practical computation and made R_3 the basis for his Table 3, but to further simplify computation, he interposed an extra step. The 21st term of the LR_3 series is $LR_3^{20} = 9900473.5780232$, showing that every term of the LR_3 series is nearly $R_4 = 0.99$ times its 20-th predecessor. The ratio R_4 is easier to iterate than R_3 , so Napier arranged Table 3 in Rows 0-20 and Columns 0-68 using a computational process equivalent to $LR_3^{row} R_4^{col}$.

Deriving logarithms for Table 3 is more complex than for Table 2. For example, the midpoint estimator gives too crude an estimate for the gap between LR_2^{50} and LR_3 . With $R = LR_3/LR_2^{50}$ the error estimate is,

$$L \cdot TMaxErr[1, R] = L \cdot \frac{(1 - R)^2}{R} < 1.50 \cdot 10^{-7}$$

This error bound is small alone, but it would be repeated 1449 times in the construction of Table 3 and be compounded with the errors of the LR_2 series and the LR_4 series that could undermine the accuracy

needed for computation of the canonical table. To improve the estimate for a gap larger than the stride of the LR_1 series, Napier “bootstrapped,” using a related interval in the scope of Table 1 that can be estimated more accurately. Roegel explains this in [36, Sec. 2.4.3, Pp. 12–15].

Here we use a procedure summarized in the following synopsis. We seek a proportionate interval $[L, X]$, where X must be calculated beforehand to locate its position in Table 1 or Table 2; by table lookup $X \in [LR_1, LR_1^2]$ in Table 1.²⁸ The gap referred to is $[LR_1, X]$, smaller than the LR_1 stride. Note that $\lambda_N(LR_1) = c$.

$$\begin{aligned}
[L, X] &\sim [LR_2^{50}, LR_3] = \frac{1}{R_2^{50}} \cdot [L, L \frac{R_3}{R_2^{50}}] \\
X &= L \cdot \frac{R_3}{R_2^{50}} = 9999998.77458341877102; \\
X &\in [LR_1, LR_1^2]; \\
\text{LogGap} &\approx 2L \cdot \left(\frac{LR_1 - X}{LR_1 + X} \right) = 0.22541660631127269995; \\
\text{Err}_3 &= L \cdot (LR_1 - X)^2 / (LR_1 X) = 5.081264640089131368 \cdot 10^{-9}; \\
\text{LogX} &= \text{LogGap} + \lambda_N(LR_1); \\
&= 1.22541665631127269995 \pm 5.08 \cdot 10^{-9}; \\
\lambda_N(R_3) &= \text{LogX} + 50 \cdot \lambda_N(R_2); \\
\text{Error}_3 &= \text{Err}_3 + 50 \cdot \text{Error}_2 = 5.08248978 \cdot 10^{-9}; \\
\lambda_N(R_3) &= \text{LogX} + 50 * 100.0005000033 \pm \text{Error}_3; \\
&= 5001.2504168213112727 \pm 5.08248978 \cdot 10^{-9}.
\end{aligned}$$

The logarithm of the k -th row in Column 0 of Table 3 are integer multiples: $k \cdot 5001.2504168213112727 \pm 5.1 \cdot 10^{-9}$, $k = 0, 1, 2, \dots, 20$:

```

L=10^7;
R1=0.9999999; R2=0.99999; R3=0.9995; R4=0.99;
c=1.00000005;
Xk = L*R3^k;
LogXk = round(k * 5001.25041682, 1);
IdealizedNapLogXk = c*ln(Xk/L)/ln(R1);
CalcLog =L * ln(L/Xk);
print( k, LogXk, IdealizedNapLogXk, CalcLog );

```

The gap between LR_3^{20} and LR_4 presents an additional complication. Its proportionate interval $[L, X]$ based at L lies beyond Table 1 in

²⁸Note that the error estimate is improved because the midpoint estimator is applied to a value smaller than the stride of LR_1 .

the 5th interval of Table 2. The following synopsis summarizes the computation, where X is backtracked to Y in the 79th interval of Table 1, and the logarithm of Y is estimated by estimating the log of the gap between it and LR_1^{78} and adding $78\lambda_N(LR_1)$. Then two prior estimations are used: $4\lambda_N(LR_2)$ is added to the log of Y to compensate for the backshift, and $20\lambda_N(LR_3)$ is added to that to find $\lambda_N(LR_4)$ at the end of the 21st interval of Table 3. Note that $\lambda_N(LR_1^{78}) = 78c$.

$$\begin{aligned}
[L, X] &\sim [LR_3^{20}, LR_4] \sim [L, L \frac{R_4}{R_3^{20}}] \implies \\
X &= LR_4/R_3^{20} = 9999521.661242208371119; \\
X \in [LR_2^4, LR_2^5] &\implies Y = \frac{X}{R_2^4} \in [L, LR_2]; \\
Y &= 9999921.65210837971463 \in [LR_1^{78}, LR_1^{79}]; \\
Gap &= [LR_1^{78}, Y]; \\
LogGap &\approx 2L \cdot \left(\frac{LR_1^{78} - Y}{LR_1^{78} + Y} \right) = 0.348194641494288; \\
Err_4 &= L \cdot \frac{(LR_1^{78} - Y)^2}{LR_1^{78}Y} = 1.21239508365336 \cdot 10^{-8}; \\
LogY &= LogGap + 78c = 78.348198541494288 \\
LogX &= LogY + 4 \cdot \lambda_N(R_2) = 478.350198554694288; \\
\lambda_N(LR_4) &= LogX + 20 \cdot \lambda_N(LR_3); \\
&= 100503.358534680919748; \\
Error_4 &= Err_4 + 4 \cdot Error_2 + 20 \cdot Error_3 = 1.13773844 \cdot 10^{-7}.
\end{aligned}$$

With the error estimates for the LR_2, LR_3 and LR_4 series in hand ($Error_2, Error_3$ and $Error_4$), we can make an estimate for error bounds for the final entry of Table 3 at Column 68 of Row 20: $20 \cdot Error_3 + 68 \cdot Error_4 = 20 \cdot 5.08122514155 \cdot 10^{-9} + 68 \cdot 1.13773844 \cdot 10^{-7} < 8.0 \cdot 10^{-6}$.

```

L=10^7; R1=0.9999999; R3=0.9995; R4=0.99;
c=1.00000005;
NLogR3 = 5001.2504167;
NLogR4 = 100503.35853468;
EstLog = Row*NLogR3 + Col*NLogR4;
Num= round(L*R3^Row*R4^Col,6);
IdealizedNapLogX =c*ln(Num/L)/ln(R1);
CalcLog =L * ln(L/Num);
print( Num, EstLog, IdealizedNapLogX, CalcLog );

```

The foregoing code calculates all the entries for Table 3, using the estimates $\lambda_N(LR_3) = 5001.2504167$ and $\lambda_N(LR_4) = 100503.35853468$

calculated in the previous paragraphs. (See these two numbers rounded in Roegel's first page of Table 3 [36, Sec. 10.3, Pp. 61].) The input parameters for the code entered by the user are Row (0 – 20) and Column (0–68). Comparative analytics are included in the output.

4.3. Short Table. [36, Sec. 2.5.4, Pp. 19–20]: The Short Table along with Tables 1–3 enabled Napier to construct his Canonical Table for the sine function. The problem with the construction tables, Tables 1, 2 and 3 is that they reach only near the halfway mark of the interval [10,000,000, 1]. The Short Table extends the reach of Table 3.

The Short Table in effect provides augmentations to logarithms that divide or multiply the number associated with the logarithm to fit it within the scope of Table 3: add to divide, subtract to multiply.

Consider a number x and its logarithm $\lambda_N(x)$, where $x = LR_1^{\lambda_N(x)/c}$.²⁹ To achieve multiplication by $a > 0$, find the augmentation α so that $a = R_1^{\alpha/c}$, then $ax = L * R_1^{\lambda_N(x)/c + \alpha/c}$, and therefore,³⁰

$$\lambda_N(ax) = \lambda_N(x) + \lambda_N(La). \quad (28)$$

The factor $\alpha = \lambda_N(La)$ (< 0) doesn't depend on x and acts as a universal multiplier, and $-\alpha$ (> 0) serves as a divider.

Equation 28 can be solved for $\lambda_N(aL)$ if both x and ax are within the scope of the construction tables, Tables 1, 2 or 3, because then $\lambda_N(ax)$ and $\lambda_N(x)$ can be found. A likely problem is that not both will be exact values of entries, and the missing logarithm(s) can be estimated with procedures like the ones demonstrated in finding $\lambda_N(LR_3)$ and $\lambda_N(LR_4)$. Let us assume that any necessary prior steps have been taken to ensure that the values $\lambda_N(x)$ and $\lambda_N(ax)$ are known. Then we have,

$$\lambda_N(La) = \lambda_N(ax) - \lambda_N(x). \quad (29)$$

And now, as a result of knowing $\lambda_N(La)$, if given *any* positive number x and its logarithm $\lambda_N(x)$ we can know the logarithm of ax and x/a by adding or subtracting $\lambda_N(La)$ to it, just as in Equation 28 but with the variable x freely chosen. Moreover, by adding $\lambda_N(La)$ any number of times, say n times, to $\lambda_N(x)$ gives $\lambda_N(a^n x)$ and subtraction gives $\lambda_N(a^{-n} x)$.

Napier's Short Table gives values $-\lambda_N(La)$ for various powers of $a = 2$ and $a = 10$ and their mixed products up to 10,000,000. See Roegel [36, Pp. 20–21]. The positive values implement division to cut down any number larger than 10,000,000 to fit the interval $[L, 0]$ and

²⁹The role of c is discussed in relation to Equation 20.

³⁰ $a = R_1^{\alpha/c} \implies La = LR_1^{\alpha/c} \implies \alpha = \lambda_N(La)$.

the negative values implement multiplication to amplify small numbers to fit. For a numerical application, see the Short Table in Section 4.3 below.

Here we follow the procedure prescribed above using Equation 29. Napier's method is discussed by Roegel [36, Sec. 2.5]

Example: The first entry for the Short Table. We will compute the first entry of the Short Table for halving/doubling a given number. Equation 29 is used with $a = 2$, $X_1 = L$ and $X_2 = L/2$ to find $\lambda_N(2000000) = -\lambda_N(X_2)$.

The next-to-last number of Table 3, at Row 19 and Column 68, is the nearest number to $X_2 = 5000000.0$, namely $L * R_3^{19} R_4^{68} \approx 5001109.956831605$. We already know the logarithm of L : $\lambda_N(L) = 0$, but we need to estimate $\lambda_N(L/2)$ for use in Equation 29. An explanation follows the synopsis:

$$\begin{aligned}
 X_2 &\in [LR_3^{19}R_4^{68}, LR_3^{20}R_4^{68}] \\
 X_2/(R_3^{19}R_4^{68}) &\in [L, R_3] \\
 Y = X_2/(R_3^{19}R_4^{68}) &= 9997780.57902908 \\
 &\in [LR_2^{22}, LR_2^{23}] = \frac{1}{R_2^{22}}[L, LR_2] \\
 Z = Y/R_2^{22} &= 9999980.34372056 \\
 &\in [LR_1^{19}, LR_1^{20}] \\
 \lambda_N(Z) &\approx 2L(LR_1^{19} - Z)/(LR_1^{19} + Z) + 19 * c \\
 &\approx 19.65629876 \\
 &\approx 2219.667298836 \\
 \lambda_N(X_2) &= -\lambda_N(2000000) \\
 &= \lambda_N(Y) + 19\lambda_N(LR_3) + 68\lambda_N(LR_4) \\
 &\approx 6931471.80559943 \approx 6931471.81
 \end{aligned}$$

To begin, we backstepped from Table 3 through Table 2 to Table 1. Beginning with X_2 , the numbers at each step backward were indicated by Y , and Z . To return to $\lambda_N(X_2)$, we compensated for the backsteps using the known values for $\lambda_N(R_1)$, $\lambda_N(R_2)$, $\lambda_N(R_3)$ and $\lambda_N(R_4)$. Because Z is within the scope of Table 1, the midpoint estimator can be used to estimate $\lambda_N(Z)$. The computations are performed in the order shown. The last number is the same as the first value in Roegel's ideal Short Table for halving a number [36, Sec. 2.5.4, p.21]. The negative of that number is used for doubling a number. Doubling and tripling the accurate value gives the second and third entries of the Short Table. The comparative analytic values are 6931471.80559943 for the Napier idealized logarithm and 6931471.80559945 for the calculus logarithm.

Example: The fourth entry for the Short Table. This is an application of the Short Table using the value we computed in the preceding example. The fourth entry of the Short Table is the number that when added to (or subtracted from) the logarithm of a number gives the logarithm of one-tenth (or ten times) the number; in this case $a = 10$, and we want to find $\lambda_N(1000000)$.

We take two steps. Begin with $X_1 = L$ and $X_2 = L/10$. The number $X_2 = 1000000$ is too small to be in any of the construction tables. But $8X_2$ is in the scope of Table 3. The strategy is to find $\lambda_N(8X_2)$ and add the third entry of the Short Table to find our number as $\lambda_N(10L)$ using Equation 29. Let $X_3 = 8X_2 = 8000000$, then:

$$\begin{aligned} X_3 &\in [LR_3^4 R_4^{22}], LR_3^4 R_4^{22}] \\ &= R_3^4 R_4^{22} [L, R_3] \\ Y &= X_3 / (R_3^4 R_4^{22}) = 9999643.38265446 \\ &\in [LR_2^3, LR_2^4] = R_2^3 [L, R_2] \\ Z &= Y / R_2^3 = 9999943.377955824819007 \\ &\in [LR_1^{56}, LR_1^{57}] \end{aligned}$$

We have stepped back to find Z in the scope of Table 1 where the midpoint estimator can be used. For the second step:

$$\begin{aligned} \lambda_N(Z) &\approx 2L(LR_1^{56} - Z) / (LR_1^{56} + Z) + 56 * c \\ &\approx 56.622204478580249 \\ \lambda_N(Y) &= \lambda_N(Z) + 3\lambda_N(LR_2) \\ &\approx 56.622204478580249 \\ \lambda_N(X_3) &= \lambda_N(8000000) = \lambda_N(Y) + 4\lambda_N(LR_3) + 22\lambda_N(LR_4) \\ &\approx 2231435.51314209 \end{aligned}$$

Adding the third entry of the Short Table to $\lambda_N(8000000)$ and using $\lambda_N(L) = 0$ in Equation 29, find:

$$\begin{aligned} \lambda_N(L/10) &\approx \lambda_N(8000000) + 3 \cdot 6931471.80559943 \\ &\approx 23025850.92994038. \end{aligned}$$

The latter, rounded to two places, is the same as the fourth value in Roegel's exact Short Table in Figure 5 [36, Sec. 2.5.4, p.21]. The comparative analytic values are 23025850.92994038 and 23025850.92994045 for the Napier ideal logarithm and the calculus logarithm. All the other values are found by adding integer multiples of the accurate versions (found above) of the first and fourth entries. For example, the entry for 8000000 should be $\lambda_N(L/8000000) = 3 \cdot 6931471.80559943 +$

$6 \cdot 23025850.92994038 = 158949520.99644057 \approx 158949521.00$ after roundoff. Also all the other entries compare precisely with the idealized and calculus analytics after roundoff.

Readers may check the entries of the short table by using the following code and rounding the results to two decimal places. To doublecheck the above, use $Entry = 8000000$ and notice the negative value:

```
L=10^7; R1=0.9999999; c=1.00000005;
IdealizedNapLogX =-c*ln(Entry)/ln(R1);
CalcLog =-L * ln(1/Entry);
print( Entry, IdealizedNapLogX, CalcLog );
```

Constructing the Canonical Table. Napier's *Canonical Table* presented logarithms for the sines of all the degrees and minutes from 1 – 90. Napier's Tables 1-3 and the Short Table were used by Napier for constructing the table and were not included with the canonical table; they were not published in Napier's lifetime.

Roegel explains Napier's method [36, Sec. 2.5]. The example of this section is worked out in a different way, following the same procedures used above. The example uses data from Roegel's Table 7 on p.25, which lists the sines of angles as multiples of $L = 10^7$ with their logarithms.

Example, $\lambda_N(\sin 3')$. Napier's value for $10^3 \sin(3')$ is given as 8727, surely not as accurate as the value he knew, as we will see. This number is not in the scope of Table 3, but its multiple by 10^3 is, so the Short Table will be used. The procedure follows. The Short Table data for 10^3 appears as an addition in the last line, for $LogX$:

$$\begin{aligned}
 X &= 8727 \\
 X3 &= 10^3 X = 8727000 \\
 &\in [LR_3^{11} R_4^{13}, LR_4^{12} R_4^{13}] = R_3^{11} R_4^{13} [L, LR_3] \\
 Y &= X3 / (R_3^{11} R_4^{13}) = 9999923.1677 \\
 &\in [LR_1^{76}, LR_1^{77}] \\
 LogY &= L \cdot 2(LR_1^{76} - Y) / (LR_1^{76} - Y) + 76c \\
 &\approx 76.832588473998395 \\
 LogX3 &= LogY + 11\lambda_N(R_3) + 13\lambda_N(R_4) \\
 &\approx 1361634.248128709584851 \\
 LogX &= LogX3 + c\lambda_N(1/10^3) \\
 &\approx 70439187.0379498
 \end{aligned}$$

Roegel's entry for the logarithm published by Napier is presented as 70439560 in Roegel's Figure 7 [36, Sec. 4, p25]. But the comparative analytics for the preceding calculation are: 70439187.0379498 for the Idealized Napier logarithm and 70439187.03795008 for the calculus logarithm, confirming the above.

This shows that Napier made arithmetic errors, as can be confirmed by using the following code to compare data in Roegel and in Bruce's "A modern generation of Napier's logarithms" [37, 11].

The code can be used to compare results in:

- (1) Napier's Table in Roegel's Figure 7 [36, Sec. 4, p25],
- (2) Roegel's reconstruction of the tables of Napier's Descriptio,
- (3) Bruce's modern generation of Napier's logarithms,
- (4) A fourth table online is referred to in Footnote 25.

Examination of Napier's logarithms reveals they are often erroneous, presumably due to arithmetic errors in the construction tables. Roegel reconstructs the ideal table Napier would have obtained, using the given (inaccurate) sine values, "assuming Napier did not round his values and made no computation error."³¹ Bruce computes values for the logarithms that ideally Napier could have achieved with accurate sine values; they are comparable in roundoff with Napier's postulated model, tagged as the Calculus Log, and they differ from Roegel's ideal reconstruction. Set *KC Mode* to Degrees, Enter Degrees and Minutes from the table, and enter X for the value given there as the sine.

```

L=10^7;
R1=0.9999999; R2=0.99999; R3=0.9995; R4=0.99;
c=1.00000005;
SinX = L*sin(D+M/60);
IdealizedNapLogX = c*ln(X/L)/ln(R1);
CalcLog =L * ln(L/X);
LogSineX = L*ln(L/SinX);
print( X, SinX);
print( IdealizedNapLogX, CalcLog, LogSineX );

```

I have examined cases for: $10k$ Deg, 3 Min. for $k = 0, 1, 2, \dots, 8$, and they confirm the distinctions noted above.

Roundoff Error estimation. In this paper, the Keisan Calculator is entrusted to perform arithmetic operations to 22 decimal places. But Napier would need to take care for himself. The following sketches

³¹Roegel states that "this table does not give the ideal Napierian logarithm."

considerations for addition and subtraction, and for use of the midpoint estimator.

When truncating a decimal number, the two possible extreme round-off errors at the final digit after truncation are: ± 0.5 , so for truncation after the n -th digit, the maximum error for one operation would be $\pm 0.5 \cdot 10^{-n}$. Napier used iterative subtraction for computing the numbers for his tables and iterative additions for his logarithms.³² There are at most $20 \cdot 68 = 1360$ in any one of these (the rows of Table 3), so the maximum total roundoff error for that one stream of computation would be $0.5 \cdot 1360 \cdot 10^{-n} = 680 \cdot 10^{-n}$; the latter is a 3-digit number, indicating that 3 digits of accuracy may be sacrificed in the final number, with a carryover of at most ± 0.5 on the $(n - 4)$ th decimal. So to *ensure* m decimals of accuracy with an error no greater than $\pm 0.5 \cdot 10^{-m}$, a conservative practice would be to compute each number to at least $m + 3$ digits, and truncate after the m th decimal.

An additional error comes into the calculation of logarithms, due to the use of the midpoint estimator because there is always a gap involved. The midpoint estimator is a quotient of the form $2(A - B)/(A + B)$, where A is a known number like LR_2^k, LR_3^k, LR_4^k and $B < A$, often of the form $B \in [LR_j^k, LR_j^{k+1}]$, $j = 2, 3, 4$ and always of the same order of magnitude L . Suppose both A and B have roundoff error $\leq e = 0.5 \cdot 10^{-n}$, then:

$$\begin{aligned} \Delta_{MPE} &\approx 2 \cdot \frac{2e + A - B}{2e + A + B} - 2 \cdot \frac{A - B}{A + B} = 2 \cdot \frac{4Be}{(A + B)(2e + A + B)} \\ &< \frac{2e}{e + B} = 10^{-n} \cdot \frac{1}{LR_j} \sim 10^{-n-7}. \end{aligned}$$

Therefore, the error due to use of the midpoint estimator is about 7 orders of magnitude smaller than the maximum roundoff error of A and B , not much to worry about so long as e is kept sufficiently small across all the subtractive multiplications used to compute the numbers in the construction tables.

To see how Napier dealt with errors, see Roegel ([36, p. 7]).

4.4. Using the Canonical Table. Napier came at the end of an era of heroic trigonometric table construction, a topic treated at length by Glen Van Brummelen. See especially his entries for the 700-page

³²See the discussion and code for subtractive multiplication in Section 4.1.

Opus palatinum of Rheticus, and the works of other authors such as Regiomontanus, and Pitiscus in which the basic definitions of the trigonometric functions and their expression in terms of angles and lengths in triangles rather than arcs of circles were established [45, 46, 47].³³

Immediately on publication of the *Descriptio*, Napier's Canonical Table of logarithms was recognized as a means to simplify the arduous arithmetic used in trigonometry for navigation and astronomy. Briggs himself developed tables for naval use. And Kepler, deep in massive computational problems in his Rudolphine Tables, came early to the logarithms of Bürgi and Napier [43, Sec. 5.3] and [47, p. 75-76]. Van Brummelen finds in Kepler an adumbration of integral calculus, a thought in harmony with Section 5 ahead [47, p. 111].

As important as this subject is — the purpose for which Napier developed his logarithms — we turn aside. This paper has been restricted to themes of early analysis (calculus) suggested by Napier's work, and to explain and demonstrate how he constructed his tables of logarithms. Samples of Napier's applications can be found in Roegel [36, Sec. 6]. Napier covered the topic more extensively in his *Descriptio* [29].

5. ANOTHER PERSPECTIVE — RIEMANN SUMS

Napier's logarithm was characterized in the "Stages of Construction" explained in Section 2.3, and specialized by Napier's choice of parameters, discussed in Section 3.

Napier characterized his logarithm pictorially as the correlation between a point moving rightward along a half-ray $[L, 0)$ at a constant speed — referred to as the traveling *log point* — and another point also moving to the right along a parallel interval of length $L = 10^7$ at an instantaneous rate such that proportional intervals are covered in equal times [29, *Descriptio*, Liber I, p.2].³⁴ Napier inferred that the speed of the second point must be equal to its distance from its terminal point. I have referred to this latter point as Napier's *model* or *calculus* traveling point.

A solution of the ideal problem, beyond the reach of mathematics in Napier's time, was derived from the Stages in Section 2.3 using differential calculus (Equation 12). Multiplying that solution by Napier's

³³Roegel has published reconstructions of trigonometric tables extant at the time of Napier at *LOCOMAT: The Loria Collection of Mathematical Tables* [39]. Van Brummelen's references thirteen of them [46, p. 351].

³⁴In his solution of the catenary problem presented as a geometric construction (1691), Leibniz defined the ordinates of his "logarithmic curve" as proportionate in this way [31, Fig. 2, p. 4].

value $L = 10^7$ to synchronize the initial speeds of the log point and the kindematic point, yields Napier's ideal solution:

$$\Lambda(x^*) = L \int_{x^*}^L \frac{dx}{x} = L \ln \frac{L}{x^*}, \quad (L = 10^7). \quad (30)$$

Here we revert to the general Napier-style logarithm and derive Equation Section 30 anew using integration, assuming the terms of Section 3. A point moves at discrete speeds over intervals defined by a progression of points defined by the quasi-geometric progression, $x_k = LR^k$, $k = 0, 1, 2, \dots$, where $L > 0$ and $R > 0$ are fixed. Define intervals $I^{(R)}(k) = [x_{k-1}, x_k]$, let $\Delta_k = x_{k-1} - x_k$, and select an intermediate point $\bar{x}_k \in I^{(R)}(k)$. We will use the approximated travel time of Equation 8, but here allow any intermediate point \bar{x}_k on each of the intervals, and so traversal time over $I^{(R)}(k)$ is approximately,

$$\Delta_t(k) = \frac{\Delta_k}{\sigma^*(\bar{x}_k)} = \frac{\Delta_k}{\bar{x}_k}, \quad (x_k \leq \bar{x}_k \leq x_{k-1}). \quad (31)$$

Therefore,

$$\frac{\Delta x_k}{x_{k-1}} \leq \Delta_t(k) \leq \frac{\Delta x_k}{x_k}$$

By summing, we can find flanking approximations for the time the discretized traveler takes to traverse the first k intervals. Pick a time $x^* \in [L, 0)$ and sum the intervals up to x^* to find the time of travel to x^* . Choose $k = K(x^*) = \ln(x^*/L)/\ln(r)$ and get,³⁵

$$\sum_1^{K(x^*)} \frac{\Delta x_k}{x_{k-1}} \leq \sum_1^{K(x^*)} \Delta_t(k) \leq \sum_1^{K(x^*)} \frac{\Delta x_k}{x_k}. \quad (32)$$

The flanking members are sums for a Riemann integral for the function $f(x) = 1/x$ with mesh controlled by R , and so as $R \rightarrow 1$ and mesh size goes to 0, we have the limiting time $T(x^*)$ for Napier's model traveler to travel from x^* to L :

$$T(x^*) = \int_{x^*}^L \frac{dx}{x}. \quad (33)$$

Multiplying by Napier's value for L again yields the ideal logarithm approximated by Napier displayed as Equation 30:

$$\Lambda(x^*) = L \ln \left(\frac{L}{x^*} \right), \quad (L = 10^7).$$

³⁵The brackets “[x]” denote here the standard “greatest integer not exceeding x .”

This same formula for Napier’s ideal logarithm was derived first by differential calculus and now by integral calculus as direct implications of his arithmetic operations. Both derivations are valid for any $x^* > 0$, validating Napier’s logarithm for any positive value beyond the interval $[L, 0)$.

More can be inferred from Equation 32 by using the freedom allowed by a Riemann sum to choose any intermediate points. Choose $\bar{x}_k = x_{k-1}$ so that discretized travel times are identically $1 - R$ for all intervals. Using the relationships between R and $K(x^*)$,

$$x^* = LR^{K(x^*)} \iff R = \left(\frac{x^*}{L}\right)^{1/K(x^*)},$$

yields,

$$\sum_{k=1}^n \Delta_t(k) = n \cdot \left(1 - \left(\frac{x^*}{L}\right)^{1/n}\right), \quad (n = K(x^*)).$$

This particular expression for total time is flanked by the first and third members of Equation 32, so by their convergence to the same integral as Riemann sums, we arrive at a root-extraction technique for estimating Napier’s logarithms:³⁶

$$\lambda_N(x^*) = L \lim_{n \rightarrow \infty} n \left(1 - \left(\frac{x^*}{L}\right)^{1/n}\right) = L \ln \frac{L}{x^*}, \quad (x > 0). \quad (34)$$

This result was obtained in a different manner in Section 2.2 for the case $L = 1$. It implies two important approximation formulas:

$$\lambda_N(x^*) \approx L \cdot n \left(1 - \left(\frac{x^*}{L}\right)^{1/n}\right), \quad (\text{large } n, x^* > 0), \quad (35)$$

which inverted yields the number n_N paired with Napier’s logarithm,

$$n_N(x^*) \approx L \left(1 - \frac{x^*}{Ln}\right)^n, \quad (\text{large } n, x^* > 0). \quad (36)$$

³⁶A more general root-extraction technique will be presented in Section 6.1.

For example, the last entry of Roegel's of Table 3 is $\lambda_N(4998609.401853) = 6934253.4$ [36, p. 83]. For comparison, using $L = 10^7$ and $n = 2^{54}$ in the two preceding formulas, the Keisan Calculator gives,³⁷

$$\begin{aligned}\lambda_N(4998609.401853) &= 6934253.388717818210612 \\ n_N(6934253.4) &= 4998609.396213478006596.\end{aligned}$$

6. TRANSITION FROM NAPIER TO BRIGGS

Goldstine: "Briggs must be viewed as one of the great figures in numerical analysis. His ideas were far in advance of his time, and he has never been accorded the honor which is his due. This is probably because of the fallacious theory which grew up that he was merely the slavery of drudge who carried out the ideas of his master, Napier. Briggs' techniques were purely arithmetical and indicate that he must have been one of the very first, if not the first, to use the calculus of finite differences with great facility. His work is, however, difficult to read since he gave no proofs." [21, Pp. xi, 13].

Briggs, if still in the shadow of Napier today, may finally be coming into the light himself. Sonar's major biography presents what is known about the early life and distinguished career of Briggs before his meetings with Napier [42].³⁸

This section is transitional because it connects algebraic formulations suggested by Napier's methods that were incorporated into techniques used by Briggs. Those techniques make it easy to derive several numbers used by Briggs. How Briggs discovered his method will be touched on in Section 8, *Briggs' Discovery of his Golden Multiplier*.

For $L = 1$ the approximation in Equation 34 gives the *negation* of the natural logarithm, so:

$$\ln(x) = \lim_{n \rightarrow \infty} n(x^{1/n} - 1), \quad (x > 0), \quad (37)$$

and its inverse,

$$E(x) = \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n, \quad (x > 0). \quad (38)$$

³⁷Briggs used progressive square roots out to the 54-th in his "golden rule" for computing base-10 logarithms of small integers, so the use of 54 squares in this example is not outlandish.

³⁸Unfortunately, not so much is known about Napier's life. Havil summarizes his life and militant Protestantism in the context of troubled times in Scotland. See Footnote 50.

Although inherent in Napier's logarithms and utilized by Briggs, these results were published first by Leonhard Euler [17, Vol 1, Ch. 7, p192]. They can be used to estimate the natural logarithm and exponential functions.³⁹

Briggs applied the expression $n(x^{1/n} - 1)$ from Equation 37 using repeated square roots of x , as many as 54 ($n = 2^{54}$). Briggs' preeminent example was $x = 10$, providing him with the key multiplier for his "golden rule," discussed in Section 7. Briggs did not recognize his root-extraction technique as producing logarithms, but we can see that it produced base- e logarithms. Code for using Equation 37 is in the following Section.

6.1. Root-extraction for increasing accuracy. Section 2.2 demonstrated a method for inserting roots of the basis of a geometric progression to improve the accuracy of a logarithm. By other routes we have come to formulas equivalent to Equation 37. The point now is that root-extraction of positive numbers, using the distance from the point $x = L$ where $\lambda(L) = 0$ as speed of travel, is deeply connected to the natural logarithm.

We start with the Fundamental Progression (Equation 6) of (Stage 2). There we assumed $R < 1$, but here set $L = 1$, change the direction of motion *away* from the origin, take *any* $R > 0$ and let $x_k = R^k$, $k = 1, 2, \dots$

Because the intervals $I_R(x_{k-1}, x_k) = [x_{k-1}, x_k)$ are proportionate, travel time over each is identical, say T , where,

$$T = 2 \cdot \frac{R - 1}{R + 1}.$$

The interval $[1, R]$ can be divided into m proportionate slices:

$$[R^{(k-1)/m}, R^{k/m}) \quad k = 1, 2, \dots, m,$$

and travel time over each of these is identical with an aggregate time to traverse all m of them:

$$T_m = 2m \cdot \frac{R^{1/m} - 1}{R^{1/m} + 1}. \quad (40)$$

³⁹Jacob Bernoulli demonstrated that,

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad (39)$$

but he only considered this to be the limiting value for compound interest and said nothing about it as an exponential function. That interpretation had to wait for Euler, but we see that fact already near the surface in Napier.

You can see accuracy improvement with increasing m by pasting the following code in the Keisan Calculator. The code extracts square roots. Try setting the upper limit (Lim) to any integer up to 54 (practical for Briggs) to see successive square roots for choices of $R > 0$. Try $R = 3, 5, 10$ and compare the results with examples in the next section. Reminder: Set Editor to *Textarea*:

```

for(j=1; j<=Lim; j=j+1) {
m=2^j;
EstLogR_m =2*m*(R^(1/m)-1)/(R^(1/m)+1);
Error=m*(R^(1/m)-1)^2/R^(1/m) ;
print( EstLogR_m );
}
print( 100000000000000001, ln(R), Error );

```

L'Hopital's rule shows that $\lim_{m \rightarrow \infty} T_m = \ln(R)$, and this is displayed numerically in the code's final result from the Keisan Calculator's analytic estimate for $\ln(R)$. An error estimate, analogous to Equation 19, is also printed at the end, showing that the arithmetic error bound is cruder than the correct error shown by the analytic result, but it does provide persuasive numerical evidence that accuracy increases indefinitely with increasing Lim .

7. TYPICAL EXAMPLES BASED ON BRIGGS' GOLDEN RULE

Briggs based his golden rule for computing base-10 logarithms on approximations using what we have expressed formally in Equation 37, where he used repetitious square roots. For example, let $n = 2^k$ for k large, then we have,

$$\ln x \approx 2^k(x^{1/2^k} - 1). \quad (41)$$

This formulation can be used to produce the natural logarithm of any number $x > 0$. Performed manually, the results come at the cost of laborious arithmetic.

Briggs began by computing $\ln 10$, using $k = 54$. We find by use of the Keisan Calculator that:

$$\ln 10 \approx 2^{54}(10^{1/2^{54}} - 1) \approx 2.302585092994045831175$$

The Keisan Calculator's approximation is bolded to mark the correct digits in Briggs' computation:

$$\ln 10 \approx \mathbf{2.302585092994045684018}.$$

Briggs used the reciprocal of this estimate as his *multiplier*. Next we will see the use Briggs makes of it for his multiplier in the golden rule.

Briggs' golden rule. The golden rule can be inferred from basic operations of logarithms for exponentiation and root taking.⁴⁰ Denote the logarithms to bases a and b , respectively, as $\log_a x$ and $\log_b x$, and define,

$$y = (\log_a b)(\log_b x). \quad (42)$$

Then,

$$a^y = x \implies y = \log_a x, \quad \text{and} \quad \log_a b = \frac{1}{\log_b a},$$

From these results, letting $a = 10$, we have the golden rule of Briggs:

$$\textit{Golden Rule: } \log_{10} c = (\log_{10} e)(\ln c),$$

where Briggs used as his multiplier the reciprocal of the number for $\ln 10$ calculated above,

$$M = \log_{10} e = \frac{1}{\log_e 10} = \mathbf{0.4342944819032517998956}. \quad (43)$$

Briggs' value for the multiplier, 0.434294481903251804, correct in the figures bolded above, can be seen in *Arithmetica Logarithmica* [7, p. 11]. To compute his base-10 logarithm of any number, he used Equation 41 to compute the *natural* logarithm of the number, then multiplied by M , his multiplier. For examples,

$$\ln 2 \approx 2^{54}(1 - 2^{1/2^{54}}) = 0.6931471805599453 \xrightarrow{\times M}$$

$$\mathbf{\log 2 = M \ln 2 = 0.30102999566398}$$

$$\ln 5 \approx 2^{54}(1 - 5^{1/2^{54}}) = 1.609437912434100 \xrightarrow{\times M}$$

$$\mathbf{\log 5 = M \ln 5 = 0.69897000433601}$$

These results can be compared with Keisan Calculator values:

$$\mathbf{\log 2 \approx 0.3010299956639811952137}$$

$$\mathbf{\log 5 \approx 0.6989700043360188047863}$$

⁴⁰For a logarithm of any base, $\log(xy) = \log(x) + \log(y) \implies \log x^{(p/q)} = (p/q)(\log x)$, where p and q are non-zero integers, and this leads to the rules for manipulating exponents and the validity of the implications following Equation 42. Apostol notes that proof of this is complicated if done from first principles [1, p. 227]. But it is apparent from the rules for manipulating logarithms. Negative and fractional exponents were used by Wallis (1685) and Newton (1712).[4, *Indices*, p. 42].

The natural logarithm is fundamental. The golden rule has a straightforward generalization:

$$\log_a c = (\log_a e)(\ln c). \quad (44)$$

In the unique case $a = e$, the multiplier $M = \log_a e = 1$, obviating the need for multiplication. This is reason enough for regarding $\ln = \log_e$ as the most natural logarithm.

Let's use Equation 38 to estimate e , the number whose natural logarithm is 1, let $x = 1$.

$$e = \lim_{n \rightarrow \infty} (1 + 1/n)^n.$$

An approximation can be carried out manually by setting $n = 54$ and progressively squaring the expression in parentheses 53 times, a task no more onerous than ones carried out by Napier and Briggs. Using the Keisan Calculator, we find directly that

$$\text{KC: } (1 + 1/2^{54})^{2^{54}} \approx \mathbf{2.7182818284590451}59913,$$

where the bolded digits are correct for e .

8. BRIGGS' DISCOVERY OF HIS GOLDEN MULTIPLIER

Briggs' method of discovering his golden rule was altogether different from the derivation given above in Section 6. This section describes the table he used to derive his golden multiplier, published in *Arithmetica Logarithmica* in 1624.⁴¹ An excerpt of the table is shown and code is provided for computing the entries of the entire table to full accuracy.

We labeled Briggs' multiplier M , the key factor used in his golden rule, and calculated it above (see Equation 43). But here we turn to a sample of Briggs' arithmetic to illustrate his method of discovery.

To compute the multiplier, Briggs' began with the number 10 and extracted 54 progressive square-roots carried out to the 40th decimal place. He wrote the results in a two-column table available online at the references in Footnote 41. In the first column, for each $n = 1, 2, \dots, 54$ he tabulated the number $10^{1/2^n}$, and in the second column he tabulated

⁴¹ For the table in context, see Briggs, *Arithmetica Logarithmica* [7, p. 10]. A clean copy can be viewed at Wikimedia [6]. Roegel provides a reconstruction of the complete Briggs logarithmic tables with a full discussion of Briggs' methodology [35]. Bruce translates the *Arithmetic* into English and includes biographical notes [9]. Goldstine and Sonar discuss Briggs' finite differencing method [21] and [43, Sec. 6.3.4, Pp. 299-310]. Sonar's tutorial is available online [41]. See Naux.[30, pp. 99-127]

the corresponding the base-10 logarithm $1/2^n$. Table 2 below shows the 1st, 2nd and 3rd and 52nd, 53rd and 54th entries. Code is provided before Table 2 for computing all the entries.

Observe that the fractional parts of the roots become more accurately halved at each extraction, which suggests re-doubling them in compensation at each root-taking. That process, shown in Table 3, converges to Briggs' golden multiplier.⁴²

The following code computes the full set of values for Table 2 shown below. Reminder: Use the *Textarea* editor, not *Ace*. **Set *Digit* to 30** to expose the significant digits Briggs calculated manually.

```
X=10; XRouted=X;
for(j=1; j<=54; j=j+1){
XRouted=sqrt(XRouted);
k=2^j;
Log=1/k;
print( j, XRouted, Log ); }
```

TABLE 2. From Briggs' table of progressive square-roots of 10. (Entries truncated to fit the page)

n	Roots of 10	Logs of Roots of 10
<i>n</i>	$10^{1/2^n}$	$1/2^n$
1	3.162277660168379331999	0.5
2	1.778279410038922801225	0.25
3	1.333521432163324025676	0.125
*	*	
*	*	
52	1.000000000000000511277	2.220446049250313080847E-16
53	1.000000000000000255638	1.110223024625156540424E-16
54	1.000000000000000127819	5.551115123125782702118E-17

Table 3 along with the results from the code below for producing its entries provide persuasive numerical evidence of convergence. But that is not a proof. We have seen modern proofs by reference to L'Hopital's rule in Sections 2.2, 3 and 6.1. But an elementary proof can be contrived from the use of the Riemann sums in Section 5.

⁴²This procedure used by Briggs — subtract 1 from the root and multiply by the exponent — is less mysterious when viewed in light of Sections 2.2 and 6.1.

```

X=10; XRrooted=X;
for(j=1; j<=54; j=j+1){
XRrooted=sqrt(XRrooted);
k=2^j;
Y=XRrooted-1;
kY=k*Y;
print( j, Y, kY);
}
print( 10000000000001 );
print( kY, ln(X), 1/kY, log(e) );

```

TABLE 3. Observation of Convergence to $1/M = \ln(10)$.
(Entries truncated to fit the page)

n	Subtract 1 from the Root	Multiply by the Power of 2
<i>n</i>	$10^{1/2^n} - 1$	$\times 2^n$
1	2.162277660168379331999	4.324555320336758663997787
2	0.778279410038922801225	3.113117640155691204901684
3	0.333521432163324025676	2.668171457306592205407453
*	*	
*	*	
52	5.11276597280129479E-16	2.302585092994046272646927
53	2.55638298640064707E-16	2.302585092994045978332459
54	1.27819149320032345E-16	2.302585092994045831175225

The last four lines produced by the code display the arithmetic and comparative analytic results with correct digits in bold. We have seen these values earlier,:

$$1/M = \ln 10 = \mathbf{2.3025850929940458311} \text{ and,}$$

$$M = \log_{10} e = \mathbf{0.4342944819032517999097}.$$

Briggs labelled the last number as a logarithm but did not identify a base. In order to normalize similar logarithms to base 10, he multiplied them by his golden multiplier M .

9. BRIGGS' GOLDEN RULE AND THE BERNOULLI NUMBERS

In the preceding sections we found by various arguments, from Napier to Briggs, formulas equivalent to,

$$\ln(x) = \lim_{k \rightarrow \infty} k \cdot (x^{1/k} - 1). \quad (45)$$

This is fundamental. It expresses what may be the simplest practical means of computing the natural logarithm by plain arithmetic in the manner described for Equation 41.

The algebraic expression on the righthand side of Equation 45 has a formal similarity to the generating function for the Bernoulli numbers. This can be seen by changing $x \rightarrow e$, $1/k \rightarrow x$ and taking the reciprocal.

$$\frac{x}{e^x - 1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j. \quad (46)$$

The generating function for the Bernoulli numbers was discovered by Euler and used to define the numbers he named in honor of Jacob Bernoulli [21, p. 127-8]. The formality is significant because the functional relationship expressed by Equation 46 can be used numerically to compute natural logarithms using the Bernoulli numbers, demonstrating close affiliation. A more obvious connection is with the exponential function, which appears in the denominator in Equation 46. These relationships are demonstrated by means of computer codes below.

The Bernoulli numbers appear throughout analysis and are the subject of an expansive literature. Kitagawa discusses their history [25]. The book by Arakawa, Ibukiyama and Kaneko, *The Bernoulli numbers and Zeta Functions* begins with history and elementary material; the bibliography contains 109 references [2]. They were introduced to European mathematics in Jacob Bernoulli's formula for summing the powers of the integers.⁴³ Euler found them as coefficients in power series for trigonometric functions and used them in his Euler-Maclaurin summation formula.⁴⁴ They appear in Euler's zeta-function where they are entwined with π at the even integers, and then in Riemann's extension of Euler's function to the complex plane, where they show up again on the negative integers.

Wherever they appear, the Bernoulli numbers are noteworthy. In the case dealt with in this article, they have a hidden role hinted at by Equation 45 as noted above. That formula for defining the natural logarithms was published by Euler as a way to define the natural logarithm [16, Chap.7, p. 192]. Euler developed a power series for natural logarithms and used the results for converting those logarithm to another base, equivalent in effect to Briggs' golden rule. Similarly he developed an expansion for the exponential function. His notation and route of deduction are different than developed in this paper. As

⁴³Kitagawa cites a contemporaneous discovery in Japan by Takakazu Seki.

⁴⁴There is a small but important difference between Bernoulli's version of the numbers and Euler's: $B_1 = -1/2$ for Euler but $+1/2$ for Bernoulli. We'll use Euler.

far as I can determine, Euler does not mention an association with the Bernoulli numbers.

9.1. Bernoulli numbers and the exponential function. The denominator of the function on the lefthand side of Equation 46 vanishes in the complex plane at $x = \pm 2k\pi i, k = 0, 1, 2, 3, \dots$, and it has a removable singularity at $x = 0$, so the series on the righthand side converges for $|x| < 2\pi$.⁴⁵ We are concerned only with real values for x that satisfy this constraint.

Execute the following code in the Keisan Calculator to estimate e^x using Bernoulli numbers. Input parameters for the code are x and the number *numB* of Bernoulli numbers to use. For the standard examples, begin with $x = 1$ and $x = -1$. (Be aware of the limitation $|x| < 2\pi$.) The formula for e^x utilizing Bernoulli numbers is derived directly from Equation 46. The number of iterations of the code is set to *numB*, and the accuracy is limited thereby. Partial sums for the exponential series and the Bernoulli series are shown to compare convergence rates. The last number produced by the code is the Keisan Calculator's value for e^x for a comparison of values.

The results below were obtained by using *numB* = 14. Higher accuracy can be achieved by using larger numbers and setting *Digits* in the KC to larger numbers, e.g, 38.

(Reminder: set the KC editor to *Textarea*.):

```

BernoulliSum_j = 0; ExponentialSum = 0;
for(j=0; j<=numB-1; j=j+1){
  BernoulliSum_j = BernoulliSum_j + x^j*bernoulli(j)/j!;
  ExpByBernoulli= 1+x/BernoulliSum_j;
  ExponentialSum = ExponentialSum +x^j/j!;
  print(j, ExpByBernoulli, ExponentialSum );
}
print(1000000000001);
print( ExpByBernoulli, ExponentialSum, e^x);

```

For $x = 1$, the final results are listed in order according to how they were calculated: e_{Bern} using Bernoulli numbers, e_{Exp} using the power series, e_{KC} using the Keisan Calculator's value, and e_{Eul} for Euler's first estimate for the number he was first to designate as e : 2.71828182845904523536028 [16, Chap.7, p. 189].

⁴⁵Freitag and Busam [18, p.149]. It is only here that we stray outside the bounds of elementary calculus.

$$\begin{aligned}
e_{Bern} &\approx \mathbf{2.718281828497580996796}, \\
e_{Exp} &\approx \mathbf{2.718281828446759002315}, \\
e_{KC} &= \mathbf{2.71828182845904523536}, \\
e_{Eul} &= \mathbf{2.718281828459045235360}.
\end{aligned}$$

I have bolded the digits that match with the Keisan Calculator's value; Euler's matches them all.

9.2. The recursion algorithm for natural logarithms. Briggs' golden multiplier was expressed in Equation 43. A method like that used by Briggs to calculate it can be encapsulate algebraically in modern language as

$$-\log_a(e) = \lim_{x \rightarrow 0} \frac{x}{a^x - 1}. \quad (47)$$

The quotient on the righthand side of Equation 47 can be expanded in term of the generating function for the Bernoulli numbers:

$$\frac{x}{a^x - 1} = \frac{1}{\ln(a)} \cdot \frac{x \ln a}{e^{x \ln a} - 1} = \frac{1}{\ln(a)} \cdot \sum_{j=0}^{\infty} \frac{B_j}{j!} (x \ln(a))^j. \quad (48)$$

The lefthand side of Equation 48 states in algebra the arithmetic method Briggs used to compute special logarithms. The righthand side gives an equivalent expression in terms of the Bernoulli numbers for $|x \ln(a)| < 2\pi$. In that equivalence, x , a^x and $\ln(a)$ seem hopelessly entangled; the following code, however, demonstrates a recursive procedure for computing $\ln(a)$ to show that they are not.

We would like to solve that equation for $\ln(a)$, but x and $\ln(a)$ are entwined. Rewriting:

$$x \ln(a) = (a^x - 1) \sum_{j=0}^{\infty} \frac{B_j}{j!} (x \ln(a))^j. \quad (49)$$

This formulation suggests a recursive algorithm for $y = x \ln(a)$:

$$y_{k+1} = (a^x - 1) \sum_{j=0}^K \frac{B_j}{j!} (y_k)^j, \quad (50)$$

We settle on a simpler formulation by putting $x = 1$ and substituting $d = a - 1$:

$$y_2 = d \sum_{n=0}^{\infty} \frac{B_n}{n!} y_1^n = d \frac{y_1}{e^{y_1} - 1}. \quad (51)$$

Notice that within the circle of convergence of the Bernoulli series, the second member is equivalent to the functional formulation of the third one; we will use this interchangeably below keeping in mind the convergence constraint $|y| < 2\pi$. To bring attention to the use of the Bernoulli numbers for estimations, we will use again in the following code (as above for the exponential function) partial sums of the Bernoulli series, where accuracy will depend on the value of the argument y and the number of terms used.

Two convergence processes are in consideration here: the *series* in Equation 51, and the *recursion* of Equation 49. Convergence criteria for recursion are developed in an ancillary paper using the functional formulation of Equation 51. A critical number D is found such that if $d \leq D$, the recursion will converge, otherwise it will diverge.⁴⁶ The critical number is within the circle of convergence of the series of Equation 51: $D < 2\pi$. This makes it possible to demonstrate convergence and divergence of recursions using approximations computed by use of the Bernoulli numbers in the series formulation.

The following code uses partial sums of the Bernoulli series to estimate $\ln(d+1)$, where, to see convergence, choose $d < 3.9$. But be aware that convergence and divergence may require thousands of iterations for values of d too close to D , and KC time limits may cause cancellation.⁴⁷

The key input parameter is d , and when convergent the code will estimate $\ln(d+1)$. Accuracy depends on the number of Bernoulli numbers used (*numB*) and the number of recursions (*recursions*). For assessment of the results, the last two values produced by the code show the final value computed for y followed by the KC's accurate calculation of $\ln(d+1)$ for assessment.

As an example, start with $d = 1$ ($a = 2$) with *numB* = 14 and *recursions* = 50 to find

$$\ln(2) = \mathbf{0.6931471805598889474459},$$

⁴⁶A *Recursive Algorithm for Calculating Natural Logarithms using Bernoulli Numbers* is in preparation. Note that all solutions of Equation 51 occur where the curve meets the line $y = x$, and there $y = \ln(d+1)$. The critical point occurs for the curve defined by $d = D$ where the intersection is orthogonal. The critical values are $D = 3.9215536345675050924567623117544787\dots$, and the value of x there is $x = 1.5936242600400400923230418758751602418\dots = \ln(D+1)$.

⁴⁷Using 50,000 iterations for the values $d = 3.92155364$ and $d = 3.92155365$ bracketing D , the upper and lower limit of the recursion values for each differed by about 1.61 and 1.68, unsatisfactory for estimation.

correct in the bold digits. Notice that full 12-decimal accuracy occurs around the 33rd recursion. The theory shows that greater accuracy can be attained using more Bernoulli numbers.

For additional trials, try $d = 1$ ($a = 2$), $d = 3$ ($a = 4$) to see how many iterations are needed to demonstrate convergence. For cases with $d = 4$ ($a = 5$) and $d = 5$ ($a = 6$), the recursion diverges (but notice the oscillation in values). Compare results for $d = 3.5$ and $d = 5$ with Panels A and B of Figures 2 and 3. If for any iteration in the code $y \geq 2\pi$, the Bernoulli series diverges and execution of the code is discontinued — a zebra strip (1010101010101) marks the cutoff:

(Reminder: set the KC Editor to *Textarea*. The 22 decimal *Digit* is sufficient but larger values when more Bernoulli numbers are used will reveal greater accuracy.)

```

for(m=1; m <= recursions; m = m+1){
Sum=0;
for(j=0; j<=numB-1; j=j+1){
Sum = Sum + bernoulli(j)/j! * (y)^(j);
if( y > 2*pi ) { print( 10101010101, j, y ); exit;}
}
y =d*Sum;
print( y );}
print(1000000000001);
print( y, ln(d+1));

```

The code is impractical. It would be difficult to test it without a computer, and its applicability is limited to logarithms for numbers $a \in (1, D + 1)$. Other logarithms outside this range can be found by combining results. For example, $\ln(10)$ can be estimated by doubling $\ln(\sqrt{10})$. Then $\ln(5)$ is the difference between $\ln(10)$ and $\ln(2)$. Adding $\ln(2)$, $\ln(3)$ and $\ln(10)$ gives $\ln(60)$, and so forth. This strategy, using practical formulations, was used by Briggs in his first computations for base-10 logarithms.

Figures 2, 3 and 4 illustrate patterns of convergence and divergence for recursion of Equation 51. The vertical value marks the value of the recursion and the horizontal value marks the count. For example, in each plot the evaluations begin at count 1, and the maximum count varies among the plots to illustrate progress of the recursion. The analytical solution was used in the Keisan Calculator for these three figures, not the truncated power series.

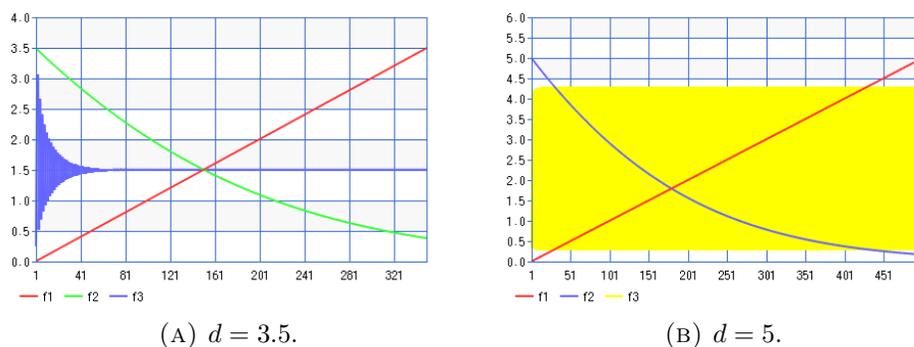


FIGURE 2. Convergence occurs only if $d \leq D$. Panel A shows the curve of the generating function from Equation 51, its recursion, and the line $x = y$. The unique solution $\ln(d + 1)$ always occurs at the intersection of the generating function and $x = y$. Panel B shows the densely colored recursive oscillation between polar values for $d > D$. See text for explanation and code to generate the figure.

The code to generate Figure 2 follows; it uses the functional form of Equation 51 because of simplicity, but the preceding code that uses only Bernoulli numbers could have been used instead because the values of d in the examples lie within the circle of convergence for the Bernoulli series.⁴⁸ Input parameters are d and an initial value for y . The values of d used were 3.5 illustrating convergence and 5 for divergence. Figure 3 shows a more detailed illustration of the oscillations, which in Figure 2 blend together in a solid color. The starting value $y = 5$ was used in both cases; this choice is arbitrary, but it will determine the shape of the convergence pattern as can be seen in comparing Panel B of Figure 3 with Figure 4. To produce examples like Figure 2, execute the code in the Keisan Calculator then press the *Chart* button:

```
iterates=int( 100*d );
for( j=1; j=<iterates; j=j+1 ) {
  x = j/100; y = d*y/(e^y-1); z = d*x/(e^x-1);
  println( j, x, z, y ); }
```

Figure 3 shows the initial portions of the plots in Figure 2 to emphasize the oscillations that occur in cases of convergence and divergence.

⁴⁸See Section 9.1 and the Bernoulli series in Equation 46.

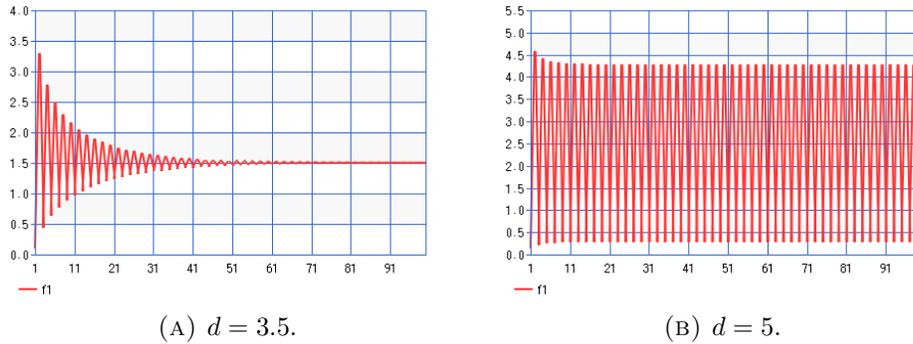


FIGURE 3. These plots show the initial behavior of recursion using Equation 51. They illustrate converging and diverging oscillatory behavior for $d \leq D$ and $d > D$.

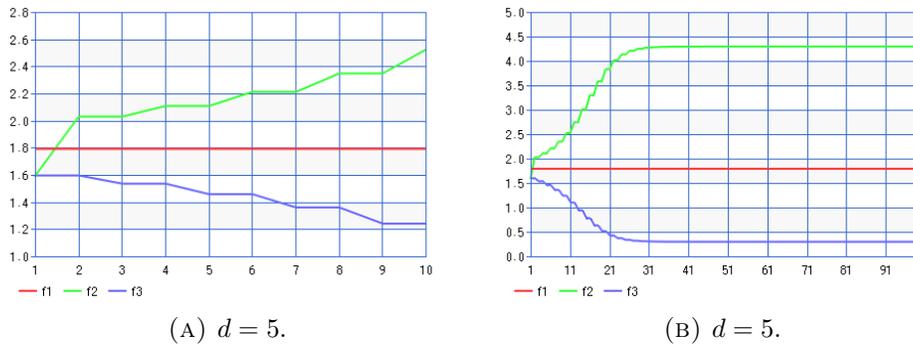


FIGURE 4. Continuing example of divergence for $d = 5$. Upper and lower oscillations are shown as they stabilize at limiting polar values above and below the line for $\ln(d + 1)$.

Figure 4 separates the alternate values of the recursion for the case $d = 5$ to illustrate convergence to the opposing polar limits.

Panel A of Figure 4 provides a hint that oscillation occurs in converging to a pole as it does in the case of convergence in Panel A of Figure 3 .

Results of Newton’s method are shown next. Polar limits were found for the case $d = 5$ for the oscillations around the line $y = \ln(5 + 1)$. It produced the limiting poles x and y shown. Their accuracy as oscillating opposites is demonstrated in computations performed by the Keisan Calculator using the standard *Digits* = 22:

$$\begin{aligned}
 x &= 4.29382403889703194825 \\
 dy/(e^y - 1) &= 4.293824038897031948244 \quad (d = 5) \\
 \ln(5 + 1) &= 1.791759469228055000813 \\
 y &= 0.29716678415392511284 \\
 dx/(e^x - 1) &= 0.297166784153925112838 \quad (d = 5)
 \end{aligned}$$

10. CONCLUSION

This work is based primarily on expositions of the logarithms of John Napier and Henry Briggs by Denis Roegel and Ian Bruce [36, 8]. Roegel describes Napier's methods and applies them to computing corrected tables, and he does the same thing for Briggs. Roegel provides an extensive bibliography. Bruce translates and annotates Napier's and Briggs' original documents and postumous editions. See the Bibliography for the numerous citations of Roegel and Bruce, where some are listed under *Napier*.⁴⁹ This work includes codes written for immediate execution on Casio's free online Keisan Calculator to produce any of the entries of the tables in Roegel's *Napier's ideal construction of the logarithms* and to demonstrate the recursion formula for computing natural logarithms using Bernoulli numbers.

I have relied on several other authors cited throughout the study. Goldstine provides a succinct modern description of Napier's logarithm as well as an homage to the pioneering numerics of Briggs [21]. Havil explains, with step-by-step arithmetic, Napier's actual methods for calculating the numbers and logarithms in his tables [22].⁵⁰ Burn gives a chronology relevant to Napier's logarithms [13]. Naux's history of

⁴⁹Roegel has published reconstructions of the important trigonometric tables extant at the time of Napier in his *Loria Collection of Mathematical Tables* [39]. Bruce is a prolific author of translations and commentaries alongside original texts in his *Mathematical Works of the 17th & 18th Centuries* [10].

⁵⁰Havil's book, written in the Year of the Quadcentenary celebration of Napier's *Descriptio*, is more than a gloss on Napier's *Constructio*. The title, *John Napier — Life Logarithms and Legacy* declares its reach, including 12 appendixes with two abbreviated histories of Scotland and the Reformation, discussions of devices like the slide rule, the Logarithms of Briggs and Bürgi, and Kepler's use of Napier in the Rudolphine Tables. The chapter "Revelation and Recognition" discusses another less-known side of Napier as apocalyptic theologian; his *Plaine Revelation* was a thoroughgoing anti-papist interpretation of the Book of Revelations predicting the end times. Circulated throughout Europe, it was indicative of the tumultuous religious upheaval of the time.

logarithms from the earliest beginnings to Napier and onward to developments in the era of analysis is encyclopedic [30]. Truemper's recent book, *The Daring Invention of Logarithm Tables: How Jost Bürgi, John Napier and Henry Briggs Simplified Arithmetic and Started the Computing Revolution* explains Bürgi's logarithm in elementary language [44].⁵¹

My purpose has been different from any of these authors. I have shown how direct algebraic formulations of Napier's and Briggs' arithmetic techniques and their limits according to elementary calculus demonstrate how remarkably their work forshadows modern analysis.⁵² In some places I have dealt freely with ideas that to me seemed close enough to the methods of Napier and Briggs to be within the spirit of their methods.

Here are brief discussions of several themes in the paper:

Napier defined his logarithm in terms of a kinematic traveling point that moves at a continuously decelerating rate equal to its distance from the endpoint. From this he devised a discretized method, resulting in a function closely related to the natural logarithm. The term "function" was not used at the time, but Napier's table of logarithms was a prime early example.

Briggs continued work after meeting with Napier by devising methods to produce the more convenient base-10 logarithm. How much of the ideas published later by Briggs were already known to Napier is a question for historians, but Napier already had the idea of a base-10 logarithm at the time he met with Briggs and expressed a desire to improve the accuracy of his tables and correct errors.

Napier's method used a summation of travel times across discretized sub-intervals of $[L, 0)$, determined by the quasi-geometric progression LR_1^k , $k = 0, 1, 2, \dots$ with $L = 10^7$, $R_1 = 0.9999999$, that can be interpreted as Riemann sums, and he calculated error bounds, implying that he could control accuracy by choice of L and R_1 . Having tied the speed of travel to the distance of the traveler to the terminal point, his

⁵¹Two additional references were brought to my attention after the research for this paper was completed: *The Life and Works of John Napier* by Brian Rice et al, provides commentary and English translations of the known works [33]; Napier's *Plaine Revelation* is covered in full. See also "Napier revisited: A new look at the computation of his logarithms," by Joachim Fischer and Bärbel Rues [24].

⁵²Truemper's Note #106, referenced in the section *Napier's Model*, uses modern calculus to estimate time by integrating from y to $y + dy$: $T(z) = \int_0^z \frac{dy}{(N - y)}$. The chain of reasoning from Equations 32 and 33 to the modern integral is very short [44, Pp. 66, 127].

numerics approximated the ideal kinematic traveler, no matter what quasi-geometric progression he used so long as the selection of points was sufficiently refined, and it allowed interpolations and adjustments consistent with the speed requirements. The implication is that Napier had essentially discovered and characterized the integral $\int dx/x$, and evaluated it in fine detail some thirty years before Gregory St. Vincent had introduced logarithms into the language of calculus.

Napier's method for computing the average speed of the traveling-point across discretized intervals was essentially a difference quotient with limiting value equal to the distance of the traveler from the terminal point. In selecting that limit as the ideal speed for his traveler, it can be said that he had defined instantaneous speed as the limit of a difference quotient. The rules for finding products, divisions and powers of numbers using logarithms allowed manipulations of the numbers as a field, and that implies associated rules for exponents not formulated in convenient symbols until the time of Wallis and Newton [4, *Indics*, p. 43].

Napier popularized the use of the decimal system by adroit usage.⁵³ He used a method of successive subtractions to duplicate and simplify progressive multiplications. He presented logarithms by arranging that the significant figures would stand before the decimal point, facilitating multiplication and division by shifts of the decimal point [36, Sec 5, *Decimal fractions*, pp. 27-28]. Napier understood he could make logarithms arbitrarily accurate by choice of two parameters denoted in the text as L and R_1 . He specified the possibility of using $L = 10^8$, $R_1 = 0.99999999$, and so doing he could increase the number of digits of accuracy arbitrarily [36, Sec. 7.1]. This implies limiting values of definite but never-ending decimal numbers, *real* numbers. Napier couldn't know that there are too many real numbers for all of them to be defined by an algorithm, but imagining infinite decimals as defining all numbers anticipated the real number system.

Napier recognized errors in his tables and discussed with Briggs how to make a better logarithm using the base 10 [28, 35]. Briggs' *golden rule* for computing base-10 logarithms is closely allied with Napier's methods. It is plausible that the golden rule was a joint discovery during their consultation. Algorithms based directly in Napier's and Briggs' work can be used to compute base- e and base-10 logarithms,

⁵³Truemper reviews the early history of the decimal system in European mathematics [44, Pp. 20–21]. See also the discussion of Napier's use of the decimal system in William Rae Macdonald's translation into English of Napier's *The Construction of the Wonderful Canon of Logarithms* with a catalog of Napier's works and appendixes listed here under Napier [27].

and for computing e and the exponential function, to high accuracy. For example:

$$\begin{aligned} e &\approx 2.7182818284590451, \\ \ln 2 &\approx 0.6931471805599453, \\ \log_{10} e &\approx 0.434294481903251. \end{aligned}$$

The latter term, computed by Briggs, was the multiplicative factor (we designated as M) he used in his golden rule. Briggs method for calculating M is based on the fact that,

$$\lim_{x \rightarrow 0} \frac{x}{10^x - 1} = \log_{10} e.$$

More generally, a conversion to base- a logarithms can use approximations for,

$$\lim_{x \rightarrow 0} \frac{x}{a^x - 1} = \ln a.$$

Setting $a = e$ in the preceding form yields the generating function for the Bernoulli numbers:

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k,$$

and we found,

$$\frac{x}{a^x - 1} = \frac{1}{\ln(a)} \cdot \sum_{j=0}^{\infty} \frac{B_j}{j!} (x \ln(a))^j. \quad (52)$$

The generating function is well known, so to see it connected with the exponential function e^x is not new. But it may come as a surprise to find the Bernoulli numbers lodged with the definition of the natural logarithm. A recursive code derived from Equation 52 for $x = 1$ was presented in the text for calculating natural logarithms using the Bernoulli numbers.

The classic calculus text by Courant and John provides examples of ideas that could be adapted straight out of Napier and Briggs. They use geometric subdivision for integrating the area under a parabola, similar to Napier's discretization for producing logarithms under a rectangular hyperbola. Their limit formula for the natural logarithm is equivalent to one we derived from the limit of Riemann sums used by Napier, as does their limit formulas for the exponential function and for e . They express Briggs golden rule for calculating base-10 logarithms without mentioning any of these connections [15, pp. 132,145,149,154].

The number of modern conceptions Napier and Briggs touched upon using arithmetic is astonishing. They probed deeply into a new world, acquiring insight through countless hours of calculations. Although their aims were practical and achievements great— to improve computational methods for astronomers and navigators in an age of naval rivalry — their work had equally important implications for theoretical analysis.

ONTOLOGY

- **ln** and **log** denote the base- e and base-10 logarithms, and λ_N and Λ are used to denote the idealized Napier logarithm and the calculus logarithm postulated by Napier. Because the theory of Napier's logarithm is set up here in a general framework — dubbed a *Napier-style* logarithm — $\lambda(x)$ without the subscript N is used to denote a general Napier-style logarithm.
- **Archimedean logarithms** index a geometric progression with its exponents: $\log_a a^k = k$, $k = 0, 1, 2, \dots$. Napier's logarithm was more complicated than an Archimedean logarithm.
- Progressions of the type LR^k , $k = 0, 1, 2, \dots$ are distinguished according to whether $L = 1$ or $L \neq 1$: the former is a conventional **geometric** progression resulting in an Archimedean logarithm, and the latter is **quasi-geometric** that yields a different type of logarithm, notably Napier's. The shorthand LR is used to denote the full progression.
- The **stride** of a progression is the length of its first interval: $L - LR$. The density of two progressions is compared by the ratio of their strides.
- Napier's **Canonical Table**: the table published originally by Napier of his logarithms for the sines of all the degrees and minutes from 1 – 90. (See Numerics, Section 4.)
- **Coordinate Values vs Name of Moving Point**: For example, x can be the value of a variable named x , and x can be the name of a moving point that has values also denoted by x : this was metaphysical for Napier because he postulated instantaneous motion without calculus.
- **Traveling points** are points that move. Napier used kinematics to define his logarithm. His model for a logarithm was defined in terms of a point that moves at a speed equal to its distance from the origin, a motion that can be determined definitively only by calculus, so it is referred to as the *calculus* point, the *model* point or *continuous* point depending on context. Napier had to use discretized motion to approximate the model arithmetically, so *discrete* traveler refers to a point that moves at constant speeds over finite intervals. A *log* point travels at a constant rate of $L = 10^7$ units of distance in 1 unit of time. The *idealized* traveler refers to a point that moves in discrete steps corresponding to Roegel's corrected logarithms.

These terms are used throughout the text, where their distinctive meanings should become clear by usage and in turn help to clarify the text.

- **Ambiguities** in the word **logarithm**: here it can mean the logarithm of a particular number to a particular base, or it can mean the entire canon of logarithms to a particular base, or it can even mean an entire category of logarithms of different kinds.
- **Discretized logarithm** or **discrete logarithm** refers to the discretization of the function $y = 1/x$ to approximate $\int dx/x$ using arithmetic. It does not refer to the discrete logarithm of cryptology.

VITA

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INDEX

- C , the basis for Napier's choice for c , 17
- D , the cutoff constant for
 - convergence of the recursion algorithm for natural logarithms, 50
- ln: reasons to call it natural, 44
- $\sigma_M(\bar{x}) = \alpha x_1 + \beta x_2$, ($\alpha + \beta = 1$), 15
- c , Napier's settled value, 17
 - determines Napier's logarithm, 19
- d , the coefficient of the generating function for the Bernoulli numbers, 49
 - see D , 49
- $e = 2.7182818284590451$
 - approximation by powers of 2, 44
 - 6,900,000, 27
- algebra, 7
- ambiguity, 59, 60
 - the word *logarithm*, 60
- Archimedes, 7
- basic notation, 11, 59
- basic ontology, 11, 59
- Bernoulli numbers, 8
 - generating function, 8, 47
 - Bernoulli series, 9, 48, 50
 - functional formulation, 50
 - in literature, 47
 - Seki, 47
- Bernoulli series, 47
 - Bernoulli numbers
 - generating function, 47
- Bernoulli, Jacob
 - see Bernoulli numbers, 8
- bootstrapping, 29
- Bower, 7
- Briggs, 7
 - $\ln 10 = 2.302585092994045$, 42
 - $\log e = 0.434294481903251$, 43
 - algebraic representation of Briggs' method, 47
 - formal similarity to generating function for Bernoulli numbers, 47
 - see generating function, 47
 - base-10 logarithms, 9
 - discovery of the golden rule, 44
 - code for generating Briggs' table, 45
 - excerpts from Briggs' table, 45, 46
 - golden rule, 8, 43
 - collaboration with Napier, 9
 - generalization, 44
 - life before meeting Napier, 40
 - multiplier, 10, 43
 - use of square roots, 9
- Bruce, 7
 - e first time in history when, 9
- Burn, 7
- calculus, 7
- calculus traveler
 - see traveling points, 11, 59
- Canonical Table, 24, 34
 - Construction, 34
 - see logarithm, 59
- Cantor, Moritz see Gauss, 7
- Casio
 - see Keisan Calculator, 20
- Coordinate value vs name of moving point, 14, 16, 59
- Decimal points, 25
- discrete traveler
 - see traveling points, 11, 59
- error approximation
 - see logarithm
 - Napier's logarithm λ_N , 19
- Estimate of e
 - Euler's, 49
 - using Bernoulli numbers, 49
 - using exponential series, 49
- Euler, 8, 9, 41, 47
 - Bernoulli numbers, 47
 - Estimate of e , 49
- Figures
 - Converging and Diverging Recursions, 51

- Oscillatory Recursions, 53
- Fundamental Interval
 - See Napier-style logarithms, 14
- Fundamental Partition
 - see Napier-style logarithms, 14
- Fundamental Progression
 - see Napier-style logarithms, 14
- Galileo, 8
- Gauss
 - interest in logarithms, 7
- generating function for Bernoulli numbers, 47
 - discovered by Euler, 47
 - see Bernoulli numbers, 8, 49
 - used to compute e^x
 - code, 48
 - used to compute natural logarithms, 49
 - code, 49
- generic logarithm λ
 - see logarithm, 12
- geometric vs quasi-geometric progression, 59
- Goldstine, 7
- Gregory St. Vincent, 8
- Guicciardini
 - Early calculations of $\log e$, 9
- Havil, 7, 24, 54
- Huygens
 - calculation of $\log e$, 9
- ideal traveler
 - see traveling points, 14
 - speed of, 14
- idealized traveler
 - see traveling points, 11, 59
- Junctures: estimated gaps between tables, 28
- Keisan Calculator, 20
- Kepler
 - Rudolphine Tables, 37, 54
 - Trigonometry, 37
- L'Hopital's Rule, 13, 42
- log traveler
 - see traveling points, 11, 59
- constant speed of, 11, 59
- logarithm
 - accuracy assessment, 26
 - basic expressions, 12–14, 17, 38–40, 42, 46
 - defined by a kinematic point, 12
 - generic model, 12
 - Kepler, 37
 - Napier's Canonical Table, 59
 - Napier's logarithm λ_N , 19
 - as difference quotient, 16
 - as Riemann sum, 37
 - Basic calculations mentioned, 36
 - Napier's logarithm defined
 - parameters $\sigma_M(\bar{x})$, 15
 - parameters L_N, R_1 , 19
 - the defining coefficient c , 19
 - Napier's tables discussed
 - Short Table, 31
 - Short Table Problem, 31
 - Table 1, 27
 - Table 2, 28
 - Table 3, 28
 - Napier-style logarithms, 14
 - Stages of Construction, 14
 - proof of convergence by Riemann integration, 45
 - stages of construction
 - Napier-stye logarithms, 14
- Napier, 7
 - discretized motion, 11
 - base-10 logarithms, 9
 - instantaneous motion, 10
 - little is known about his life, 40
 - militant theology in troubled times in Scotland, 40
 - see logarithm, 7
 - antecedents, 7
 - base-10 logarithms, 9
 - The Construction of the Wonderful Canon of Logarithms, 8
 - The Description of the Wonderful Canon of Logarithms, 8
- Napier-style logarithms
 - see logarithm, 14

- Napier-style multiplication:
 - subtraction using decimal arithmetic, 25
- Naux, 7
- Newton
 - calculation of $\log e$, 9
- partition refinement
 - see root extraction, 13
 - taken to the limit, 13
- Pitiscus
 - Trigonometry, 37
- Probst, Siegmund
 - Leibniz calculation of e , 9
- proportionate intervals
 - see traversal times, 19
- quasi-geometric vs geometric progression, 59
- Raugh, Michael
 - Leibniz calculation of e , 9
- Readers Guide
 - see Numerics
 - codes: reference tips for reading, 5
- recursion algorithm for computing
 - natural logarithms, 49
 - functional formulation, 50
 - see generating function for Bernoulli numbers, 49
- Regiomontanus
 - Trigonometry, 37
- Reich, Karin
 - see Gauss, 7
- Rheticus
 - Trigonometry, 37
- Riemann Sums, 8
- Roegel, 7
- root extraction
 - accuracy improvement by, 14
- Short Table problem, 31
 - see logarithm
 - Napier's tables discussed, 31
- speed of discretized traveler — for Napier, $\alpha = \beta = 0.5$.
 - see discretized traveler, 15
- stages of construction
 - Napier-style logarithms, 14
- Symbols for ratios: R and $r = 1 - R$, 25
- synopsis, 29, 30, 32
- traveling points
 - continuous traveler. calculus traveler, discrete traveler, idealized Napier traveler, log traveler, 11, 59
- traversal times
 - identical on proportionate intervals, 17, 19
- Trigonometry, 37
 - Kepler, 37
 - Pitiscus, 37
 - Regiomontanus, 37
 - Rheticus, 37
- Van Brummelen, 37
- Variable Names vs Coordinate Values, 59

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